Trends in Mathematics

Piotr Kielanowski
Anatol Odzijewicz
Emma Previato
Editors

Geometric Methods in Physics XXXVI

Workshop and Summer School, Białowieża, Poland, 2017

Birkhäuser

Birkhäuser

Trends in Mathematics

Trends in Mathematics is a series devoted to the publication of volumes arising from conferences and lecture series focusing on a particular topic from any area of mathematics. Its aim is to make current developments available to the community as rapidly as possible without compromise to quality and to archive these for reference.

Proposals for volumes can be submitted using the Online Book Project Submission Form at our website www.birkhauser-science.com.

Material submitted for publication must be screened and prepared as follows:

All contributions should undergo a reviewing process similar to that carried out by journals and be checked for correct use of language which, as a rule, is English. Articles without proofs, or which do not contain any significantly new results, should be rejected. High quality survey papers, however, are welcome.

We expect the organizers to deliver manuscripts in a form that is essentially ready for direct reproduction. Any version of TeX is acceptable, but the entire collection of files must be in one particular dialect of TeX and unified according to simple instructions available from Birkhäuser.

Furthermore, in order to guarantee the timely appearance of the proceedings it is essential that the final version of the entire material be submitted no later than one year after the conference.

More information about this series at http://www.springer.com/series/4961

Piotr Kielanowski • Anatol Odzijewicz
Emma Previato
Editors

Geometric Methods in Physics XXXVI

Workshop and Summer School, Białowieża, Poland, 2017

Birkhäuser

Editors
Piotr Kielanowski
Departamento de Física
CINVESTAV
Ciudad de México, Mexico

Anatol Odzijewicz
Institute of Mathematics
University of Białystok
Białystok, Poland

Emma Previato
Department of Mathematics and Statistics
Boston University
Boston, MA, USA

ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-3-030-01155-0 ISBN 978-3-030-01156-7 (eBook)
https://doi.org/10.1007/978-3-030-01156-7

Library of Congress Control Number: 2018968410

Mathematics Subject Classification (2010): 01-06, 01A70, 20N99, 58A50, 58Z05, 81P16, 33D80, 51P05

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Contents

Part V: Differential Geometry and Physics

Part VI: Topics in Spectral Theory

Part VII: Representation Theory

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, ix–x

Preface

This book contains a selection of papers presented during the Thirty-Sixth "Workshop on Geometric Methods in Physics" (WGMPXXXVI) and abstracts of lectures given during the Sixth "School on Geometry and Physics", both of which took place in Białowieża, Poland during the summer of 2017. These two coordinated activities constitute an annual event. Information on previous and upcoming schools and workshops, and related materials, can be found at the URL: http://wgmp.uwb.edu.pl.

The volume opens with a chapter containing papers presented at the special session organized by A. Odzijewicz, G. Goldin, J.-P. Antoine, T. Bhattachryya, J.P. Gazeau, J. Harnad, and F. Schroeck, dedicated to the memory of S. Twareque Ali. Professor Ali, who died suddenly in 2016, was an active member of the Organizing Committee of our workshop for many years. There follow chapters on "Noncommutative Geometry", "Quantization", "Integrable Systems", "Differential Geometry and Physics", "Topics in Spectral Theory", "Representation Theory" and "Special Topics", with papers based on the talks and posters presented at the workshop. The final chapter contains extended abstracts of the lecture series given during the "Sixth School on Geometry and Physics".

The WGMP is an international conference organized each year by the Department of Mathematical Physics in the Faculty of Mathematics and Computer Science of the University of Białystok, Poland. The main subject of the workshops, consistent with their title, is the application of geometric methods in mathematical physics. They frequently include studies of noncommutative geometry, Poisson geometry, completely integrable systems, quantization, infinite-dimensional groups, supergroups and supersymmetry, quantum groups, Lie groupoids and algebroids, and related topics. Participation in the workshops is open; the participants typically consist of physicists and mathematicians from countries across several continents, who have a wide spectrum of interests.

The Workshop and School are held in Białowieża, a village located in the east of Poland near the border with Belarus. Białowieża is situated in the center of the renowned Białowieża Forest. This forest, shared between Poland and Belarus, is one of the last remnants of the primeval forest that covered the European Plain before human settlement. It has been designated a UNESCO World Heritage Site. The peaceful atmosphere of a small village, together with natural beauty, affords a unique environment for learning, cooperation, and creative work. As a result the core participants in the WGMPs have become a strong scientific community, as reflected in this series of Proceedings.

The Organizing Committee of the 2017 WGMP gratefully acknowledges the financial support of the University of Białystok and the Centre de Recherche Mathématique (Canada). Last but not least, credit is due to early-career scholars and students from the University of Białystok, who contributed limitless time and effort to setting up and hosting the event as well as participating actively in the scientific activities.

The Editors

Participants of the XXXVI WGMP
(Photo by Tomasz Goliński)

Part I

Quantum Mechanics and Mathematics
Twareque Ali in Memoriam

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 3–8

In Memory of S. Twareque Ali

Gerald A. Goldin

Abstract. We remember a valued colleague and dear friend, S. Twareque Ali, who passed away unexpectedly in January 2016.

Prefatory note
I wrote the following tribute in 2016 to my close friend and colleague S. Twareque Ali, for publication in the Proceedings of the 35th Workshop on Geometric Methods in Physics. It seems appropriate to reprint it here, in connection with the Special Session at the 36th Workshop devoted to his memory. – *Gerald Goldin*

S. Twareque Ali in Białowieża.

1. Remembering Twareque

Syed Twareque Ali, whom we all knew as Twareque, was born in 1942, and died in January 2016. This brief tribute is the second one I have prepared for him in a short period of time. With each sentence I reflect again on his extraordinary personality, his remarkable career – and, of course, on the profound influence he had in my life. Twareque was more than a colleague – he was a close friend, a confidant, and a teacher in the deepest sense.

When I remember Twareque, the first thing that comes to mind is his laughter. He found humor in his early changes of nationality: born in the British Empire, a subject of George VI, Emperor of India, he lived in pre-independence India, became a citizen of Pakistan, and then of Bangladesh – all without moving from home. Eventually he became a Canadian citizen, residing with his family in Montreal for many years.

Twareque's laughter was a balm. In times of sadness or disappointment, he was a source of optimism to all around him. His positive view of life was rooted in deep, almost unconsciously-held wisdom. Although he personally experienced profound nostalgia for those lost to him, he knew how to live with joy. He could laugh at himself, never taking difficulties too seriously.

And he loved to tell silly, inappropriate jokes – which, of course, cannot be repeated publicly. He introduced me to the clever novels by David Lodge, *Changing Places*, and *Small World*, which satirize the academic world mercilessly. In Lodge's characters, Twareque and I saw plenty of similarities to academic researchers we both knew in real life – especially, to ourselves.

Twareque was fluent in several languages, a true "citizen of the world." He loved poetry, reciting lengthy passages from memory in English, German, Italian, or Bengali. In Omar Khayyam's *Rubaiyat*, translated by Edward Fitzgerald, he found verses that spoke to him. These are among them:

> ...
>
> Come, fill the Cup, and in the Fire of Spring
> The Winter Garment of Repentance fling:
>   The Bird of Time has but a little way
> To fly – and Lo! the Bird is on the Wing.
>
> ...
>
> A Book of Verses underneath the Bough,
> A Jug of Wine, a Loaf of Bread – and Thou
>   Beside me singing in the Wilderness
> Oh, Wilderness were Paradise enow!
>
> ...
>
> The Moving Finger writes, and, having writ,
> Moves on; nor all your Piety nor Wit
>   Shall lure it back to cancel half a Line,
> Nor all your Tears wash out a Word of it.

2. A short scientific biography

Twareque obtained his M.Sc. in 1966 in Dhaka (which is now in Bangladesh). He received his Ph.D. from the University of Rochester, New York, USA, in 1973, where he studied with Gérard Emch. Professor Emch remained an inspiration to him for the rest of his life, and Twareque expressed his continuing gratitude. In 2007, together with Kalyan Sinha, he edited a volume in honor of Emch's 70th birthday [1]; and in 2015, he organized a memorial session for Emch at the 34th Workshop on Geometric Methods in Physics in Białowieża.

After earning his doctorate, Twareque held several research positions: at the International Centre for Theoretical Physics (ICTP) in Trieste, Italy; at the University of Toronto and at the University of Prince Edward Island in Canada; and at the Technical University of Clausthal, Germany in the Arnold Sommerfeld Institute for Mathematical Physics with H.-D. Doebner. He joined the mathematics faculty of Concordia University in Montreal as an assistant professor in 1981, becoming an associate professor in 1983 and a full professor in 1990.

During his career as a mathematical physicist, Twareque achieved wide recognition for his scientific achievements. He was known for his studies of quantization methods, coherent states and symmetries, and wavelet analysis. A short account cannot do justice to his accomplishments; the reader is referred for more detail to two published obituaries from which I have drawn [2, 3], and asked to forgive the many omissions. I cannot do better than to quote the summary in another tribute I wrote [4]:

"During the 1980s, Twareque worked on measurement problems in phase space, and on stochastic, Galilean, and Einsteinian quantum mechanics [5, 6] Then he began to study coherent states for the Galilei and Poincaré groups, and collaborated with Stephan de Bièvre on quantization on homogeneous spaces for semidirect product groups.

"There followed his extensive, long-term, and indeed famous collaboration with Jean-Pierre Antoine and Jean Pierre Gazeau, focusing on square integrable group representations, continuous frames in Hilbert space, coherent states, and wavelets. Their joint work culminated in publication of the second edition of their book in 2014 – a veritable treasure trove of mathematical and physical ideas [7–10].

"Twareque's work on quantization methods and their meaning is exemplified by the important review he wrote with M. Engliš [11], and his work on reproducing kernel methods with F. Bagarello and Gazeau [12]."

Twareque's contributions of time and effort helped bring a number of scientific conference series to international prominence. Foremost among these was the Workshop on Geometric Methods in Physics (WGMP) in Białowieża (organized by Anatol Odzijewicz). Twareque attended virtually every meeting from 1991 to 2015, where we would see each other each summer. He was a long-time member of the local organizing committee, and co-edited the *Proceedings* volumes. Other conference series to which he contributed generously of his energy included the University of Havana International Workshops in Cuba (organized by Reinaldo Rodriguez

Ramos), and the Contemporary Problems in Mathematical Physics (Copromaph) series in Cotonou, Benin (organized by M. Norbert Hounkonnou).

He was also an active member of the Standing Committee of the International Colloquium on Group Theoretical Methods in Physics (ICGTMP) series. Twareque and his wife Fauzia came together to the 29th meeting of ICGTMP in Tianjin, China in 2012. She attended the special session where Twareque (to his surprise) was honored on the occasion of his 70th birthday. Their son Nabeel, of whom he always spoke with great pride, practices pediatric medicine in Montreal.

Twareque was a deep thinker, who sought transcendence through ideas and imagination. The truths of science and the elegance of mathematics in the quantum domain were part of the mysterious beauty for which he longed – a longing shared by many great scientists, a longing that we, too, share.

S. Twareque Ali in thought at WGMP XXXIII,
July 2, 2014. Photograph by G.A. Goldin.

As profoundly as Twareque cared about understanding the meanings of scientific ideas, he cared equally about inspiring his students to succeed. He helped them with personal as well as professional issues. As Anna Krasowska and Renata Deptula, two of his more recent students who came from Poland to work with him, wrote [2], "If anything in our lives became too complicated it was a clear sign we needed to talk to Dr. Ali. Every meeting with him provided a big dose of encouragement and new energy, never accompanied with any criticism or judgment." This was Twareque's gift – to understand, to inspire, to give of himself.

Twareque died suddenly and unexpectedly January 24, 2016 in Malaysia, after participating actively in the 8th Expository Quantum Lecture Series (EqualS8) – indeed, doing the kind of thing he loved most.

3. Concluding thoughts

Twareque believed passionately in world peace, in service to humanity, and in international cooperation. He understood the broad sweep of history. His tradition was Islam, as mine is Judaism, and although neither of us adhered to all the rituals of our traditions, we shared an interest in their history, their commonalities, and their contributions to world culture. We even researched correspondences between the roots of words in Arabic and Hebrew. On a first visit to Israel for a conference in 1993, we visited Jerusalem together. Twareque did much to aid the less privileged and less fortunate – in the best of our traditions, often anonymously.

Often one closes a retrospective on someone's life with a sunset, marking the ending of day and the beginning of night. My choice for Twareque is different. He is someone who joined a scientific mind with a spiritual heart, and for Twareque, the park and the forest in Białowieża were at the center of his spirituality. So I imagine him looking at us, even now, and marveling at the beauty of heavenly clouds reflected in the water.

Reflection of the heavens in Białowieża Park, July 4, 2013.
Photograph by G.A. Goldin.

Acknowledgment

I am deeply indebted to Twareque's family, friends, students, and colleagues. Thanks to the organizers of the 35th Workshop on Geometric Methods in Physics for this opportunity to honor and remember him.

References

[1] S. Twareque Ali, Kalyan B. Sinha (eds.), *A Tribute to Gérard G. Emch*. Hindustan Book Agency, New Delhi, 2007.

[2] J.-P. Antoine, J.P. Gazeau, G. Goldin, J. Harnad, M. Ismail, A. Krasowska, R. Deptula, R.R. Ramos. In memoriam, S. Twareque Ali (1942–2016). Le Bulletin, Centre de Recherches Mathématiques **22** (1) (2016), 9–12.

[3] F. Schroeck, J.-P. Antoine, G. Goldin, R. Benaduci. Syed Twareque Ali: Obituary. Journal of Geometry and Symmetry in Physics **41** (2016), 105–111.

[4] G.A. Goldin, *In Memoriam: Syed Twareque Ali. Proceedings of the* 31*st International Colloquium on Group Theoretical Methods in Physics*. Springer, 2017 (to be published).

[5] S.T. Ali, Stochastic localization, quantum mechanics on phase space and quantum space-time. La Rivista del Nuovo Cimento **8** (11) (1985), 1.

[6] S.T. Ali, J.A. Brooke, P. Busch, R. Gagnon, F. Schroeck, Current conservation as a geometric property of space-time. Canadian J. Phys. **66** (3) (1988), 238–244.

[7] S.T. Ali, J.-P. Antoine, J.P. Gazeau, Square integrability of group representations on homogeneous spaces I. Reproducing triples and frames. Ann. Inst. H. Poincaré **55** (1991), 829–856.

[8] S.T. Ali, J.-P. Antoine, J.P. Gazeau, Continuous frames in Hilbert space. Annals of Physics **222**, 1–37; Relativistic quantum frames, Ann. of Phys. **222** (1993), 38–88.

[9] S.T. Ali, J.-P. Antoine, J.P. Gazeau, U.A. Müller, Coherent states and their generalizations: A mathematical overview. Reviews in Math. Phys. **7** (1995), 1013–1104.

[10] S.T. Ali, J.-P. Antoine, J.P. Gazeau *Coherent States, Wavelets and their Generalizations* (and references therein). Springer, 2000; 2nd edn., 2014.

[11] S.T. Ali, M. Engliš, Quantization Methods: A Guide for Physicists and Analysts. Rev. Math. Phys. **17** (2005), 391–490.

[12] S.T. Ali, F. Bagarello, J.P. Gazeau, Quantizations from reproducing kernel spaces. Ann. Phys. **332** (2013), 127–142.

Gerald A. Goldin
Rutgers University
New Brunswick, NJ 08901, USA
e-mail: geraldgoldin@dimacs.rutgers.edu

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 9–19

Two-dimensional Noncommutative Swanson Model and Its Bicoherent States

Fabio Bagarello, Francesco Gargano and Salvatore Spagnolo

This paper is dedicated to the memory of Syed Twareque Ali

Abstract. We introduce an extended version of the Swanson model, defined on a two-dimensional noncommutative space, which can be diagonalized exactly by making use of pseudo-bosonic operators. Its eigenvalues are explicitly computed and the biorthogonal sets of eigenstates of the Hamiltonian and of its adjoint are explicitly constructed. We also show that it is possible to construct two displacement-like operators from which a family of bi-coherent states can be obtained. These states are shown to be eigenstates of the deformed lowering operators, and their projector allows to produce a suitable resolution of the identity in a dense subspace of $\mathcal{L}^2(\mathbb{R}^2)$.

Mathematics Subject Classification (2010). 81Q12; 81R30.

Keywords. Pseudo-bosons; coherent states; Swanson model.

1. Introduction

In the past twenty years or so a lot of interest arose on the so-called PT-quantum mechanics. This was mainly due to the paper in [1] where the authors introduced a manifestly non self-adjoint, but PT-symmetric, Hamiltonian with purely real (and discrete) eigenvalues. Here P and T are the parity and the time-reversal operators. The main point was that having a physical, rather than a mathematical, condition which guarantees the reality of the spectrum would be quite interesting and more natural for the physicists community. One of the very famous examples of this situation was later introduced in [2], with the Hamiltonian

$$H_\nu = \frac{1}{2}\left(p^2 + x^2\right) - \frac{i}{2}\,\tan(2\nu)\left(p^2 - x^2\right).$$

Here ν is a real parameter taking value in $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\setminus\{0\}$. This model has relevant mathematical and physical implications, and was discussed in terms of the so-called $\mathcal{D}$-pseudo bosons, [3, 4]. Here we consider a two-dimensional version of this model living in a noncommutative plane, and we show how this model can again be understood in terms of pseudo-bosons. Also, we briefly discuss what changes if we do not assume ν to be strictly real. Finally, we construct bicoherent states associated to the model and we check some of their properties.

2. Noncommutative two-dimensional harmonic oscillator with linear terms

The Hamiltonian we want to consider here, depending on two parameters ν and θ, is the following

$$H_{\nu,\theta}=\frac{1}{2\cos(2\nu)}\left\{\hat{p}_1^2\left(e^{-2i\nu}+\frac{\theta^2}{4}\,e^{2i\nu}\right)+\hat{x}_1^2e^{2i\nu}+\hat{p}_2^2\left(e^{-2i\nu}+\frac{\theta^2}{4}\,e^{2i\nu}\right)\right.$$
$$\left.+\hat{x}_2^2e^{2i\nu}+2\theta\left(\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1\right)\right\},\tag{1}$$

where the operators $\hat{x}_j$ and $\hat{p}_j$ satisfy the following commutation rules:

$$[\hat{x}_j,\hat{p}_k]=i\delta_{j,k}1\!\!1,\qquad[\hat{x}_1,\hat{x}_2]=i\theta1\!\!1,\qquad[\hat{p}_j,\hat{p}_k]=0.\tag{2}$$

The Hamiltonian $H_{\nu,\theta}$ can be seen as a reasonable two-dimensional version of the one-dimensional Swanson model discussed in [2, 3, 5], defined in a noncommutative two-dimensional plane. Here θ is the *noncommutativity parameter*, while ν is a real[1] *non self-adjointness parameter*, taking values in $I:=\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$. Whenever $\nu\in I$ is not zero, $H_{\nu,\theta}\neq H_{\nu,\theta}^\dagger$. On the other hand, $H_{\nu=0,\theta}=H_{\nu=0,\theta}^\dagger$. Moreover, if we take $\theta=0$, i.e., if we go back to a commuting plane, we see that

$$H_{\nu,\theta}=\frac{1}{2\cos(2\nu)}\left\{\hat{p}_1^2e^{-2i\nu}+\hat{x}_1^2e^{2i\nu}+\hat{p}_2^2e^{-2i\nu}+\hat{x}_2^2e^{2i\nu}\right\},$$

which is exactly the two-dimensional version of the Hamiltonian considered in [3, 5]: removing the noncommutativity (by sending θ to zero) returns the standard Swanson model, in two dimensions and without interactions. Finally, if we take $\nu=\theta=0$, $H_{\nu,\theta}$ is nothing but the Hamiltonian of a two-dimensional harmonic oscillator.

Despite of its apparently complicated expression, the operator $H_{\nu,\theta}$ can be diagonalized in a rather simple way, by making use of the $\mathcal{D}$-pseudo bosons introduced by one of us (F.B.), and widely analyzed in [4]. In fact, let:

$$\begin{cases}A_1=\frac{1}{\sqrt{2}}\left(\hat{x}_1e^{i\nu}+\frac{\theta}{2}\,\hat{p}_2e^{i\nu}+i\hat{p}_1e^{-i\nu}\right),\ A_2=\frac{1}{\sqrt{2}}\left(\hat{x}_2e^{i\nu}-\frac{\theta}{2}\,\hat{p}_1e^{i\nu}+i\hat{p}_2e^{-i\nu}\right)\\ B_1=\frac{1}{\sqrt{2}}\left(\hat{x}_1e^{i\nu}+\frac{\theta}{2}\,\hat{p}_2e^{i\nu}-i\hat{p}_1e^{-i\nu}\right),\ B_2=\frac{1}{\sqrt{2}}\left(\hat{x}_2e^{i\nu}-\frac{\theta}{2}\,\hat{p}_1e^{i\nu}-i\hat{p}_2e^{-i\nu}\right).\end{cases}\tag{3}$$

[1] In some part of the paper we will remove the assumption of ν being real, and see what happens when we put an imaginary part in it.

First of all, it is clear that, for $\nu \neq 0$, $B_j \neq A_j^\dagger$, $j=1,2$. Moreover, it is easy to check using (2) that

$$[A_j, B_k] = \delta_{j,k} 1\!\!1, \qquad [A_j, A_k] = [B_j, B_k] = 0. \tag{4}$$

Then these operators satisfy the two-dimensional pseudo-bosonic rules, [4]. More important, in terms of them our Hamiltonian $H_{\nu,\theta}$ in (1) acquires a much simpler form:

$$H_{\nu,\theta} = \frac{1}{\cos(2\nu)} \left(B_1 A_1 + B_2 A_2 + 1\!\!1\right), \tag{5}$$

which is manifestly non self-adjoint for $\nu \neq 0$. Indeed we have

$$H_{\nu,\theta}^\dagger = \frac{1}{2\cos(2\nu)} \left(A_1^\dagger B_1^\dagger + A_2^\dagger B_2^\dagger + 1\!\!1\right), \tag{6}$$

which is different from $H_{\nu,\theta}$ when $\nu \neq 0$.

Once the Hamiltonian has been written as in (5), we can use the general settings described in details in [4]: we have to look first for the vacua $\varphi_{0,0}$ and $\Psi_{0,0}$ of A_j and $B_j^\dagger$, $j=1,2$, and identify a set $\mathcal{D}$, dense in the Hilbert space, such that $\varphi_{0,0}, \Psi_{0,0} \in \mathcal{D}$ and $\mathcal{D}$ is left stable under the action of A_j, B_j and their adjoints. Then, we act on $\varphi_{0,0}$ and $\Psi_{0,0}$ with B_j and $A_j^\dagger$, respectively, producing two biorthogonal sets of eigenstates of $H_{\nu,\theta}$ and $H_{\nu,\theta}^\dagger$. The procedure here is particularly simple if we adopt the so-called Bopp shift to represent the commutation rules in (2). In fact, let us introduce two pairs of self-adjoint operators (x_j, p_j), $j=1,2$, satisfying $[x_j, p_k] = i\delta_{j,k} 1\!\!1$, $[x_j, x_k] = [p_j, p_k] = 0$. Then (2) are recovered if we assume that

$$\hat{x}_1 = x_1 - \frac{\theta}{2} p_2, \quad \hat{x}_2 = x_2 + \frac{\theta}{2} p_1, \quad \hat{p}_1 = p_1, \quad \hat{p}_2 = p_2. \tag{7}$$

In terms of these operators A_j and B_j can be rewritten as

$$\begin{cases} A_1 = \frac{1}{\sqrt{2}} \left(x_1 e^{i\nu} + e^{-i\nu} \frac{d}{dx_1}\right), \quad A_2 = \frac{1}{\sqrt{2}} \left(x_2 e^{i\nu} + e^{-i\nu} \frac{d}{dx_2}\right) \\ B_1 = \frac{1}{\sqrt{2}} \left(x_1 e^{i\nu} - e^{-i\nu} \frac{d}{dx_1}\right), \quad B_2 = \frac{1}{\sqrt{2}} \left(x_2 e^{i\nu} - e^{-i\nu} \frac{d}{dx_2}\right), \end{cases} \tag{8}$$

which shows that, in terms of (x_j, p_j), the two pairs (A_1, B_1) and (A_2, B_2) are completely independent. Hence, the construction of the set of eigenvectors of $H_{\nu,\theta}$, $\mathcal{F}_\varphi = \{\varphi_{n_1,n_2}(x_1,x_2)\}$, and the set of eigenvectors of $H_{\nu,\theta}^\dagger$, $\mathcal{F}_\Psi = \{\Psi_{n_1,n_2}(x_1,x_2)\}$, can be carried out simply considering tensor products of the one-dimensional construction already considered in for instance in [3, 7]. In particular, the two vacua of A_j and $B_j^\dagger$ are easily found:

$$\varphi_{0,0}(x_1,x_2) = \varphi_0(x_1)\varphi_0(x_2) = N_1 \exp\left\{-\frac{1}{2} e^{2i\nu} (x_1^2 + x_2^2)\right\},$$

$$\Psi_{0,0}(x_1,x_2) = \Psi_0(x_1)\Psi_0(x_2) = N_2 \exp\left\{-\frac{1}{2} e^{-2i\nu} (x_1^2 + x_2^2)\right\},$$

where N_1 and N_2 are normalization constants satisfying $\overline{N_1} N_2 = (\pi e^{-2i\nu})^{-1}$, to ensure that $\langle \varphi_{0,0}, \Psi_{0,0} \rangle = 1$.

Notice that, since $\Re(e^{\pm 2i\nu}) = \cos(2\nu) > 0$ for all $\nu \in I$, both $\varphi_{0,0}(x_1, x_2)$ and $\Psi_{0,0}(x_1, x_2)$ belong to $\mathcal{S}(\mathbb{R}^2)$, and therefore to $\mathcal{L}^2(\mathbb{R}^2)$. Now, if we define

$$\varphi_{n_1,n_2} = \frac{1}{\sqrt{n_1! n_2!}} B_1^{n_1} B_2^{n_2} \varphi_{0,0},$$
$$\Psi_{n_1,n_2} = \frac{1}{\sqrt{n_1! n_2!}} (A_1^\dagger)^{n_1} (A_2^\dagger)^{n_2} \Psi_{0,0},$$

we get, see [3],

$$\begin{cases} \varphi_{n_1,n_2}(x_1, x_2) = \frac{N_1}{\sqrt{2^{n_1+n_2}\, n_1!\, n_2!}} H_{n_1}\left(e^{i\nu} x_1\right) H_{n_2}\left(e^{i\nu} x_2\right) \\ \qquad\qquad \times \exp\left\{-\frac{1}{2} e^{2i\nu}\left(x_1^2 + x_2^2\right)\right\}, \\ \Psi_{n_1,n_2}(x_1, x_2) = \frac{N_2}{\sqrt{2^{n_1+n_2}\, n_1!\, n_2!}} H_{n_1}\left(e^{-i\nu} x_1\right) H_{n_2}\left(e^{-i\nu} x_2\right) \\ \qquad\qquad \times \exp\left\{-\frac{1}{2} e^{-2i\nu}\left(x_1^2 + x_2^2\right)\right\}, \end{cases} \tag{9}$$

where $H_n(x)$ is the nth Hermite polynomial. We see from these formulas that, for all $n_j \geq 0$, $\frac{1}{N_1} \varphi_{n_1,n_2}(x_1, x_2)$ coincides with $\frac{1}{N_2} \Psi_{n_1,n_2}(x_1, x_2)$, with ν replaced by $-\nu$. Moreover, they all belong to $\mathcal{S}(\mathbb{R}^2)$, and therefore to $\mathcal{L}^2(\mathbb{R}^2)$, which is a clear indication that $\varphi_{0,0}(x_1, x_2) \in D^\infty(B_j)$ and $\Psi_{0,0}(x_1, x_2) \in D^\infty(A_j^\dagger)$, $j = 1, 2$. Also, they are biorthogonal $\langle \varphi_{n_1,n_2}, \Psi_{m_1,m_2} \rangle = \delta_{n_1,m_1} \delta_{n_2,m_2}$, for all $n_j, m_j \geq 0$, and the following equations are satisfied:

$$\begin{cases} A_1 \varphi_{n_1,n_2} = \sqrt{n_1} \varphi_{n_1-1,n_2}, \quad A_2 \varphi_{n_1,n_2-1} = \sqrt{n_2} \varphi_{n_1,n_2-1}, \\ B_1^\dagger \Psi_{n_1,n_2} = \sqrt{n_1} \Psi_{n_1-1,n_2}, \quad B_2^\dagger \Psi_{n_1,n_2} = \sqrt{n_2} \Psi_{n_1,n_2-1}, \\ B_1 A_1 \varphi_{n_1,n_2} = n_1 \varphi_{n_1,n_2}, \quad B_2 A_2 \varphi_{n_1,n_2} = n_2 \varphi_{n_1,n_2}, \\ (B_1 A_1)^\dagger \Psi_{n_1,n_2} = n_1 \Psi_{n_1,n_2}, \quad (B_2 A_2)^\dagger \Psi_{n_1,n_2} = n_2 \Psi_{n_1,n_2}. \end{cases} \tag{10}$$

Following the same arguments as in [4], it is possible to check that the norm of these vectors, $\|\varphi_{n_1,n_2}\|$ and $\|\Psi_{n_1,n_2}\|$, diverge with n_j. Then, $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are not Riesz bases, and not even bases. We are still left with the possibility that they are $\mathcal{G}$-quasi bases, for a suitable set $\mathcal{G}$ dense in $\mathcal{L}^2(\mathbb{R}^2)$, see below. Indeed, this is the case, as we can check extending, once again, what was done in [4] in the one-dimensional case. We don't give the details here, since they do not differ significantly from what is done in [3, 4]. We only stress that the crucial ingredient is provided by the operator

$$T_\nu = e^{i\frac{\nu}{2}\left(x_1 \frac{d}{dx_1} + \frac{d}{dx_1} x_1\right)} e^{i\frac{\nu}{2}\left(x_2 \frac{d}{dx_2} + \frac{d}{dx_2} x_2\right)},$$

which maps (except for a normalization constant) the orthonormal basis of a two-dimensional harmonic oscillator, $\mathcal{F}_e$, into $\mathcal{F}_\varphi$. In the same way $(T^{-1})^\dagger$ maps (again, except for a normalization constant) the same basis into $\mathcal{F}_\Psi$. Then, calling $\mathcal{D}_e$ the linear span of $\mathcal{F}_e$, which is obviously dense in $\mathcal{L}^2(\mathbb{R}^2)$, it turns out that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{D}_e$-quasi bases. This means that, for all $f, g \in \mathcal{D}_e$, the following resolution of the identity holds true:

$$\langle f, g\rangle = \sum_{n_1,n_2} \langle f, \varphi_{n_1,n_2}\rangle \langle \Psi_{n_1,n_2}, g\rangle = \sum_{n_1,n_2} \langle f, \Psi_{n_1,n_2}\rangle \langle \varphi_{n_1,n_2}, g\rangle \tag{11}$$

Notice that both T_ν and T_ν^{-1} are unbounded. This can be easily understood easily, since both these operators are not everywhere defined on $\mathcal{L}^2(\mathbb{R}^2)$.

Finally, the metric operator can now be explicitly deduced: $\Theta := \frac{1}{\pi\,|N_1|^2}\, T_\nu^{-2}$, which is unbounded, with unbounded inverse. Moreover, $(A_j, B_j^\dagger)$ are Θ-conjugate in the sense of [8], and $H_{\nu,\theta}$ is similar to a self-adjoint Hamiltonian: $h_{\nu,\theta} f = T_\nu\, H_{\nu,\theta}\, T_\nu^{-1} f$, for all f in a suitable dense domain of $\mathcal{L}^2(\mathbb{R}^2)$, where

$$h_{\nu,\theta} = \frac{1}{\cos 2\nu}\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\!\!1\right), \quad a_j = \frac{1}{\sqrt{2}}(x_j + ip_j).$$

3. Bi-coherent states

We now consider the two pairs of pseudo-bosonic operators $(A_j, B_j), \quad j = 1, 2$, behaving as in the previous section, in order to construct a generalized version of the canonical coherent states. First of all we introduce $\forall z, w \in \mathbb{C}$ the two displacement-like operators

$$\mathcal{U}(z,w) = e^{zB_1 - \bar{z}A_1} e^{wB_2 - \bar{w}A_2}, \quad \mathcal{V}(z,w) = e^{zA_1^\dagger - \bar{z}B_1^\dagger} e^{wA_2^\dagger - \bar{w}B_2^\dagger}. \tag{12}$$

Of course these operators are not unitary and they are possibly not even bounded. Hence, at the moment, they should be understood as formal objects.

If we assume that the Baker–Campbell–Hausdorff relation can be applied to $\mathcal{U}(z,w)$ and $\mathcal{V}(z,w)$, due to the commutation relations

$$[A_j, [A_j, B_j]] = [B_j, [A_j, B_j]] = 0, \;\; j = 1, 2,$$

we obtain the following alternative representations:

$$\begin{aligned}\mathcal{U}(z,w) &= e^{-\frac{|z|^2+|w|^2}{2}} e^{zB_1} e^{-\bar{z}A_1} e^{wB_2} e^{-\bar{w}A_2} = e^{\frac{|z|^2+|w|^2}{2}} e^{-\bar{z}A_1} e^{zB_1} e^{-\bar{w}A_2} e^{wB_2} \\ \mathcal{V}(z,w) &= e^{-\frac{|z|^2+|w|^2}{2}} e^{zA_1^\dagger} e^{-\bar{z}B_1^\dagger} e^{wA_2^\dagger} e^{-\bar{w}B_2^\dagger} = e^{\frac{|z|^2+|w|^2}{2}} e^{-\bar{z}B_1^\dagger} e^{zA_1^\dagger} e^{-\bar{w}B_2^\dagger} e^{wA_2^\dagger},\end{aligned} \tag{13}$$

so that

$$\mathcal{U}(z,w)^{-1} = \mathcal{U}(-z,-w) = \mathcal{V}(z,w)^\dagger, \quad \mathcal{V}(z,w)^{-1} = \mathcal{V}(-z,-w) = \mathcal{U}(z,w)^\dagger,$$

Now, bi-coherent states could be constructed in the following way:

$$\varphi(z,w) = \mathcal{U}(z,w)\varphi_{0,0}, \quad \Psi(z,w) = \mathcal{V}(z,w)\Psi_{0,0}, \tag{14}$$

where $\varphi_{0,0}$, $\Psi_{0,0}$ are the two vacua introduced in the previous section. However, it is more convenient to define $\varphi(z,w)$ and $\Psi(z,w)$ via the following series representations:

$$\varphi(z,w) = e^{-\frac{|z|^2+|w|^2}{2}} \sum_{n_1,n_2\geq 0} \frac{z^m w^n}{\sqrt{n_1!n_2!}} \varphi_{n_1,n_2}, \tag{15}$$

$$\Psi(z,w) = e^{-\frac{|z|^2+|w|^2}{2}} \sum_{n_1,n_2\geq 0} \frac{z^m w^n}{\sqrt{n_1!n_2!}} \Psi_{n_1,n_2}. \tag{16}$$

This is because, if we are able to prove that the series converge, then we don't need to take care of all the mathematical subtleties appearing if $\mathcal{U}(z,w)$ and $\mathcal{V}(z,w)$ are unbounded. On the other hand, it is not hard to prove that the above series converge $\forall z,w \in \mathbb{C}$, and that the states they define have interesting properties. For that, it is convenient to prove first a rather general result on bi-coherent states, which in a sense unifies and extend the results described in many papers recently, [9–14].

3.1. A general theorem

Here we work with two biorthogonal families of vectors, $\mathcal{F}_\varphi = \{\varphi_n,\, n\geq 0\}$ and $\mathcal{F}_\Psi = \{\Psi_n,\, n\geq 0\}$ which are $\mathcal{D}$-quasi bases for some dense subset of $\mathcal{H}$, see (11). Consider an increasing sequence of real numbers α_n satisfying the inequalities $0=\alpha_0<\alpha_1<\alpha_2<\cdots$. We call $\overline{\alpha}$ the limit of α_n for n diverging, which coincides with $\sup_n \alpha_n$. We further consider two operators, a and $b^\dagger$, which act as lowering operators respectively on $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ in the following way:

$$a\,\varphi_n = \alpha_n \varphi_{n-1}, \qquad b^\dagger\,\Psi_n = \alpha_n \Psi_{n-1}, \tag{17}$$

for all $n\geq 1$, with $a\,\varphi_0 = b^\dagger\,\Psi_0 = 0$.

Theorem 1. *Assume that four strictly positive constants A_φ, A_Ψ, r_φ and r_Ψ exist, together with two strictly positive sequences $M_n(\varphi)$ and $M_n(\Psi)$ for which*

$$\lim_{n\to\infty} \frac{M_n(\varphi)}{M_{n+1}(\varphi)} = M(\varphi), \qquad \lim_{n\to\infty} \frac{M_n(\Psi)}{M_{n+1}(\Psi)} = M(\Psi), \tag{18}$$

where $M(\varphi)$ and $M(\Psi)$ could be infinity, such that, for all $n\geq 0$,

$$\|\varphi_n\| \leq A_\varphi\, r_\varphi^n M_n(\varphi), \qquad \|\Psi_n\| \leq A_\Psi\, r_\Psi^n M_n(\Psi). \tag{19}$$

Then the following series:

$$N(|z|) = \left(\sum_{k=0}^\infty \frac{|z|^{2k}}{(\alpha_k!)^2}\right)^{-1/2}, \tag{20}$$

$$\varphi(z) = N(|z|)\sum_{k=0}^\infty \frac{z^k}{\alpha_k!}\varphi_k, \qquad \Psi(z) = N(|z|)\sum_{k=0}^\infty \frac{z^k}{\alpha_k!}\Psi_k, \tag{21}$$

are all convergent inside the circle $C_\rho(0)$ *centered in the origin of the complex plane and of radius* $\rho = \overline{\alpha}\min\left(1, \frac{M(\varphi)}{r_\varphi}, \frac{M(\Psi)}{r_\Psi}\right)$. *Moreover, for all* $z \in C_\rho(0)$,

$$a\varphi(z) = z\varphi(z), \qquad b^\dagger\Psi(z) = z\Psi(z). \tag{22}$$

Suppose further that a measure $d\lambda(r)$ *does exist such that*

$$\int_0^\rho d\lambda(r) r^{2k} = \frac{(\alpha_k!)^2}{2\pi}, \tag{23}$$

for all $k \geq 0$. *Then, for all* $f, g \in \mathcal{D}$, *calling* $d\nu(z,\overline{z}) = d\lambda(r)d\theta$, *we have*

$$\begin{aligned}\int_{C_\rho(0)} & N(|z|)^{-2}\langle f, \Psi(z)\rangle\langle\varphi(z), g\rangle\, d\nu(z,\overline{z}) \\ &= \int_{C_\rho(0)} N(|z|)^{-2}\langle f, \varphi(z)\rangle\langle\Psi(z), g\rangle\, d\nu(z,\overline{z}) = \langle f, g\rangle\end{aligned} \tag{24}$$

The proof of the theorem is simple and will not be given here. Rather than this, there are few comments which are in order: first of all, we see from (19) that the norms of the vectors φ_n and Ψ_n need not being uniformly bounded, as it happened to be in [11]. On the contrary, they can diverge rather fastly with n. To see this, we just consider $r_\varphi, r_\Psi > 1$ and $M_n(\varphi)$ and $M_n(\Psi)$ constant sequences.

To apply the above theorem to the Swanson model we need to construct a two-dimensional version of it. This can be done in a natural way: suppose again we have two biorthogonal families of vectors, $\mathcal{F}_\varphi = \{\varphi_{n_1,n_2},\, n_j \geq 0\}$ and $\mathcal{F}_\Psi = \{\Psi_{n_1,n_2},\, n_j \geq 0\}$ which are $\mathcal{D}$-quasi bases for some dense subset of $\mathcal{H}$. As we can see, these vectors depend on two sequences of natural numbers. Let now $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ and $0 = \beta_0 < \beta_1 < \beta_2 < \cdots$. We call $\overline{\alpha}$ and $\overline{\beta}$ their limits. We further consider four operators, a_j and $b_j^\dagger$, $j = 1, 2$, which act as lowering operators respectively on $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ [2] in the following way:

$$a_1\,\varphi_{n_1,n_2} = \alpha_{n_1}\varphi_{n_1-1,n_2}, \qquad a_2\,\varphi_{n_1,n_2} = \beta_{n_2}\varphi_{n_1,n_2-1}, \tag{25}$$

$$b_1^\dagger\,\Psi_{n_1,n_2} = \alpha_{n_1}\Psi_{n_1-1,n_2}, \qquad b_2^\dagger\,\Psi_{n_1,n_2} = \beta_{n_2}\Psi_{n_1,n_2-1}, \tag{26}$$

for all $n_j \geq 0$. As before, we assume that the norms of the vectors are bounded in a very *mild* way:

$$\|\varphi_{n_1,n_2}\| \leq A_\varphi r_{1,\varphi}^{n_1} r_{2,\varphi}^{n_2} M_{n_1}(1,\varphi) M_{n_2}(2,\varphi), \tag{27}$$

$$\|\Psi_{n_1,n_2}\| \leq A_\Psi r_{1,\Psi}^{n_1} r_{2,\Psi}^{n_2} M_{n_1}(1,\Psi) M_{n_2}(2,\Psi), \tag{28}$$

for some real constants A_Φ, $r_{k,\Phi}$ and some sequences $M_j(k,\Phi)$, Φ is both φ or Ψ, $k = 1, 2$, $j \geq 0$. Then we require that

$$\lim_{j\to\infty} \frac{M_j(k,\Phi)}{M_{j+1}(k,\Phi)} = M(k,\Phi),$$

[2] For instance these operators can be those satisfying (10).

which can also be divergent. Hence, generalizing Theorem 1, we can define

$$\rho_1 = \overline{\alpha} \min\left(1, \frac{M(1,\varphi)}{r_{1,\varphi}}, \frac{M(1,\Psi)}{r_{1,\Psi}}\right), \quad \rho_2 = \overline{\beta} \min\left(1, \frac{M(2,\varphi)}{r_{2,\varphi}}, \frac{M(2,\Psi)}{r_{2,\Psi}}\right),$$

and the two related circles $C_{\rho_j}(0)$, $j = 1, 2$, as well as the following quantities:

$$N(z, w) = \left(\sum_{k=0}^{\infty} \frac{|z|^{2k}}{(\alpha_k!)^2}\right)^{-\frac{1}{2}} \left(\sum_{l=0}^{\infty} \frac{|w|^{2k}}{(\beta_k!)^2}\right)^{-\frac{1}{2}}, \tag{29}$$

$$\varphi(z, w) = N(z, w) \sum_{n_1,n_2 \geq 0} \frac{z^{n_1} w^{n_2}}{\alpha_{n_1}! \beta_{n_2}!} \varphi_{n_1,n_2}, \tag{30}$$

$$\Psi(z, w) = N(z, w) \sum_{n_1,n_2 \geq 0} \frac{z^{n_1} w^{n_2}}{\alpha_{n_1}! \beta_{n_2}!} \Psi_{n_1,n_2}. \tag{31}$$

They are all well defined for $z \in C_{\rho_1}(0)$ and $w \in C_{\rho_2}(0)$, and satisfy, for all such (z, w), the normalization condition $\langle \varphi(z, w), \Psi(z, w)\rangle = 1$. Also:

$$a_1\varphi(z, w) = z\varphi(z, w), \quad a_2\varphi(z, w) = w\varphi(z, w),$$

and

$$b_1^\dagger \Psi(z, w) = z\Psi(z, w), \quad b_2^\dagger \Psi(z, w) = w\Psi(z, w).$$

Concerning the resolution of the identity, this time we have to solve two moment problems: suppose that we can find two measures, $d\lambda_j(r)$, $j = 1, 2$, such that

$$\int_0^{\rho_1} d\lambda_1(r) r^{2k} = \frac{(\alpha_k!)^2}{2\pi}, \qquad \int_0^{\rho_2} d\lambda_2(r) r^{2k} = \frac{(\beta_k!)^2}{2\pi}.$$

for all $k \geq 0$. Then, calling $d\nu_1(z, \overline{z}) = d\lambda_1(r)\, d\theta$ and $d\nu_2(w, \overline{w}) = d\lambda_1(r')\, d\theta'$, we can prove the following: for all $f, g \in \mathcal{D}$ we have, for instance,

$$\int_{C_{\rho_1}(0)} d\nu_1(z, \overline{z}) \int_{C_{\rho_2}(0)} d\nu_2(w, \overline{w}) N(z, w)^{-2} \langle f, \Psi(z, w)\rangle \langle \varphi(z, w), g\rangle = \langle f, g\rangle,$$

and a similar formula with $\Psi(z, w)$ and $\varphi(z, w)$ exchanged.

Remark. If, in particular, $\alpha_n = \sqrt{n} = \beta_n$, as is the case for the Swanson model, it is clear that $\overline{\alpha} = \overline{\beta} = \infty$ and, because of their definitions, $\rho_1 = \rho_2 = \infty$. Moreover, $N(z, w) = e^{-\frac{|z|^2+|w|^2}{2}}$ and $\varphi(z, w), \Psi(z, w)$ reduce to (15) and (16). This means that convergence of the bi-coherent states is guaranteed in all $\mathbb{C}^2$.

3.2. Back to Swanson

To apply the previous results to our modified Swanson model we need now to find a relevant estimate for the norms of the vectors in $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$. For that we use the formula ([15, pag. 502]):

$$\int_0^\infty \int_0^\infty e^{-p(x^2+y^2)} H_{n_1}(ax) H_{n_1}(bx) H_{n_2}(cy) H_{n_2}(fy) dxdy$$
$$= \frac{2^{n_1+n_2-2} n_1! n_2! \pi}{p^{(n_1+n_2+2)/2}} (a^2+b^2-p)^{n_1/2} (c^2+f^2-p)^{n_2/2}$$
$$\times P_{n_1}\left(\frac{ab}{\sqrt{p(a^2+b^2-p)}}\right) P_{n_2}\left(\frac{cf}{\sqrt{p(c^2+f^2-p)}}\right),$$

where P_n is the Legendre polynomial of order n. The above formula is valid for all p having a nonnegative real part, and in our context $p = \cos(2\nu) > 0, \forall \nu \in I \setminus \{0\}$ [3]. Straightforward computations finally lead to

$$\|\varphi_{n_1,n_2}\|^2 = \frac{\pi |N_1|^2}{\cos(2\nu)} P_{n_1}\left(\frac{1}{\cos(2\nu)}\right) P_{n_2}\left(\frac{1}{\cos(2\nu)}\right),$$

and using the estimate in [16] for $P_n(x)$ we deduce that

$$\|\varphi_{n_1,n_2}\|^2 \leq A_\nu r_\nu^{n_1} t_\nu^{n_2}, \quad r_\nu = t_\nu = \sqrt{\frac{1}{\cos(2\nu)} + \left(\frac{1}{\cos(2\nu)} - 1\right)^{1/2}},$$

with A_ν a non-relevant positive constant. Then, it is clear that the assumption in (27) is satisfied, taking for instance $M_n(1,\varphi) = M_n(2,\varphi) = 1$, for all $n \geq 0$. Similar considerations can be repeated for $\Psi(z,w)$, so that all the results deduced before apply here. In particular $\varphi(z,w)$ are eigenstates of A_j, $\Psi(z,w)$ are eigenstates of $B_j^\dagger$ and, solving the above moment problems (which collapse to a single one), they produce a resolution of the identity.

3.3. What if ν is complex?

In the literature on Swanson model, ν is always taken to be real. We will briefly show now that this is not really essential, at least if its real part still belongs to the set I introduced before. For that, let us assume that $\nu = \nu_r + i\nu_i$, with $\nu \in I$ and $\nu_i \in \mathbb{R}$. Then, formulas (1)–(5) are still valid. However, (6) should be replaced with

$$H_{\nu,\theta}^\dagger = \frac{1}{\cos(2\overline{\nu})} \left(A_1^\dagger B_1^\dagger + A_2^\dagger B_2^\dagger + 1\!\!1\right).$$

Also, while the analytic expression of $\varphi_{n_1,n_2}(x_1,x_2)$ in (9) does not change, that of $\Psi_{n_1,n_2}(x_1,x_2)$ can be deduced from $\varphi_{n_1,n_2}(x_1,x_2)$ by replacing ν with $-\overline{\nu}$. Again, we deduce that $\varphi_{n_1,n_2}(x_1,x_2)$ and $\Psi_{n_1,n_2}(x_1,x_2)$ are all in $\mathcal{S}(\mathbb{R}^2)$, and therefore in $\mathcal{L}^2(\mathbb{R}^2)$. And again, also in this extended case, it is possible to check that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are not Riesz bases. In fact we find that

$$\|\varphi_{n_1,n_2}\|^2 = \frac{\pi |N_1|^2}{e^{-2\nu_i}\cos(2\nu_r)} P_{n_1}\left(\frac{1}{\cos(2\nu_r)}\right) P_{n_2}\left(\frac{1}{\cos(2\nu_r)}\right),$$

[3]We still assume that $\nu \neq 0$ as we are interested in the non-Hermitian case.

where ν_i explicitly appears. A similar estimate, with N_1 replaced by N_2, also holds for $\|\Psi_{n_1,n_2}\|^2$. Both these norms diverge when n_1 and n_2 diverge, see [6]. Hence, see [4], $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ cannot be Riesz bases, also for complex ν. For this reason, no major differences are expected with respect to our previous results.

4. Conclusions

In this paper we have proposed a noncommutative, two-dimensional, version of the Swanson model and we have shown that its Hamiltonian can be rewritten in terms of $\mathcal{D}$-pseudo-bosonic operators. In this way, the eigenvalues and the eigenvectors can be easily deduced. We have also considered the bi-coherent states attached to the model, analyzing some of their properties. In particular, the fact that they resolve the identity has been proved.

Acknowledgment

F.B. and F.G. acknowledge the support by the University of Palermo and GNFM of INdAM.

References

[1] C.M. Bender and S. Boettcher, *Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry*, Phys. Rev. Lett., **80**, 5243–5246 (1998).

[2] M.S. Swanson, Transition elements for a non-Hermitian quadratic Hamiltonian, J. Math. Phys., **45**, 585, (2004).

[3] F. Bagarello, *Examples of Pseudo-bosons in quantum mechanics*, Phys. Lett. A, **374**, 3823–3827 (2010).

[4] F. Bagarello, *Deformed canonical (anti-)commutation relations and non hermitian Hamiltonians*, in Non-selfadjoint operators in quantum physics: Mathematical aspects, F. Bagarello, J.P. Gazeau, F.H. Szafraniec and M. Znojil Eds., John Wiley and Sons Eds., Hoboken, New Jersey (2015).

[5] J. da Providência, N. Bebiano, J.P. da Providência, *Non hermitian operators with real spectrum in quantum mechanics*, ELA, **21**, 98–109 (2010).

[6] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Springer-Verlag, Berlin Heidelberg (1966).

[7] F. Bagarello, F. Gargano, D. Volpe *$\mathcal{D}$-Deformed Harmonic Oscillator*, Int. J. Theor. Phys., **54**(11), 4110–4123 (2015).

[8] F. Bagarello, *More mathematics on pseudo-bosons*, J. Math. Phys., **54**, 063512 (2013).

[9] F. Bagarello *Pseudo-bosons, Riesz bases and coherent states*, J. Math. Phys., **50**, 023531 (2010) (10 pp.).

[10] S.T. Ali, F. Bagarello, J.P. Gazeau, *$\mathcal{D}$-pseudo-bosons, Complex Hermite Polynomials, and Integral Quantization*, SIGMA, **11** (2015), 078, 23 pp.

[11] F. Bagarello, *Pseudo-bosons and Riesz bi-coherent states*, spedito ai Proceedings of the WGMP Conferences, 2015, Białowieża, Polonia.

[12] F. Bagarello, *Intertwining operators for non self-adjoint Hamiltonians and bicoherent states*, J. Math. Phys., **57**, 103501 (2016); doi: 10.1063/1.4964128.

[13] F. Bagarello, *kq-representation for pseudo-bosons, and completeness of bi-coherent states*, JMAA, **450**, 631–643 (2017).

[14] F. Bagarello, *Deformed quons and bi-coherent states*, Proc. Roy. Soc. A, in stampa.

[15] A.P. Prudnikov, Yu.A. Bryehkov, O.I. Marichev, *Integrals and series*, vol. 2, Special functions, Opa, Amsterdam, (1986).

[16] G. Szegő, *Orthogonal polynomials*, American Mathematical Society, Colloquium Publications, Vol. 23, 4th ed., Amer. Math. Soc., Providence, R.I., 1975.

Fabio Bagarello
DEIM – Dipartimento di Energia
Ingegneria dell'Informazione e Modelli Matematici
Scuola Politecnica
Università di Palermo
I-90128 Palermo, Italy

and

INFN, Sezione di Napoli
e-mail: fabio.bagarello@unipa.it

Francesco Gargano and Salvatore Spagnolo
DEIM – Dipartimento di Energia
Ingegneria dell'Informazione e Modelli Matematici
Scuola Politecnica
Università di Palermo
I-90128 Palermo, Italy
e-mail: francesco.gargano@unipa.it
salvatore.spagnolo@unipa.it

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 21–29

Universal Markov Kernels for Quantum Observables

Roberto Beneduci

To the memory of my friend Twareque Ali

Abstract. We prove the existence of a universal Markov kernel, i.e., a Markov kernel μ such that every commutative POVM F is the smearing of a self-adjoint operator A^F with the smearing realized through μ. The relevance of the smearing is illustrated in connection with the problem of the joint measurability of two quantum observables. Also the connections with phase space quantum mechanics is outlined.

Mathematics Subject Classification (2010). Primary 81P15, 28B15; Secondary 46N50, 81P45.

Keywords. Positive operator-valued measures, Markov kernels, joint measurability, phase space quantum mechanics.

1. Introduction

In the modern formulation of quantum mechanics positive operator-valued measures play a key role since they are a very important tool for the mathematical representation of quantum observables [11, 13, 15, 17]. For example, they are used to define a localization observable for the photon [1] overcoming the problem of the photon localization. They can also be used to define a time observable [11] overcoming the problem of the existence of a time observable. Another very relevant feature of POVMs is that two POVMs can be jointly measurable also if they do not commute while it is well known that joint measurability and commutativity coincide in the case of self-adjoint operators [11]. That makes it possible to describe, by means of a rigorous mathematical approach, the joint measurability of two incompatible observables (see below) and is at the roots of the phase space formulation of quantum mechanics [15, 17]. Before we focus on these topics, let us recall some of the main definitions and properties of POVMs.

In what follows, $\mathcal{B}(X)$ denotes the Borel σ-algebra of a topological space X and $\mathcal{L}_s^+(\mathcal{H})$ the space of all bounded positive self-adjoint linear operators acting in a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$.

Definition 1. A Positive Operator-valued measure (for short, POVM) is a map $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ such that:

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n \right) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(X)$ and the series converges in the weak operator topology. It is said to be normalized if

$$F(X) = \mathbf{1}$$

where $\mathbf{1}$ is the identity operator.

Definition 2. A POVM is said to be commutative if

$$\big[F(\Delta_1), F(\Delta_2)\big] = \mathbf{0}, \quad \forall\, \Delta_1\,, \Delta_2 \in \mathcal{B}(X). \tag{1}$$

Definition 3. A POVM is said to be orthogonal if $\Delta_1 \cap \Delta_2 = \emptyset$ implies

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \tag{2}$$

where $\mathbf{0}$ is the null operator.

Definition 4. A Spectral measure or Projection-Valued measure (for short, PVM) is an orthogonal, normalized POVM.

Note that the image of an orthogonal POVMs are projection operators. In quantum mechanics, non-orthogonal normalized POVMs are also called **generalized** or **unsharp** observables while PVMs are called **standard** or **sharp** observables.

We recall that $\langle \psi, F(\Delta)\psi \rangle$ is interpreted as the probability that a measurement of the observable represented by F gives a result in Δ.

The following theorem gives a characterization of commutative POVMs as smearings of spectral measures with the smearing realized by means of Markov kernels.

Definition 5. Let Λ be a topological space. A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

In the following the symbol $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by the POVM F, i.e., the von Neumann algebra generated by the set $\{F(\Delta)\}_{\mathcal{B}(X)}$. Hereafter, we assume that X is a Hausdorff, locally compact, second countable topological space.

Theorem 6 ([4, 6]). *A POVM $F : \mathcal{B}(X) \to \mathcal{L}_s^+(\mathcal{H})$ is commutative if and only if there exists a bounded self-adjoint operator $A = \int \lambda\, dE_\lambda$ with spectrum $\sigma(A) \subset [0, 1]$, and a Markov Kernel $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ such that*

1) $F(\Delta) = \int_\Lambda \mu_\Delta(\lambda)\, dE_\lambda, \quad \Delta \in \mathcal{B}(X)$.
2) $\mathcal{A}^W(F) = \mathcal{A}^W(A)$.

The operator A introduced in Theorem 6 is called the **sharp version** of F and is unique up to almost everywhere bijections [4]. The POVM F is said to be a smearing of the PVM E or equivalently a smearing of A. It can be interpreted as a noisy version of E [6].

A characterization of POVMs not necessarily commutative is due to Ali [2] who obtained a Choquet type of an integral representation for POVMs. In particular, F is represented as an integral over the space of PVMs endowed with a Baire measure.

2. Joint Measurability

Now we outline the relevance of POVMs to the problem of joint measurability of two quantum observables. First we need to recall Naimark's dilation theorem and the definition of joint measurability.

Definition 7. Two POVMs $F_1 : \mathcal{B}(X_1) \to \mathcal{L}_s^+(\mathcal{H})$, $F_2 : \mathcal{B}(X_2) \to \mathcal{L}_s^+(\mathcal{H})$ are compatible (or jointly measurable) if they are the marginals of a joint POVM $F : \mathcal{B}(X_1 \times X_2) \to \mathcal{L}_s^+(\mathcal{H})$.

Theorem 8 (Naimark [14]). *Let F be a POVM. Then, there exist an extended Hilbert space $\mathcal{H}^+$ and a PVM E^+ on $\mathcal{H}^+$ such that*

$$F(\Delta)\psi = PE^+(\Delta)\psi, \quad \forall \psi \in \mathcal{H}$$

where P is the operator of projection onto $\mathcal{H}$.

Two PVMs are jointly measurable if and only if they commute. In particular, two spectral measures are jointly measurable if and only if the corresponding self-adjoint operators commute. The relationships between commutativity and joint measurability is weaker in the case of POVMs: the first implies the second by the converse is not true in general. A characterization of the joint measurability comes from Naimark's dilation theorem.

Theorem 9 ([5]). *Two POVMs $F_1 : \mathcal{B}(X_1) \to \mathcal{L}_s^+(\mathcal{H})$ and $F_2 : \mathcal{B}(X_2) \to \mathcal{L}_s^+(\mathcal{H})$ are compatible if and only if there are two Naimark extensions $E_1^+ : \mathcal{B}(X_1) \to \mathcal{L}_s^+(\mathcal{H})$ and $E_2^+ : \mathcal{B}(X_2) \to \mathcal{L}_s^+(\mathcal{H})$ such that $[E_1^+, E_2^+] = \mathbf{0}$.*

In the case of commutative POVMs, the previous theorem can be expressed in terms of the relationships between the sharp versions and their Naimark's dilations.

Theorem 10 ([5]). *Let F_1 and F_2 be two commutative POVMs such that the operators in their ranges are discrete. They are compatible if and only if the corresponding sharp versions A_1 and A_2 can be dilated to two compatible self-adjoint operators A_1^+, A_2^+ such that $P\chi_\Delta(A_i^+)P = F_i(f_i^{-1}(\Delta))$, $i = 1, 2$, with f_i one-to-one.*

In order to illustrate the previous theorems and their connections with the phase space formulation of quantum mechanics, we focus on the relevant physical example of position and momentum observables, $Q = \int q\, dQ(q)$ and $P = \int p\, dP(p)$ on the space $\mathcal{H} = L^2(\mathbb{R})$. We recall that $(Q\psi)(q) = q\,\psi(q)$ while $(P\psi)(q) = -i\frac{\partial \psi}{\partial q}(q)$.

Let us consider the POVM [15]

$$F^\eta(\Delta \times \Delta') = \int_{\Delta\times\Delta'} U_{q,p}\,\eta\, U^*_{q,p}\, dq\, dp = \int_{\Delta\times\Delta'} P_{q,p}\, dq\, dp,$$

where, $U_{q,p} = e^{-iqP}e^{ipQ}$, $\eta := P_g$ is the projector on the subspace generated by $g \in L^2(\mathbb{R})$, $\|g\|_2 = 1$ and $P_{q,p} = U_{q,p}\,\eta\, U^*_{q,p}$. The marginals

$$F^Q_\eta(\Delta) := F(\Delta \times \mathbb{R}) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |g|^2)(q)\, dQ(q), \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{3}$$

$$F^P_\eta(\Delta) := F(\mathbb{R} \times \Delta) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |\hat{g}|^2)(p)\, dP(p), \quad \Delta \in \mathcal{B}(\mathbb{R}) \tag{4}$$

are the unsharp position and momentum observables respectively [5, 6]). Note that the maps $\mu_\Delta(q) := (\mathbf{1}_\Delta * |g|^2)(q)$ and $\hat{\mu}_\Delta(p) := (\mathbf{1}_\Delta * |\hat{g}|^2)(p)$ define two Markov kernels [5–7]) so that F^Q_η and F^P_η are smearings of Q and P respectively. Now, we can define the isometry

$$\begin{aligned} W^\eta : \mathcal{H} &\to L^2(\Gamma, \mu) \\ \psi &\mapsto \langle U_{q,p}\, g, \psi\rangle \end{aligned}$$

where, μ is the Lebesgue measure on $\Gamma = \mathbb{R} \times \mathbb{R}$. The map W^η embeds $\mathcal{H}$ as a subspace of $L^2(\Gamma, \mu)$. The projection operator $\widetilde{P}^\eta$ from $L^2(\Gamma, \mu)$ to $W^\eta(\mathcal{H})$ is defined as follows

$$(\widetilde{P}^\eta f)(q, p) = \int_\Gamma \langle U_{q,p}\, g, U_{q',p'}\, g\rangle f(q', p')\, dq'\, dp'.$$

In the phase space formulation of quantum mechanics [17], the function $f^\eta_\psi(q, p) := |\langle U_{q,p}\, g, \psi\rangle|^2$ is the phase space representation of the pure quantum state ψ while

$$F^\eta(f) = \int_{\mathbb{R}\times\mathbb{R}} f(q, p)\, |U_{q,p}\, g\rangle\langle U_{q,p}\, g|\, dq\, dp$$

defines a quantization procedure since to any real measurable function f on the phase space Γ associates a self-adjoint operator $F^\eta(f)$ which is positive whenever f is positive. Moreover, the expectation value of the quantum observable corresponding to f is

$$\langle F^\eta(f)\rangle_\psi = \langle \psi, F^\eta(f)\psi\rangle = \int_{\mathbb{R}\times\mathbb{R}} f(q, p)\, f^\eta_\psi(q, p)\, dq\, dp$$

and

$$\langle \psi, F^\eta(\Delta_q \times \Delta_p)\psi\rangle = \langle \psi, F^\eta(\chi_{\Delta_q\times\Delta_p})\psi\rangle = \int_{\Delta_q\times\Delta_p} f^\eta_\psi(q, p)\, dq\, dp.$$

Next, we prove the existence of two commuting Naimark's dilations for F^Q_η and F^P_η. It is sufficient to consider the following two PVMs

$$(\widetilde{E}^+_Q(\Delta)f)(q,p) = \chi_\Delta(q)f(q,p), \quad f \in L^2(\Gamma,\mu)$$
$$(\widetilde{E}^+_P(\Delta)f)(q,p) = \chi_\Delta(p)f(q,p), \quad f \in L^2(\Gamma,\mu)$$

They commute since they are multiplications by characteristic functions. Moreover, for any $f \in W^\eta(\mathcal{H})$,

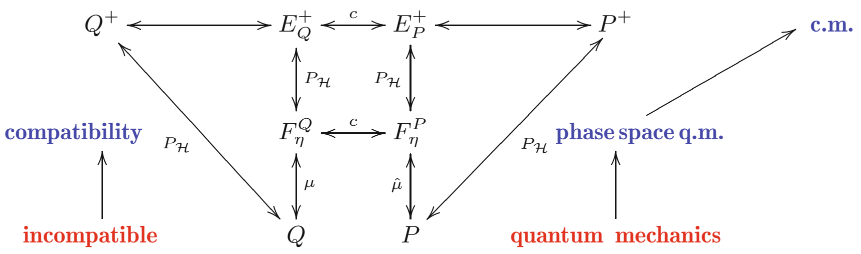
$$\begin{aligned}(\widetilde{P}^\eta \widetilde{E}^+_Q(\Delta)f)(q,p) &= \int_\Gamma \langle U_{q,p}g, U_{q',p'}g\rangle \chi_\Delta(q') f(q',p')\, dq'\, dp' \\ &= W^\eta \int_{\Delta\times\mathbb{R}} U_{q',p'}\, \eta\, U^*_{q',p'}\psi\, dq'\, dp' = [W^\eta F^Q_\eta(\Delta)(W^\eta)^{-1} f](q,p).\end{aligned}$$

which proves that $\widetilde{E}^+_Q$ is Naimark's dilation of $W^\eta\, F^Q_\eta\, (W^\eta)^{-1}$. An analogous argument holds for $\widetilde{E}^+_P$ and $W^\eta\, F^Q_\eta\, (W^\eta)^{-1}$.

Now, if we specialize ourselves to the case $g = \frac{1}{l\sqrt{2\pi}}\, e^{(-\frac{x^2}{2\,l^2})}$, $l \in \mathbb{R} - \{0\}$, we get, (see, e.g., Ref. [5])

$$\widetilde{P}^\eta \Big(\int t\, d\widetilde{E}^+_Q(t) \Big) \widetilde{P}^\eta = W^\eta \int t\, dF^Q_\eta(t)\, (W^\eta)^{-1} = W^\eta\, Q\, (W^\eta)^{-1}$$
$$\widetilde{P}^\eta \Big(\int t\, d\widetilde{E}^+_P(t) \Big) \widetilde{P}^\eta = W^\eta \int t\, dF^P_\eta(t)\, (W^\eta)^{-1} = W^\eta\, P\, (W^\eta)^{-1}.$$

Therefore, the compatible operators $Q^+ := \int t\, d\widetilde{E}^+_Q(t)$ and $P^+ := \int t\, d\widetilde{E}^+_P(t)$ are dilations of $W^\eta\, Q\, (W^\eta)^{-1}$ and $W^\eta\, P\, (W^\eta)^{-1}$ respectively. All that is summarized (up to isometry) in the following commutative diagram.



We can observe the following transitions: 1) from the position and momentum operators to their compatible smearings. That corresponds to the transition from incompatibility to compatibility and from quantum mechanics to phase space quantum mechanics [15, 17], 2) the transition from the non-commuting position and momentum operators in $\mathcal{H}$ to the commuting position and momentum operators in $\mathcal{H}^+$. That corresponds to the transition from quantum mechanics to classical mechanics [9, 10].

Finally note that in the standard formalism of quantum mechanics, where observables are represented by self-adjoint operators, it is nonsense to speak about the joint measurement of Q and P since they do not commute. That is instead possible once F_η^Q and F_η^P are introduced into the formalism.

3. Universal Markov kernel

In the previous sections we have shown the relevance of commutative POVMs in the phase space approach and in the transition from incompatibility to compatibility. Such a transition is realized by the smearing procedure which, by Theorem 6, is realized through a Markov kernel.

In the present section we analyze the structure of the smearing in general and prove the existence of a Markov kernel μ such that for any real commutative POVM F there is a sharp version E^F of F such that $F(\Delta) = \int \mu_\Delta(\lambda)\, dE^F$. In other words, we prove the existence of a universal Markov kernel.
In the following, the set of commutative POVMs with spectrum in $[0,1]$ is denoted by $\mathcal{D}$.

We need some results obtained in References [3, 12] which we briefly recall. There exists an algorithmic procedure (which is an extension of a procedure developed by Riesz, see Ref. [16, page 356]) for the construction of a family of set functions $\{\omega_{(\cdot)}(\lambda)\}_{\lambda\in[0,1]}$ with the property that, for any commutative POVM $F : \mathcal{B}[0,1] \to \mathcal{L}_s^+(\mathcal{H})$, there exists a self-adjoint operator A^F with spectrum $\sigma(A^F) \subset [0,1]$ such that

$$F(\Delta) = \int_{[0,1]} \omega_\Delta(\lambda)\, dE_\lambda^F = \omega_\Delta(A^F), \quad \Delta \in \mathcal{B}[0,1], \tag{5}$$

where, E^F is the spectral resolution corresponding to A^F. Moreover, there exists a countable semi-ring $\mathcal{S}$ which generates the σ-algebra $\mathcal{B}[0,1]$, such that, for each $\lambda \in \sigma(A^F)$, $\omega_{(\cdot)}(\lambda)$ is additive on the ring $\mathcal{R}(\mathcal{S})$ generated by $\mathcal{S}$. For each $\Delta \in \mathcal{B}[0,1]$, the function ω_Δ is Borel measurable.

Now we introduce some technicalities. In what follows we need the set $I := \cup_{F\in\mathcal{D}}\, \sigma(A^F) \subset [0,1]$ to be measurable. Thus, if I is not a Borel set we enlarge the Borel σ-algebra in order to include I. In particular, we consider the σ-algebra $\mathfrak{S}$ generated by I and $\mathcal{B}([0,1])$.

Since, $\forall F \in \mathcal{D}$, $[0,1] \setminus I \subset [0,1] \setminus \sigma(A^F)$ and $E^F([0,1] \setminus \sigma(A^F)) = \mathbf{0}$, the set $[0,1]/I$ is a subset of a E^F-null set for any $F \in \mathcal{D}$. Then, each PVM E^F can be extended to $\mathfrak{S}$. The extension $\widetilde{E}^F : \mathfrak{S} \to \mathcal{L}_s(\mathcal{H})$ satisfies the following relations:

$$\widetilde{E}^F(\sigma(A^F)) = \widetilde{E}^F(I) = \mathbf{1}$$

$$\widetilde{E}^F(I \cap \Delta) = E^F(\Delta), \quad \forall \Delta \in \mathcal{B}[0,1]$$

$$A^F = \int_{[0,1]} \lambda\, d\widetilde{E}_\lambda^F.$$

The space $([0,1],\mathfrak{S})$ is a measurable space and I is a measurable subset of $\mathfrak{S}$. Moreover, for each $\Delta\in\mathcal{B}([0,1])$, the function $\omega_\Delta:([0,1],\mathfrak{S})\to([0,1],\mathcal{B}[0,1])$ is measurable and

$$\int_{[0,1]}\omega_\Delta(\lambda)\,d\widetilde{E}^F_\lambda=\int_{[0,1]}\omega_\Delta(\lambda)\,dE^F_\lambda=F(\Delta),\quad\forall F\in\mathcal{D}. \tag{6}$$

Theorem 11. *There is a Markov kernel* $\widetilde{\mu}:([0,1],\mathfrak{S})\times\mathcal{B}[0,1]\to([0,1],\mathcal{B}[0,1])$ *such that*

$$F(\Delta)=\int_{[0,1]}\widetilde{\mu}_\Delta(\lambda)\,d\widetilde{E}^F_\lambda,$$

for any $\Delta\in\mathcal{B}[0,1]$ *and* $F\in\mathcal{D}$.

Proof. Let $\mathcal{D}$ be the set of the commutative POVMs with spectrum in $[0,1]$, $\{\omega_\Delta\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ the family of functions whose existence is proved in Ref. [12] and $I:=\cup_{F\in\mathcal{D}}\sigma(A^F)\subset[0,1]$. For each $\lambda\in I$, the set function $\omega_{(\cdot)}(\lambda)$ is additive on $\mathcal{R}(\mathcal{S})$ [3]. Now, the map

$$\widetilde{\omega}:([0,1],\mathfrak{S})\times\mathcal{B}[0,1]\to([0,1],\mathcal{B}[0,1])$$
$$\widetilde{\omega}_\Delta(\lambda)=\begin{cases}\omega_\Delta(\lambda), & \text{if }\lambda\in I,\\ 0, & \text{if }\lambda\in[0,1]/I\end{cases}$$

is such that $\widetilde{\omega}_{(\cdot)}(\lambda)$ is additive for any $\lambda\in[0,1]$ and $\widetilde{\omega}_\Delta(\cdot)$ is measurable for any $\Delta\in\mathcal{B}[0,1]$. Moreover, by the definition of the integral and by (6),

$$\int_{[0,1]}\widetilde{\omega}_\Delta(\lambda)\,d\widetilde{E}^F_\lambda=\int_{[0,1]}\omega_\Delta(\lambda)\,d\widetilde{E}^F_\lambda=F(\Delta),\quad F\in\mathcal{D}.$$

By Corollary 1 in [4][1], there is a Markov kernel $\widetilde{\mu}$ such that, for every $F\in\mathcal{D}$, $\widetilde{\mu}_\Delta(\lambda)=\widetilde{\omega}_\Delta(\lambda)$, $\widetilde{E}^F$-a.e.. Therefore, for every pair (F,E^F), the triplet $(F,\widetilde{E}^F,\widetilde{\mu})$ is a von Neumann triplet, i.e.,

$$\int_{[0,1]}\widetilde{\mu}_\Delta(\lambda)\,d\widetilde{E}^F_\lambda=F(\Delta)$$

and $\widetilde{\mu}$ does not depend on F. □

Note that $\widetilde{\mu}_\Delta$ coincide with ω_Δ up to null sets. Indeed,

$$\int_{[0,1]}\widetilde{\mu}_\Delta(\lambda)\,d\widetilde{E}^F_\lambda=F(\Delta)=\int_{[0,1]}\omega_\Delta(\lambda)\,d\widetilde{E}^F_\lambda$$

so that $\widetilde{\mu}_\Delta(\lambda)=\omega_\Delta(\lambda)$, $\widetilde{E}^F$-a.e.. Therefore, $\widetilde{\mu}_\Delta(\lambda)=\omega_\Delta(\lambda)$, E^F-a.e. since E^F is the restriction of $\widetilde{E}^F$.

[1] Note that the proof of corollary 1 in Ref. [4] does not depend on the POVM F. It depends only on the set functions $\omega_{(\cdot)}(\lambda)$. Moreover, the change from the function $\omega_\Delta:([0,1],\mathcal{B}[0,1])\to([0,1],\mathcal{B}[0,1])$ to the function $\widetilde{\omega}_\Delta:([0,1],\mathfrak{S})\to([0,1],\mathcal{B}[0,1])$ does not affect the proof of the corollary since it does not affect the set functions $\omega_{(\cdot)}(\lambda):\mathcal{B}[0,1]\to\mathcal{B}[0,1]$ for any $\lambda\in I$.

Remark 12. The result in Theorem 11 holds also for the set of POVMs with spectrum in $\mathbb{R}$. Indeed, for every POVM $F : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s^+(\mathcal{H})$, there is a POVM $F^0 : \mathcal{B}([0,1]) \to \mathcal{L}_s^+(\mathcal{H})$, $F^0(\Delta) = F(g(\Delta))$ where $g : [0,1] \to \mathbb{R}$ is an arbitrary bijective measurable map. Then, the Markov kernel $\mu_{g^{-1}(\Delta)}(\lambda)$ is a universal Markov kernel, i.e.,

$$F(\Delta) = F^0(g^{-1}(\Delta)) = \int \mu_{g^{-1}(\Delta)}(\lambda)\, d\widetilde{E}^F_\lambda, \quad \Delta \in \mathcal{B}(\mathbb{R}),$$

for every real commutative POVM F.

We then have the following generalization of Theorem 11.

Theorem 13. *There is a Markov kernel* $\widetilde{\mu} : ([0,1], \mathfrak{S}) \times \mathcal{B}(\mathbb{R}) \to ([0,1], \mathcal{B}[0,1])$ *such that*

$$F(\Delta) = \int_{\mathbb{R}} \widetilde{\mu}_\Delta(\lambda)\, d\widetilde{E}^F_\lambda, \quad \Delta \in \mathcal{B}(\mathbb{R}),$$

for any real commutative POVM F.

Acknowledgment

The present work was performed under the auspices of the GNFM (Gruppo Nazionale di Fisica Matematica).

References

[1] S.T. Ali, G.G. Emch, *Fuzzy observables in quantum mechanics*, J. Math. Phys. **15** (1974), 176.

[2] S.T. Ali, *A geometrical property of POV-measures and systems of covariance*, Lecture Notes in Mathematics **905** (1982), 207–228.

[3] R. Beneduci, G. Nisticó, *Sharp reconstruction of unsharp quantum observables*, J. Math. Phys. **44** (2003), 5461.

[4] R. Beneduci, *A geometrical characterizations of commutative positive operator valued measures*, J. Math. Phys. **47** (2006), 062104.

[5] R. Beneduci, *Joint measurability through Naimark's dilation theorem*, Rep. Math. Phys. **79** (2017), 197–213.

[6] R. Beneduci, *Positive Operator Valued Measures and Feller Markov Kernels*, J. Math. Anal. Appl. **442** (2016), 50–71.

[7] R. Beneduci, *Unsharp number observable and Neumark theorem*, Il Nuovo Cimento B **123** (2008), 43.

[8] R. Beneduci, *Unsharpness, Naimark Theorem and Informational Equivalence of Quantum Observables*, Int. J. Theor. Phys. **49** (2010), 3030.

[9] R. Beneduci, J. Brooke, R. Curran, and F.E. Schroeck, *Classical Mechanics in Hilbert Space, part I*, Int. J. Theor. Phys. **50** (2011), 3682–3696.

[10] R. Beneduci, J. Brooke, R. Curran, and F.E. Schroeck, *Classical Mechanics in Hilbert Space, part II*, Int. J. Theor. Phys. **50** (2011), 3697–3723.

[11] Busch P., Grabowski M. and Lahti P., *Operational quantum physics*, Lecture Notes in Physics **31** (1995).

[12] G. Cattaneo, G. Nisticò, *From unsharp to sharp quantum observable: The general Hilbert space case*, J. Math. Phys. **41** (200), 4365.

[13] E.B. Davies, *Quantum mechanics of Open Systems*, Academic Press, London, 1976.

[14] M.A. Naimark, Izv. Akad. Nauk SSSR Ser. Mat. **4** (1940), 277–318.

[15] E. Prugovečki, *Stochastic Quantum Mechanics and Quantum Spacetime*, D. Reidel Publishing Company, Dordrecht, Holland, 1984.

[16] F. Riesz, B.Sz. Nagy, Functional Analysis, Dover, 1990.

[17] F.E. Schroeck, Jr., *Quantum Mechanics on Phase Space*, Kluwer Academic Publishers, Dordrecht, 1996.

Roberto Beneduci
Dipartimento di Fisica
Università della Calabria

and

Istituto Nazionale di Fisica Nucleare
Gruppo c. Cosenza
I-87036 Arcavacata di Rende (Cs), Italy
e-mail: roberto.beneduci@unical.it

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 31–37

Coherent States Associated to the Jacobi Group and Berezin Quantization of the Siegel–Jacobi Ball

Stefan Berceanu

In memory of Syed Twareque Ali

Abstract. Coherent states are associated to the Jacobi group. The metric obtained from the scalar product of coherent states based on the Siegel–Jacobi ball is a balanced metric. Several geometric properties of the Siegel–Jacobi ball are obtained via the methods of coherent states. We insist on geometric properties of the Siegel–Jacobi ball specific to Berezin quantization.

Mathematics Subject Classification (2010). 81R30, 32Q15, 81S10.

Keywords. Coherent states, Berezin quantization, balanced metric, Jacobi group, Siegel–Jacobi ball.

1. Introduction

The quantization problem is important for establishing a correspondence between quantum and classical systems [1]. A quantization method was proposed by Berezin [2–5]. Initially, Berezin applied his method to quantization of the Kähler manifolds $\mathbb{C}^n$ and the Hermitian symmetric spaces, using the supercomplete set of vectors verifying the Parceval overcompletness identity. We have investigated holomorphic discrete series representations based on hermitian symmetric spaces [6–8] and then [9] on coherent state (CS) manifolds [10, 11], using Perelomov coherent state method [12]. Loi and Mossa have extended Berezin quantization to homogeneous bounded domains [13] and then to homogenous Kähler manifolds [14]. On the other side, Rawnsley [15] and Rawnsley, Cahen and Gutt [16] have globalized Berezin construction for homogeneous Kähler manifolds to non-homogeneous Kähler manifolds. Using the so-called ϵ-function, it was underlined that Berezin quantization via coherent states is a particular realization of geometric quantization [17], and the manifold with $\epsilon = ct$ are quantizable manifolds. An essential object in this approach is the notion of balanced metric, introduced firstly by Donaldson for compact manifolds [18], and later extended to non-compact manifolds by Arezzo and Loi [19].

We have constructed coherent states based on the Siegel–Jacobi ball $\mathcal{D}^J_n \approx \mathbb{C}^n \times \mathcal{D}_n$ [20–23], a homogeneous space associated to the Jacobi group $G^J_n = \mathrm{H}_n \rtimes \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$ [24, 25], where $\mathcal{D}_n$ denotes the Siegel ball, and H_n denotes the $(2n+1)$-dimensional Heisenberg group [22, 23, 26]. The relevance of the Jacobi group for Physics and Mathematics was mentioned in the references [20, 22, 23]. Here are some more recent references on applications of the Jacobi group in Physics and Mathematics [27–31]. It was underlined in [32] that the homogeneous Kähler two-form $\omega_{\mathcal{D}^J_n}$ calculated in [22, 23] it is associated with the balanced metric on the Siegel–Jacobi ball.

In this paper we mention several geometric properties of the Siegel–Jacobi ball obtained via the coherent state method, relevant for Berezin quantization. The interest of this investigation comes from the fact that the Siegel–Jacobi ball is a partially bounded domain [33, 34]. In Section 2 we collect several results on the balanced metric of the Siegel–Jacobi ball, extracted from [21–23, 32]. The main results of this article are contained in Remark 5 and Proposition 6 of Section 3. Preliminary results have been given in [35], while details on the proofs can be found in [32].

2. Balanced metric on the Siegel–Jacobi ball via coherent states

Let $M = G/H$ be an n-dimensional homogeneous Kähler manifold endowed with a G-invariant Kähler two-form obtained from a Kähler potential f

$$\omega_M(z) = \mathrm{i}\sum_{\alpha,\beta=1}^{n} h_{\alpha\bar\beta}(z)\,\mathrm{d}\,z_\alpha \wedge \mathrm{d}\,\bar z_\beta,\ \ h_{\alpha\bar\beta} = \bar h_{\beta\bar\alpha} = h_{\bar\beta\alpha},\ \ h_{\alpha\bar\beta} = \frac{\partial^2 f}{\partial z_\alpha \partial \bar z_\beta}. \tag{1}$$

The balanced metric corresponds to the Kähler potential equal with the logarithm of the scalar product of two Perelomov coherent state vectors

$$f(z,\bar z) = \ln K_M(z,\bar z),\ \ K_M(z,\bar z) = (e_{\bar z}, e_{\bar z}). \tag{2}$$

The ϵ-function [15, 16] is defined as

$$\epsilon(z) = \mathrm{e}^{-f(z)} K_M(z,\bar z). \tag{3}$$

The Jacobi algebra is the semi-direct sum $\mathfrak{g}^J_n := \mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}}$, where the Heisenberg algebra $\mathfrak{h}_n$ is generated by the boson creation (respectively, annihilation) operators $a^\dagger_i$ (a_i), while the symplectic algebra $\mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}}$ is generated by $K^{\pm,0}_{ij}$, $i,j = 1,\dots,n$, as in [22, 23].

Let $g \in \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$ be of the form (4), (5)

$$g = \begin{pmatrix} p & q \\ \bar q & \bar p \end{pmatrix}, \quad p,q \in M(n,\mathbb{C}), \tag{4}$$

$$pp^* - qq^* = \mathbb{1}_n, \quad pq^t = qp^t; \tag{5a}$$

$$p^*p - q^t\bar q = \mathbb{1}_n, \quad p^t\bar q = q^*p, \tag{5b}$$

and let also $\alpha, z \in \mathbb{C}^n$. The (transitive) action $(g,\alpha)\times(W,z) = (W_1, z_1)$ of the Jacobi group $G^J_n = \mathrm{H}_n \rtimes \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$ on the Siegel–Jacobi ball $\mathcal{D}^J_n \approx \mathcal{D}_n \times \mathbb{C}^n$ is given by the formulas [22]

$$W_1 = (pW+q)(\bar{q}W+\bar{p})^{-1} = (Wq^*+p^*)^{-1}(q^t+Wp^t), \tag{6a}$$

$$z_1 = (Wq^*+p^*)^{-1}(z+\alpha-W\bar{\alpha}). \tag{6b}$$

The Siegel (open) ball $\mathcal{D}_n$ – the non-compact Hermitian symmetric space $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}/\,\mathrm{U}(n)$ – admits a matrix realization as the bounded homogeneous domain

$$\mathcal{D}_n := \{W \in M(n,\mathbb{C}) : W = W^t, N > 0, N := \mathbb{1}_n - W\bar{W}\}. \tag{7}$$

Perelomov coherent state vectors [12] associated to the group G^J_n, based on the Siegel–Jacobi ball $\mathcal{D}^J_n$, are [21, 22]

$$e_{z,W} = \exp(\boldsymbol{X})e_0,\ \boldsymbol{X} := \sqrt{\mu}\sum_{i=1}^{n} z_i\boldsymbol{a}^{\dagger}_i + \sum_{i,j=1}^{n} w_{ij}\boldsymbol{K}^{+}_{ij},\ z\in\mathbb{C}^n; W\in\mathcal{D}_n, \tag{8}$$

where the extremal weight vector e_0 is chosen such that

$$\boldsymbol{a}_i e_o = 0,\ \boldsymbol{K}^{+}_{ij}e_0 \neq 0,\ \boldsymbol{K}^{-}_{ij}e_0 = 0,\ \boldsymbol{K}^{0}_{ij}e_0 = \frac{k}{4}\delta_{ij}e_0,\ i,j = 1,\dots,n. \tag{9}$$

μ in (8) indexes representations of the Heisenberg group, while k in (9) parametrizes the holomorphic discrete series representation of $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$.

Using the coherent state vectors (8), the reproducing kernel $K(z,W) = (e_{z,W}, e_{z,W})_{k\mu}$, $z\in\mathbb{C}^n, W\in\mathcal{D}_n$ was calculated in [21–23] as

$$K(z,W) = \det(M)^{\frac{k}{2}}\exp\mu F, M = (\mathbb{1}_n - W\bar{W})^{-1}, \tag{10a}$$

$$2F = 2\bar{z}^t M z + z^t\bar{W}Mz + \bar{z}^t MW\bar{z}. \tag{10b}$$

With formulas (1), (2), we have obtained in [21, 22]

Theorem 1. *The Kähler two-form $\omega_{\mathcal{D}^J_n}$, G^J_n-invariant to the action* (6), *is*

$$\begin{aligned}-\mathrm{i}\omega_{\mathcal{D}^J_n}(z,W) &= \tfrac{k}{2}\mathrm{Tr}(\mathcal{B}\wedge\bar{\mathcal{B}}) + \mu\mathrm{Tr}(\mathcal{A}^t\bar{M}\wedge\bar{\mathcal{A}}),\\ \mathcal{B} &= M\,\mathrm{d}W,\ \mathcal{A} = \mathrm{d}z + \mathrm{d}W\bar{\eta},\ \eta = M(z+W\bar{z}).\end{aligned} \tag{11}$$

The fact that the Kähler two-form (11) is associated with the balanced metric on $\mathcal{D}^J_n$ was underlined in [32], where we have split the matrix h of the metric into four blocks. This allowed us to calculate a kind of inverse of h, which takes into account the fact that W is a symmetric matrix, as in (7). As a consequence, we were able to calculate in [32] the Ricci form, the scalar curvature, the determinant $\mathcal{G}$ of the matrix h of the metric and the Laplace–Beltrami operator on $\mathcal{D}^J_n$, while in [36] we have studied geodesics on $\mathcal{D}^J_n$. In [23] we have pointed out the significance of the change of variables in (11) $z \to \eta$ in the context of coherent states and also in the context of the fundamental conjecture of homogeneous Kähler manifolds [37, 38]. We have proved:

Proposition 2.

i) *The Jacobi group* G_n^J *is a unimodular, non-reductive, algebraic group of Harish-Chandra type.*

ii) *The Siegel–Jacobi domain* $\mathcal{D}_n^J$ *is a homogeneous reductive, non-symmetric manifold associated to the Jacobi group* G_n^J *by the generalized Harish-Chandra embedding.*

iii) *The Siegel–Jacobi ball* $\mathcal{D}_n^J$ *is not an Einstein manifold with respect to the balanced metric attached to the Kähler two-form* (11)*, but it is one with respect to the Bergman metric corresponding to the Bergman Kähler two-form* $\mathrm{i}\partial\bar{\partial}\ln(\mathcal{G}_{\mathcal{D}_n^J})$.

iv) *The scalar curvature of* $\mathcal{D}_n^J$ *is constant and negative.*

The Harish-Chandra embedding of the Siegel–Jacobi ball is explained in [39].

3. Geometric characterization of $\mathcal{D}_n^J$ and Berezin quantization

Firstly we recall the mentioned new results about Berezin quantization obtained in [13, 14], which we apply to Berezin quantization on the Siegel–Jacobi ball.

Theorem 3. *Let* (M, ω) *be a simply-connected homogeneous Kähler manifold such that the associated Kähler two-form* ω *is integral. Then there exists a constant* $\mu_0 > 0$ *such that* M *equipped with* $\mu_0\omega$ *is projectively induced.*

The notion of projectively induced manifolds in the context of coherent states for compact manifolds is used in [40, 41], while [42] deals with projectively induced noncompact manifolds. We recall that the proof of Theorem 3 in the case of compact manifolds M was given in [41].

Using the results of Rosenberg-Vergne [43], the proof of the fundamental conjecture of homogeneous Kähler manifolds of Vinberg and Gindikin [37] by Dorfmeister–Nakajima [38] and the sufficient conditions of Berezin quantization on bounded domains obtained by Engliš [44], Loi and Mossa have proved [14] the following:

Theorem 4. *Let* (M, ω) *be a homogeneous Kähler manifold. Then the following are equivalent:*

a) M *is contractible.*

b) (M, ω) *admits a global Kähler potential.*

c) (M, ω) *admits a global diastasis* $D_M : M \times M \to \mathbb{R}$.

d) (M, ω) *admits a Berezin quantization.*

The notion of diastasis was introduced in [45].

As a consequence of Theorem 3, it was proven in [32]:

Remark 5. Let $M = G/H$ be a simply-connected homogeneous Kähler manifold. Then the following assertions are equivalent:

A) M is a quantizable Kähler manifold.

B) M admits a balanced metric.
C) M is CS-type manifold and G is a CS-type group.
D) M is projectively induced.

The notion of CS-group is explained in [10, 11].

Putting together Theorems 1, 3, 4, Remark 5, and Proposition 4 in [32], it follows in the particular case of the Jacobi group:

Proposition 6.

i) *The homogeneous Kähler manifold* $\mathcal{D}_n^J$ *is contractible.*
ii) *The Kähler potential of the Siegel–Jacobi ball is global.* $\mathcal{D}_n^J$ *is a Lu Qi-Keng manifold, with nowhere vanishing diastasis.*
iii) *The manifold* $\mathcal{D}_n^J$ *is a quantizable Kähler manifold.*
iv) *The manifold* $\mathcal{D}_n^J$ *is projectively induced, and the Jacobi group* G_n^J *is a CS-type group.*

In [42] we have used the denomination *Lu Qi-Keng manifold* for manifolds for which the polar divisor [40, 41] of all points of the manifold is zero, extending to manifolds the notion introduced for domains in $\mathbb{C}^n$ [46].

Acknowledgment

I am indebted to the Organizing Committee of the XXXVI Workshop on Geometric Methods in Physics for the opportunity to report results at the 2017 Białowieża meeting on Geometric methods in Physics. The communication was included in the special session dedicated to the memory of S. Twareque Ali, a distinguished physicist and mathematician, a wonderful friend, who encouraged me to investigate the Jacobi group through coherent states. This investigation was supported by the Nucleus Program PN 16420101/2016, the Phase 5/2017.

References

[1] S.T. Ali and M. Engliš, *Quantization: a guide for physicists and analysts*, Rev. Math. Phys. Vol. 17, No. 4, (2005) 391–490.

[2] F.A. Berezin, *Quantization in complex bounded domains* (Russian), Dokladi Akad. Nauk SSSSR, Ser. Mat. **211** (1973) 1263–1266.

[3] F.A. Berezin, *Quantization* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974) 1116–1175.

[4] F.A. Berezin, *Quantization in complex symmetric spaces* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975) 363–402.

[5] F.A. Berezin, *The general concept of quantization*, Commun. Math. Phys. **40** (1975) 153–174.

[6] S. Berceanu and C.A. Gheorghe, *On the construction of perfect Morse functions on compact manifolds of coherent states*, J. Math. Phys. **28** (1987) 2899–2907.

[7] S. Berceanu and A. Gheorghe, *On equations of motion on Hermitian symmetric spaces*, J. Math. Phys. **33** (1992) 998–1007.

[8] S. Berceanu and L. Boutet de Monvel, *Linear dynamical systems, coherent state manifolds, flows and matrix Riccati equation*, J. Math. Phys. **34** (1993) 2353–2371.

[9] S. Berceanu, Realization of coherent state algebras by differential operators, in *Advances in Operator Algebras and Mathematical Physics*, F. Boca, O. Bratteli, R. Longo and H. Siedentop, editors, The Theta Foundation, Bucharest, 1–24, 2005; arXiv:math.DG/0504053.

[10] W. Lisiecki, *Coherent state representations. A survey*, Rep. Math. Phys. **35** (1995) 327–358.

[11] K.-H. Neeb, *Holomorphy and Convexity in Lie Theory*, de Gruyter Expositions in Mathematics **28**, Walter de Gruyter, Berlin-New York, 2000.

[12] A.M. Perelomov, *Generalized Coherent States and their Applications*, Springer, Berlin, 1986.

[13] A. Loi and R. Mossa, *Berezin quantization of homogenous bounded domains*, Geom. Dedicata **161** (2012) 119–128.

[14] A. Loi and R. Mossa, *Some remarks on homogeneous Kähler manifolds*, Geom. Dedicata **179** (2015) 377–383.

[15] J.H. Rawnsley, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford Ser. **28** (1977) 403–415.

[16] M. Cahen, S. Gutt and J. Rawnsley, *Quantization of Kähler manifolds I: geometric interpretation of Berezin's quantization*, J. Geom. Phys. **7** (1990) 45–62.

[17] B. Kostant, *Quantization and unitary representations I. Prequantization*, in Lecture Notes in Mathematics **170**, C.T. Tam, editor, Springer-Verlag, Berlin, 87–208, 1970.

[18] S. Donaldson, *Scalar curvature and projective embeddings, I*, J. Diff. Geom. **59** (2001) 479–522.

[19] C. Arezzo and A. Loi, *Moment maps, scalar curvature and quantization of Kähler manifolds*, Commun. Math. Phys. **246** (2004) 543–549.

[20] S. Berceanu, *A holomorphic representation of the Jacobi algebra*, Rev. Math. Phys. **18** (2006) 163–199; *Errata*, Rev. Math. Phys. **24** (2012) 1292001 (2 pages); arXiv:math/0408219.

[21] S. Berceanu, *A holomorphic representation of the semidirect sum of symplectic and Heisenberg Lie algebras*, J. Geom. Symmetry Phys. **5** (2006) 5–13.

[22] S. Berceanu, A holomorphic representation of Jacobi algebra in several dimensions, in *Perspectives in Operator Algebra and Mathematical Physics*, F.-P. Boca, R. Purice and S. Stratila, editors, Theta Ser. Adv. Math. **8**, Theta, Bucharest, 1–25, 2008; arXiv: math.DG/060404381.

[23] S. Berceanu, *A convenient coordinatization of Siegel–Jacobi domains*, Rev. Math. Phys. **24** (2012) 1250024, 38 pages; arXiv: 1204.5610v1.

[24] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55** Birkhäuser, Boston, MA, 1985.

[25] R. Berndt and R. Schmidt, *Elements of the representation theory of the Jacobi group*, Progress in Mathematics **163**, Birkhäuser, Basel, 1998.

[26] J.-H. Yang, *The method of orbits for real Lie groups*, Kyungpook Math. J. **42** (2) (2002) 199–272.

[27] G. Chiribella and G. Adesso, *Quantum benchmarks for pure single-mode Gaussian states*, Phys. Rev. Lett. **112** (2014) 010501, 6 pages.

[28] M. Molitor, *Gaussian distributions, Jacobi group, and Siegel–Jacobi space*, J. Math. Phys. **55** (2014) 122102, 41 pages.

[29] G. Marmo, P.W. Michor and Y.A. Neretin, *The Lagrangian Radon transform and the Weil representation*, J. Fourier Anal. Appl. **20** (2014) 321–361.

[30] B. Cahen, *Stratonovich–Weyl correspondence for the Jacobi group*, Commun. Math. **22** (2014) 31–48.

[31] B. Cahen, *Berezin transform and Stratonovich–Weyl correspondence for the multi-dimensional Jacobi group*, Rend. Semin. Mat. Univ. Padova **136** (2016) 69–93.

[32] S. Berceanu, *Balanced metric and Berezin quantization on the Siegel–Jacobi ball*, SIGMA **12** (2016) 064, 28 pages; arXiv:1512.00601.

[33] J.-H. Yang, *A partial Cayley transform of Siegel–Jacobi disk*, J. Korean Math. Soc. **45** (2008) 781–794.

[34] J.-H Yang, *Invariant metrics and Laplacians on the Siegel–Jacobi disk*, Chin. Ann. Math. **31B** (2010) 85–100.

[35] S. Berceanu, Remarks on Berezin quantization on the Siegel–Jacobi ball, talk at "31st International Colloquium on Group Theoretical Methods in Physics", Rio de Janeiro, Brazilia, 19–25 June 2016, in "Physical and Mathematical Aspects of Symmetries", S. Duarte, J.P. Gazeau, S. Faci, T. Micklitz, R. Scherer, F. Toppan, editors, Springer, Cham, 105–110, 2018.

[36] S. Berceanu, *Geodesics associated to the balanced metric on the Siegel–Jacobi ball*, Romanian J. Phys. **61** (2016) 1137–1160; arXiv: 1605.02962

[37] E.B. Vinberg and S.G. Gindikin, *Kählerian manifolds admitting a transitive solvable automorphism group* (Russian), Math. Sb. 74 (116) (1967) 333–351.

[38] J. Dorfmeister and K. Nakajima, *The fundamental conjecture for homogeneous Kähler manifolds*, Acta Math. **161** (1988) 23–70.

[39] S. Berceanu and A. Gheorghe, *On the geometry of Siegel–Jacobi domains*, Int. J. Geom. Methods Mod. Phys. **8** (2011) 1783–1798; arXiv:1011.3317v1.

[40] S. Berceanu, *Coherent states and geodesics: cut locus and conjugate locus*, J. Geom. Phys. **21** (1997) 149–168.

[41] S. Berceanu and M. Schlichenmaier, *Coherent state embeddings, polar divisors and Cauchy formulas*, J. Geom. Phys. **34** (2000) 336–358.

[42] S. Berceanu, *Bergman representative coordinates on the Siegel–Jacobi disk*, Romanian J. Phys. **60** (2015) 867–896; arXiv: 1409.0368.

[43] J. Rosenberg and M. Vergne, *Harmonically induced representations of solvable Lie groups*, J. Funct. Anal. **62** (1985) 8–37.

[44] M. Engliš, *Berezin quantization and reproducing kernels on complex domains*, Trans. Am. Math. Soc. **348** (1996) 411–479.

[45] E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. **58** (1953) 1–23.

[46] Q.-K. Lu, *On Kähler manifolds with constant curvature*, Acta. Math. Sin. **66** (1966) 269–281.

Stefan Berceanu
Horia Hulubei National Institute for Physics and Nuclear Engineering
Department of Theoretical Physics
P.O.B. MG-6
RO-077125 Magurele, Romania
e-mail: Berceanu@theory.nipne.ro

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 39–45

1D & 2D Covariant Affine Integral Quantizations

Jean Pierre Gazeau and Romain Murenzi

Abstract. Covariant affine integral quantization of the half-plane $\mathbb{R} \times \mathbb{R}_*^+$ is presented. We examine the consequences of different quantizer operators built from weight functions on the half-plane. One of these weights yields the usual canonical quantization and a quasi-probability distribution (affine Wigner function) which is real, marginal in both position and momentum vectors. An extension to the phase space for the motion of a particle in the punctured plane and its application to the quantum rotating frame are mentioned.

Mathematics Subject Classification (2010). 81S30; 83C45,81R30.

Keywords. Integral quantization, affine symmetry, wavelets, singularities.

This contribution is an outline of recent developments [1–4] of what we call *affine covariant integral quantization* and its applications. This affine quantization is useful in physics, for instance when we deal with the motion of a particle on the half-line, more generally for the dynamics of a pair (q, p), $q > 0$, $p \in \mathbb{R}$. The origin of the half-line is a singularity and the phase space is the open half-plane, which has the group structure of $\mathrm{Aff}_+(\mathbb{R})$, with its two square integrable unitary irreducible representations (UIR's). Any consistent quantization must respect this symmetry. Another example [4] concerns the motion in the punctured plane (i.e., deleted from its origin), for instance in a rotating plane frame, or an infinitely thin solenoid perpendicular to the plane at the origin. In this case, the phase space is $\mathbb{R}_*^2 \times \mathbb{R}^2$, which has the group structure of $\mathrm{SIM}(2)$, the group of similitudes of $\mathbb{R}^2$, with its unique square integrable UIR. Again, any consistent quantization must respect this symmetry. Respecting those symmetries yields an automatic regularization of the singularity in the quantum model in the following sense

$$\text{classical kinetic } p^2 \longrightarrow \text{ quantum kinetic } P^2 + \frac{K}{Q^2}\,, \tag{1}$$

with $K > 0$ is issued from affine quantization and essentially self-adjointness is insured if $K \geq 3/4$. Here we restrict the presentation to the simplest $\mathrm{Aff}_+(\mathbb{R})$ quantization. For $\mathrm{SIM}(2)$ nothing really new except more elaborate formulae, and details are found in [4]. For the motion in the punctured $\mathbb{R}^d_*$, $d \geq 3$, the phase space is in general a coset of SIM(d), and one can deal with square-integrable representations of SIM(d) with respect to a subgroup [5].

1. Covariant integral quantization(s) for a group

Let G be a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation of G in a Hilbert space $\mathcal{H}$. Let M be a bounded operator on $\mathcal{H}$. Suppose that the operator

$$R := \int_G \mathsf{M}(g)\,\mathrm{d}\mu(g)\,, \quad \mathsf{M}(g) := U(g)\,\mathsf{M}\,U^\dagger(g)\,, \tag{2}$$

is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$ the operator R commutes with all operators $U(g)$, $g \in G$, and so, from Schur's Lemma, $R = c_{\mathsf{M}} I$. Suppose that there exists a density (i.e., unit trace non-negative) operator ρ_0 such that the integral $\int_G \mathrm{tr}\,(\rho_0\,\mathsf{M}(g))\,\mathrm{d}\mu(g) := c_{\mathsf{M}}$ is convergent. Then the resolution of the identity follows:

$$\int_G \mathsf{M}(g)\,\mathrm{d}\nu(g) = I\,, \quad \mathrm{d}\nu(g) := \mathrm{d}\mu(g)/c_{\mathsf{M}}\,. \tag{3}$$

Now, suppose that the UIR U is *square-integrable* in the sense that there exists a density operator ρ such that $c_\rho = \int_G \mathrm{d}\mu(g)\,\mathrm{tr}\,\left(\rho U(g)\rho U^\dagger(g)\right) < \infty$. The resolution of the identity is then obeyed by the family of U-transported density operators $\rho(g) = U(g)\rho U^\dagger(g)$, and this allows *covariant* integral quantization of complex-valued functions on the group

$$f \mapsto A_f = \frac{1}{c_\rho}\int_G \rho(g)\,f(g)\,\mathrm{d}\mu(g)\,. \tag{4}$$

The covariance of this quantization is a straightforward consequence of the above construction

$$U(g)A_f U^\dagger(g) = A_{\mathfrak{U}(g)f}\,, \tag{5}$$

where $(\mathfrak{U}(g)f)(g') := f(g^{-1}g')$ is the regular representation if $f \in L^2(G, \mathrm{d}\mu(g)))$.

Furthermore, the map (4) can be completed with a generalization of the Berezin or heat kernel transform on G yielding a *semi-classical portrait* of A_f.

$$f(g) \mapsto \check{f}(g) := \int_G \mathrm{tr}(\rho(g)\,\rho(g'))\,f(g')\,\mathrm{d}\nu(g')\,. \tag{6}$$

2. Covariant affine integral quantization

2.1. The group framework

The above procedure is now implemented in the context of affine symmetry. As the Euclidean plane is viewed as the phase space for the motion of a particle on the line, the half-plane is viewed as the phase space for the motion of a particle on the half-line. One equips the upper half-plane $\Pi_+ := \{(q,p) \,|\, q > 0\,, p \in \mathbb{R}\}$ with the measure $\mathrm{d}q\,\mathrm{d}p$. Together with

(i) the multiplication law

$$(q,p)(q_0,p_0) = \left(qq_0, \frac{p_0}{q} + p\right), \; q \in \mathbb{R}_+^*, \; p \in \mathbb{R}\,,$$

(ii) the unity $(1,0)$,

(iii) and the inverse

$$(q,p)^{-1} = \left(\frac{1}{q}, -qp\right)\,,$$

Π_+ is viewed as the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line. The measure $\mathrm{d}q\,\mathrm{d}p$ is left-invariant with respect to this action. The affine group $\mathrm{Aff}_+(\mathbb{R})$ has two non-equivalent UIR $U_\pm$ ($\sim$ carried on by Hardy spaces). Both are square integrable and this is the rationale backing the *continuous wavelet analysis* resulting from a resolution of the identity. The UIR $U_+ \equiv U$ is realized in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+^*, \mathrm{d}x)$ as

$$U(q,p)\psi(x) = (e^{\mathrm{i}px}/\sqrt{q})\psi(x/q)\,. \tag{7}$$

Given a weight function $\varpi(q,p)$ on Π_+, one defines the operator

$$\int_{\Pi_+} \mathsf{C}_{\mathrm{DM}}^{-1} U(q,p) \mathsf{C}_{\mathrm{DM}}^{-1}\, \varpi(q,p)\, \mathrm{d}q\,\mathrm{d}p := \mathsf{M}^\varpi\,. \tag{8}$$

The following assumptions are imposed on $\varpi(q,p)$:

(i) The weight function $\varpi(q,p)$ is C^∞ on Π_+.

ii) It defines a tempered distribution with respect to the variable p for all $q > 0$.

(iii) The operator M^ϖ is bounded self-adjoint on $\mathcal{H}$.

The appearance of the positive self-adjoint and invertible Duflo–Moore operator $\mathsf{C}_{\mathrm{DM}} := \sqrt{2\pi/Q}$ is due to the non-modularity of the affine group. This operator is needed to establish the square-integrability of the UIR U

$$\int_{\Pi_+} \mathrm{d}q\,\mathrm{d}p \langle U(q,p)\psi|\phi\rangle\, \overline{\langle U(q,p)\psi'|\phi'\rangle} = \langle \mathsf{C}_{\mathrm{DM}}\psi|\mathsf{C}_{\mathrm{DM}}\psi'\rangle\, \langle\phi'|\phi\rangle\,, \tag{9}$$

for any pair (ψ,ψ') of admissible vectors, i.e., which obey $\|\mathsf{C}_{\mathrm{DM}}\psi\| < \infty$, $\|\mathsf{C}_{\mathrm{DM}}\psi'\| < \infty$, and any pair (ϕ,ϕ') of vectors in $L^2(\mathbb{R}_+^*, \mathrm{d}x)$. The operator M^ϖ is symmetric if $\varpi(q,p)$ obeys $\varpi(q,p) = (1/q)\overline{\varpi\,(1/q,-qp)}$.

2.2. Affine quantization

The corresponding integral quantization reads as

$$f \mapsto A_f^\varpi = \int_{\Pi_+} \frac{\mathrm{d}q\,\mathrm{d}p}{c_{\mathsf{M}^\varpi}}\, f(q,p)\, \mathsf{M}^\varpi(q,p)\,, \tag{10}$$

with $\mathsf{M}^\varpi(q,p) = U(q,p)\mathsf{M}^\varpi U^\dagger(q,p)$, and where the constant c_{M^ϖ} is given by

$$c_{\mathsf{M}^\varpi} = \sqrt{2\pi}\int_0^{+\infty} \frac{\mathrm{d}q}{q}\, \hat{\varpi}_p\,(1,-q)\,.$$

Here, $\hat{\varpi}_p$ is the partial Fourier transform of ϖ with respect to the variable p. The resolution of the identity holds for $c_{\mathsf{M}^\varpi} < \infty$, and the covariance reads as

$$U(q_0, p_0) A^\varpi_f U^\dagger(q_0, p_0) = A\varpi_{\mathfrak{U}(q_0,p_0)f} \,.$$

The practical calculations rest on the following result.

Proposition 1. *The action on ϕ in $\mathcal{H}$ of the operator A^ϖ_f defined by the integral quantization map is given by*

$$(A^\varpi_f \phi)(x) = \int_0^{+\infty} \mathcal{A}^\varpi_f(x, x')\, \phi(x')\, \mathrm{d}x' \,, \tag{11}$$

where the kernel $\mathcal{A}^\varpi_f$ is defined as

$$\mathcal{A}^\varpi_f(x, x') = \frac{1}{c_{\mathsf{M}^\varpi}} \frac{x}{x'} \int_0^{+\infty} \frac{\mathrm{d}q}{q}\, \hat{\varpi}_p\left(\frac{x}{x'}, -q\right)\, \hat{f}_p\left(\frac{x}{q}, x' - x\right) . \tag{12}$$

2.3. Some formulae

Having in hand Eqs. (11) and (12), one can easily derive the quantum counterparts of some particular functions. For functions depending on q only, $f(q,p) \equiv u(q)$, one gets the multiplication operator

$$A^\varpi_{u(q)} = \frac{\sqrt{2\pi}}{c_{\mathsf{M}^\varpi}} \int_0^{+\infty} \frac{\mathrm{d}q}{q}\, \hat{\varpi}_p(1, -q)\, u\left(\frac{Q}{q}\right) , \tag{13}$$

i.e., the multiplication by the convolution on the multiplicative group $\mathbb{R}^*_+$ of $u(x)$ with $\sqrt{2\pi}\, \hat{\varpi}_p(1, -x)/c_{\mathsf{M}^\varpi}$. An interesting more specific case is when u is a simple power of q, say $u(q) = q^\beta$. Then we have

$$A^\varpi_{q^\beta} = \frac{\sqrt{2\pi}}{c_{\mathsf{M}^\varpi}} \int_0^{+\infty} \frac{\mathrm{d}q}{q^{1+\beta}}\, \hat{\varpi}_p(1, -q)\, Q^\beta \equiv \frac{d_\beta}{d_0}\, Q^\beta \,, \tag{14}$$

where $d_\beta = \int_0^{+\infty} \frac{\mathrm{d}q}{q^{1+\beta}}\, \hat{\varpi}_p(1, -q)$

For momentum-dependent functions $f(q,p) \equiv v(p)$ one finds

$$\begin{aligned} \mathcal{A}^\varpi_{v(p)}(x, x') &= \frac{1}{c_{\mathsf{M}^\varpi}}\, \hat{v}(x' - x)\, \frac{x}{x'} \int_0^{+\infty} \frac{\mathrm{d}q}{q}\, \hat{\varpi}_p\left(\frac{x}{x'}, -q\right) \\ &\equiv \frac{1}{c_{\mathsf{M}^\varpi}}\, \hat{v}(x' - x)\, \frac{x}{x'}\, \Omega\left(\frac{x}{x'}\right) . \end{aligned} \tag{15}$$

As a simple but important example, let us examine the case $v(p) = p^n$, $n \in \mathbb{N}$. From distribution theory

$$\hat{v}(x' - x) = \sqrt{2\pi}\, \mathrm{i}^n\, \delta^{(n)}(x' - x) \,, \tag{16}$$

we derive the differential action of the operator $A^\varpi_{p^n}$ in $\mathcal{H}$ as the polynomial in $P = -\mathrm{i}d/dx$

$$A^\varpi_{p^n} = \frac{\sqrt{2\pi}}{c_{\mathsf{M}^\varpi}} \sum_{k=0}^{n} \binom{n}{k} \left(-\mathrm{i}\frac{d}{dx'}\right)^{n-k} \frac{x}{x'}\, \Omega\left(\frac{x}{x'}\right)\Big|_{x'=x} P^k = P^n + \cdots \,. \tag{17}$$

In particular,

$$A_p^{\varpi} = P + \frac{\mathrm{i}}{Q}\left[1 + \frac{\Omega'(1)}{\Omega(1)}\right]. \tag{18}$$

This operator is symmetric but has no self-adjoint extension. The commutation rule $[A_q, A_p] = \frac{d_1}{d_0}\,\mathrm{i}I$ is canonical up to a factor which can be easily put equal to one through a rescaling of the weight function.

For the kinetic energy we have

$$A_{p^2}^{\varpi} = P^2 + \frac{2\mathrm{i}}{Q}\left[1 + \frac{\Omega'(1)}{\Omega(1)}\right] P - \frac{1}{Q^2}\left[2 + 4\,\frac{\Omega'(1)}{\Omega(1)} + \frac{\Omega''(1)}{\Omega(1)}\right]. \tag{19}$$

This symmetric operator is essentially self-adjoint or not, depending on the strength of the (attractive or repulsive) potential $1/x^2$. With the choice of a weight function such that $-2 - 4\,\frac{\Omega'(1)}{\Omega(1)} - \frac{\Omega''(1)}{\Omega(1)} \geq 3/4$, it is essentially self-adjoint and so quantum dynamics of the free motion on the half-line is unique. For separable functions $f(q,p) \equiv u(q)\,v(p)$

$$\mathcal{A}_{u(q)v(p)}^{\varpi}(x,x') = \frac{1}{c_{\mathsf{M}^{\varpi}}}\,\hat{v}(x'-x)\,\frac{x}{x'}\int_0^{+\infty}\frac{\mathrm{d}q}{q}\,\hat{\varpi}_p\left(\frac{x}{x'},-q\right)\,u\left(\frac{x}{q}\right). \tag{20}$$

The elementary example is the quantization of the function qp which produces the integral kernel and its corresponding operator

$$\begin{aligned}\mathcal{A}_{qp}^{\varpi}(x,x') &= \frac{\sqrt{2\pi}}{c_{\mathsf{M}^{\varpi}}}\,\mathrm{i}\,\delta'(x'-x)\,\frac{x^2}{x'}\int_0^{+\infty}\frac{\mathrm{d}q}{q^2}\,\hat{\varpi}_p\left(\frac{x}{x'},-q\right),\\ A_{qp}^{\varpi} &= \frac{\Omega_1(1)}{\Omega(1)}\,D + \mathrm{i}\left[\frac{3}{2}\,\frac{\Omega_1(1)}{\Omega(1)} + \frac{\Omega_1'(1)}{\Omega(1)}\right],\end{aligned} \tag{21}$$

where $D = \frac{1}{2}(QP + PQ)$ is the essentially self-adjoint dilation generator. Here

$$\Omega_\beta(u) = \int_0^{+\infty}\frac{\mathrm{d}q}{q^{1+\beta}}\,\hat{\varpi}_p\,(u,-q)\,, \quad \Omega_0(u) = \Omega(u)\,. \tag{22}$$

2.4. Semi-classical portraits

Given a weight function $\varpi(q,p)$ yielding a symmetric unit trace operator M^{ϖ}, we define the semi-classical or lower symbol of an operator A in $\mathcal{H}$ as the function

$$\check{A}(q,p) := \mathrm{Tr}\left(A\,U(q,p)\,\mathsf{M}^{\varpi}\,U^{\dagger}(q,p)\right) = \mathrm{Tr}\left(A\,\mathsf{M}^{\varpi}(q,p)\right). \tag{23}$$

When the operator A is the affine integral quantized version of a classical $f(q,p)$ with the same weight ϖ, we get the transform

$$f(q,p) \mapsto \check{f}(q,p) = \int_{\Pi_+}\frac{\mathrm{d}q'\,\mathrm{d}p'}{c_{\mathsf{M}^{\varpi}}}\,f\left(qq', \frac{p'}{q} + p\right)\,\mathrm{Tr}\left(\mathsf{M}^{\varpi}(q',p')\mathsf{M}^{\varpi}\right). \tag{24}$$

Of course, this expression has the meaning of an averaging of the classical f if the function

$$\begin{aligned}(q,p) \mapsto & \frac{1}{c_{\mathsf{M}^{\varpi}}} \mathrm{Tr}\left(\mathsf{M}^{\varpi}(q,p)\mathsf{M}^{\varpi}\right) \\ &= \frac{1}{c_{\mathsf{M}^{\varpi}}} \frac{1}{2\pi q} \int_0^{+\infty} \mathrm{d}x \int_0^{+\infty} \mathrm{d}y \, e^{-\mathrm{i}p(y-x)} \, \hat{\varpi}_p\left(\frac{x}{y}, -\frac{x}{q}\right) \, \hat{\varpi}_p\left(\frac{y}{x}, -y\right)\end{aligned} \tag{25}$$

is a true probability distribution on the half-plane.

3. Affine Wigner integral quantization

With the specific weight $\varpi_{a\mathcal{W}}(q,p) = \frac{e^{-\mathrm{i}\sqrt{q}p}}{\sqrt{q}}$ we obtain twice the affine inversion operator $(\mathcal{I}\psi)(x) := (1/x)\,\psi(1/x)$, $\mathcal{I}^2 = I$,

$$\mathsf{M}^{a\mathcal{W}} \equiv 2\mathcal{I} = \int_{\Pi_+} U(q,p)\, \varpi_{a\mathcal{W}}(q,p)\, \mathrm{d}q\, \mathrm{d}p\,. \tag{26}$$

This operator is the affine counterpart of the operator yielding the Weyl–Wigner integral quantization when the phase space is $\mathbb{R}^2$, i.e., when one deals with Weyl–Heisenberg symmetry.

Proposition 2. *The integral kernel of the quantization of a function $f(q,p)$ through the weight function has the following expression,*

$$\mathcal{A}_f^{a\mathcal{W}}(x,x') = \frac{1}{\sqrt{2\pi}}\, \hat{f}_p\left(\sqrt{\frac{x'}{x}}, x'-x\right)\,. \tag{27}$$

The application of this procedure to particular cases gives the following results.

Proposition 3.

(i) *The quantization of a function of q, $f(q,p) = u(q)$ provided by the weight $\varpi_{a\mathcal{W}}$ is $u(Q)$.*

(ii) *Similarly, the quantization of a function of p, $f(q,p) = v(p)$ provided by the weight $\varpi_{a\mathcal{W}}$ is $v(P)$ (in the pseudo-differential sense).*

(iii) *More generally, the quantization of a separable function $f(q,p) = u(q)\,v(p)$ provided by the weight $\varpi_{a\mathcal{W}}$ is the integral operator*

$$\left(A^{a\mathcal{W}}_{u(q)v(p)}\,\psi\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}x'\, \hat{v}(x'-x)\, u\left(\sqrt{x\,x'}\right)\, \psi(x')\,.$$

(iv) *In particular, the quantization of $u(q)\,p^n$, $n \in \mathbb{N}$, yields the symmetric operator,*

$$A^{a\mathcal{W}}_{u(q)p^n} = \sum_{k=0}^{n} \binom{n}{k} (-\mathrm{i})^{n-k} u^{(n-k)}(Q)\, P^k\,,$$

and for the dilation, $A^{a\mathcal{W}}_{qp} = D$

Therefore, this affine integral quantization is the true counterpart of the Weyl–Wigner integral quantization.

4. Conclusion

The procedure presented here has been applied to questions pertaining to early cosmology in its quantum aspects (see [6] and references therein), with the choice of projector quantizer, $\mathsf{M}^{\varpi} = |\psi\rangle\langle\psi|$ ("affine coherent states"). It allowed us to set up a consistent quantum dynamics of isotropic, anisotropic non-oscillatory and anisotropic models, yielding singularity resolution for the pair volume-rate through a smooth bouncing and a unitary dynamics without boundary conditions, and offering a consistent semi-classical description of involved quantum dynamics. Another recent application points up the possible existence of an extra quantum centrifugal effect in rotating frame [4].

Remark. The ideas and opinions expressed in this article are those of the authors and do not necessarily represent the view of UNESCO.

References

[1] H. Bergeron and J.-P. Gazeau Integral quantizations with two basic examples, *Annals of Physics (NY)*, **344** 43–68 (2014) arXiv:1308.2348 [quant-ph, math-ph]

[2] H. Bergeron, A. Dapor, J.-P. Gazeau, and P. Małkiewicz, Smooth big bounce from affine quantization, *Phys. Rev. D* **89**, 083522 (2014); arXiv:1305.0653 [gr-qc]

[3] J.-P. Gazeau and R. Murenzi, Covariant Affine Integral Quantization(s), *J. Math. Phys.*, **57**, 052102 (2016); arXiv:1512.08274 [quant-ph]

[4] J.-P. Gazeau, T. Koide and R. Murenzi, More quantum centrifugal effect in rotating frame; arXiv:1704.02832v1 [quant-ph], EPL 118, 50004 (2017)

[5] S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*, 2d edition, Springer, 2014

[6] H. Bergeron, E. Czuchry, J.-P. Gazeau, and P. Małkiewicz, Spectral properties of the quantum Mixmaster universe, *Phys. Rev. D* **96**, 043521 (2017); arXiv: 1703.08462 [gr-qc]

Jean Pierre Gazeau
Astroparticle and Cosmology APC
Univ Paris Diderot, Sorbonne Paris Cité
F-75205 Paris, France
e-mail: gazeau@apc.in2p3.fr

Romain Murenzi
The World Academy of Sciences
TWAS/ICTP
Via Costiera 1
I-34151 Trieste, Italy
e-mail: rmurenzi@ictp.it

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 47–56

Diffeomorphism Group Representations in Relativistic Quantum Field Theory

Gerald A. Goldin and David H. Sharp

It has been an honor to present this talk in memory of my friend and colleague S. Twareque Ali, who passed away in January 2016*. – G.A. Goldin*

Abstract. We explore the role played by the diffeomorphism group and its unitary representations in relativistic quantum field theory. From the quantum kinematics of particles described by representations of the diffeomorphism group of a space-like surface in an inertial reference frame, we reconstruct the local relativistic neutral scalar field in the Fock representation. An explicit expression for the free Hamiltonian is obtained in terms of the Lie algebra generators (mass and momentum densities). We suggest that this approach can be generalized to fields whose quanta are spatially extended objects.

Mathematics Subject Classification (2010). Primary 81R10; Secondary 81T10.

Keywords. Diffeomorphism groups, relativistic quantum fields, unitary group representations.

1. Introduction

The focus of this article is on the role played by diffeomorphism groups and their representations in relativistic quantum field theory (QFT).

We highlight the important fact that diffeomorphism group representations arise naturally if one starts with the well-known Fock representation of the free, neutral relativistic scalar field, describing non-interacting bosons [1, 2]. We show how this occurs. The mass density and the momentum flux density obtained from the relativistic quantum field are reference-frame-dependent constructs; they are neither local in spacetime nor Lorentz covariant. However, they are essential to the description of what one actually *measures* – particle locations at particular times, and/or trajectories. Indeed, such measurements always take place in a specific inertial reference frame.

Our main idea is to reverse the direction of this construction. That is, we propose to start with a fixed frame of reference, and with respect to that frame, to obtain the quantum kinematics described by unitary representations of the group of diffeomorphisms of space and its semidirect product with the group of scalar functions. Then we identify hierarchies of such representations, and introduce intertwining creation and annihilation fields. Only at that point do we introduce the spacetime symmetry group (the Poincaré group in this case) which provides the information needed to construct local relativistic fields out of the intertwining fields.

In Section 2 of this article, we briefly review results on diffeomorphism group representations in non-relativistic quantum theory. Section 3 highlights our earlier characterization of hierarchies of representations. Section 4 is about obtaining current algebra and the corresponding diffeomorphism group representations from the field theory of relativistic neutral scalar bosons. In Section 5, we discuss more generally how one can begin with "non-relativistic" representations of the diffeomorphism group and the current algebra, and obtain a relativistic quantum field from them. We also express the free relativistic Hamiltonian explicitly in terms of the original particle density and current operators. Section 6 mentions some possible directions opened up by our approach.

2. Diffeomorphism groups in Galilean quantum theory

Diffeomorphism groups and their unitary representations play a fundamental role in non-relativistic (Galilean) quantum theory [3]. To set the stage, we review this briefly; for details, see [4] and references therein.

Let Σ be the manifold of physical space. Of course Σ can be regarded as a submanifold of spacetime, with (for example) $t = 0$. In Galilean theory, Σ is independent of the velocity of the observer. Let $\mathcal{D} = C_0^\infty(\Sigma)$ be the group of compactly-supported smooth real-valued functions on Σ, under pointwise addition, and let $\mathcal{K} = \mathit{Diff}_0(\Sigma)$ be the group of compactly supported C^∞ diffeomorphisms of Σ, under composition. Let $G = \mathcal{D} \times \mathcal{K}$ be the natural semidirect product of these groups. Then the irreducible, continuous unitary representations of G describe the quantum kinematics of a wide variety of physical systems.

For $(f, \phi) \in \mathcal{D} \times \mathcal{K}$, let us write a continuous unitary representation (CUR) of G as $U(f)V(\phi)$. Under very general conditions, one may realize the representation in a Hilbert space $\mathcal{H} = L^2_\mu(\Gamma, \mathcal{M})$, with

$$[U(f)\Psi](\gamma) = \exp i\langle\gamma, f\rangle\Psi(\gamma),$$
$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}\,; \tag{1}$$

where: Γ is a configuration space whose elements (denoted γ) are continuous linear functionals on $\mathcal{D}$; $\langle\gamma, f\rangle$ denotes the value of γ at f; $\mathcal{M}$ is a complex inner product space (accommodating vector-valued wave functions), and $\Psi \in \mathcal{H}$ takes values in $\mathcal{M}$; $\phi\gamma$ denotes the natural group action of a diffeomorphism $\phi \in \mathcal{K}$ on Γ; μ

is a measure on Γ quasiinvariant under diffeomorphisms; $d\mu_\phi/d\mu$ is the Radon–Nikodym derivative of the transformed measure with respect to the original one; and $\chi_\phi(\gamma)$ is a unitary 1-cocycle acting in $\mathcal{M}$. Each of these has a fairly direct physical interpretation in quantum mechanics.

The Lie algebra of self-adjoint local current operators is defined in a continuous unitary representation $U(f)V(\phi)$ of G by:

$$\rho(f) = m \lim_{s\to 0} \frac{U(sf) - I}{is}, \quad J(\mathbf{g}) = \hbar \lim_{s\to 0} \frac{V(\phi_s^{\mathbf{g}}) - I}{is} \tag{2}$$

where: $f \in \mathcal{D}$, $\mathbf{g} \in \mathrm{vect}_0^\infty(\Sigma)$ [the Lie algebra of compactly-supported C^∞ vector fields on Σ], and $\phi_s^{\mathbf{g}}$ is the flow generated by $\mathbf{g}$ under the real parameter s; m is a unit mass, and $\hbar$ is Planck's constant (over 2π). Here ρ describes the (space-averaged) mass density, and J the (space-averaged) momentum flux density. Then:

$$\begin{gathered}[\rho(f_1), \rho(f_2)] = 0, \qquad [\rho(f), J(\mathbf{g})] = i\hbar\rho(L_{\mathbf{g}}f) \\ [J(\mathbf{g_1}), J(\mathbf{g_2})] = -i\hbar J([\mathbf{g_1}, \mathbf{g_2}])\,,\end{gathered} \tag{3}$$

where $L_{\mathbf{g}}$ is the Lie derivative and $[\mathbf{g_1}, \mathbf{g_2}]$ is the Lie bracket of vector fields.

This framework unifies descriptions of the quantum kinematics of a wide variety of systems. Particular families of representations describe configuration spaces and exchange statistics for N-particle systems, including systems of indistinguishable particles satisfying Bose statistics, Fermi statistics, and parastatistics. In two space dimensions one obtains anyon (braid group) statistics [5–9] and non-abelian braid statistics [10]. Tightly-bound composite particles (quantum dipoles, quadrupoles, etc.) are also described [11].

One further obtains configuration spaces of infinite but locally finite particle systems, as in statistical mechanics [12–14], as well as infinite systems with accumulation points [15, 16]. Systems of extended configurations, such as vortex patches, filaments, and tubes, are also described by diffeomorphism group representations [17–19]. See also [20].

3. Hierarchies of representations

The irreducible unitary diffeomorphism group representations fall naturally into *hierarchies*, whose intertwining operators (satisfying a natural commutator bracket with the densities and currents) create and annihilate objects of the same kind. These intertwining operators have an interpretation as "second-quantized" fields, which are general enough to describe not only point particles but also extended objects [21, 22].

For example, consider a family of N-particle representations, where $N = 0, 1, 2, \ldots$. We have the Hilbert spaces $\mathcal{H}_N$, and the unitary representations $U_N(f)V_N(\phi)$. We are entitled to call the family a *hierarchy* if for each N there exists an operator-valued distribution $\psi_N^* : \mathcal{H}_N \to \mathcal{H}_{N+1}$ (the creation field), such that for all $h \in \mathcal{D}$, $f \in \mathcal{D}$, and $\phi \in \mathcal{K}$,

$$U_{N+1}(f)\psi_N^*(h) = \psi_N^*(U_{N=1}(f)h)U_N(f)\,, \tag{4}$$

$$V_{N+1}(\phi)\psi_N^*(h) = \psi_N^*(V_{N=1}(\phi)h)V_N(\phi)\,. \tag{5}$$

Physically, if we begin with an N-particle state, create a new particle in state h and then transform the resulting $(N+1)$-particle state by the unitary group representation U_{N+1} (or respectively, V_{N+1}), the result is the same as if we first transform the N-particle state by U_N (resp. V_N), and then create the new particle in the state obtained by transforming the 1-particle state h by U_1 (resp. V_1). The construction is nontrivial because creation and annihilation within a hierarchy must respect the particle statistics (bosonic, fermionic, or anyonic). When $\mathcal{H}_1 = L^2(\mathbb{R}^3, \mathbb{C})$, as in the 1-particle representation of $\mathcal{D} \times \mathcal{K}$, we write $\psi^*(h) = \int \psi^*(\mathbf{x})h(\mathbf{x})d^3x$. We call $\psi^*(\mathbf{x})$ an *intertwining field* for the hierarchy. In effect, $\psi^*(\mathbf{x})$ creates a particle at $\mathbf{x} \in \Sigma$, and its adjoint $\psi(\mathbf{x})$ annihilates a particle at $\mathbf{x}$. We earlier used these ideas to derive the q-commutation relations for anyon fields [21].

Consequently ψ^* and ψ also obey natural brackets with the local current algebra generating the group representations. Furthermore they obey a bracket with each other: canonical commutation relations, when they intertwine N-particle Bose representations; anticommutation relations when they intertwine N-particle Fermi representations; and q-commutation relations when they intertwine N-anyon representations (in 2-space), where q is the anyonic phase. They are non-relativistic fields.

In terms of ψ^* and ψ, we may write the current algebra generators (formally) as operator-valued distributions:

$$\rho(\mathbf{x}) = \psi^*(\mathbf{x})\psi(\mathbf{x}), \quad \mathbf{J}(\mathbf{x}) = (1/2i)\{\psi^*(\mathbf{x})\nabla\psi(\mathbf{x}) - [\nabla\psi^*(\mathbf{x})]\psi(\mathbf{x})\}\,. \tag{6}$$

We thus have a rather beautiful unifying, current-algebraic description of a wide variety of distinct non-relativistic quantum systems as representations of a local symmetry group. It is then natural to ask if there is a role to be played by the diffeomorphism group and its representations in *relativistic* quantum field theories.

4. The diffeomorphism group and the free relativistic neutral scalar Bose field

Suppose we begin with the free neutral scalar relativistic field in the Fock representation. We use the following notational conventions to write the main equations of interest.

In Minkowski spacetime, $x^\mu = (x^0, \mathbf{x})$, with $\mu = 0, 1, 2, 3$; and where $x^0 = ct$; the metric tensor $g_{\mu\nu} = \mathrm{diag}\,[1, -1, -1, -1]$. The covariant momentum 4-vector is $p_\mu = (p_0, \mathbf{p})$, where $p_0 = E/c$, E being the energy. The wave number 4-vector is $k = (k_0, \mathbf{k})$, where $E = \hbar\omega$ and $k_0 = \omega/c = E/\hbar c$, and $\mathbf{p} = \hbar\mathbf{k}$. Since $E^2 = p^2c^2 + m^2c^4$, we have $\omega^2 = \mathbf{k}^2c^2 + (m^2c^4/\hbar^2)$. For a given value of $\mathbf{k}$, we thus also write $k_0 = \omega_{\mathbf{k}}/c$, where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2c^2 + (m^2c^4/\hbar^2)}$.

Let $a^*_{\mathbf{k}}$ and $a_{\mathbf{k}}$ be (respectively) creation and annihilation operators for the free *relativistic* neutral scalar field in the Fock representation, for a particle with energy $E = \hbar\omega_{\mathbf{k}}$ and 3-momentum $\mathbf{p} = \hbar\mathbf{k}$. Then we have,

$$[a_{\mathbf{k}_1}, a^*_{\mathbf{k}_2}] = \omega_{\mathbf{k}_1}\delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2). \tag{7}$$

From these (relativistic) operators, one constructs the (non-relativistic) fields from which particle measurement operators are obtained [2]:

$$\phi_1(x) = \int_{k_0>0} \frac{d^3k}{k_0}\, \frac{1}{(2\pi)^{3/2}}\, (k_0)^{1/2}\, e^{-ikx} a_{\mathbf{k}}\,, \tag{8}$$

and setting $t = 0$ and $k_0 = \omega_{\mathbf{k}}$, we have

$$\phi_1(\mathbf{x}, 0) = \int \frac{d^3k}{\omega_{\mathbf{k}}^{1/2}}\, \frac{1}{(2\pi)^{3/2}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}}\,. \tag{9}$$

Then ϕ_1 and ϕ_1^* satisfy the equal-time canonical commutation relations of the intertwining fields ψ and ψ^* discussed above; i.e., without any coefficient in the δ-function:

$$[\phi_1(x^0, \mathbf{x}), \phi_1^*(y^0, \mathbf{y})]_{x^0=y^0} = \delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{10}$$

Next ρ and $\mathbf{J}$ can be defined in terms of these field operators on each N-particle subspace of the Fock representation, using Eq. (6). They in turn satisfy the local current algebra (3). When exponentiated, we obtain the N-boson representations of the semidirect product group G – i.e., "non-relativistic" quantum mechanics existing within relativistic quantum field theory. We also highlight the remarkable fact that ρ and $\mathbf{J}$ (unlike the relativistic fields) are actually defined as distributions over space at a fixed time.

This construction is the one whose direction we want to reverse. That is, having obtained ψ and ψ^* by intertwining diffeomorphism group representations at $t = 0$ in a specific inertial reference frame, we write relativistic creation and annihilation operators $a^*_{\mathbf{k}}$ and $a_{\mathbf{k}}$ using the inverse transform of the equations above. From these, we construct the relativistic quantum field.

5. Relativistic QFT from diffeomorphism group representations: General approach

5.1. Motivation

The spacetime symmetry group informs us how to relate one inertial frame of reference to another. Traditionally, relativistic QFT begins with the introduction of fields assumed to be covariant with respect to the Poincaré group, encoding the physics of special relativity into the theory right from the start.

But the group of diffeomorphisms of spacetime is, in a sense, incompatible with the Poincaré symmetry. With the exception of $(1 + 1)$-dimensional spacetime, general diffeomorphisms disrupt the Minkowskian causal structure. More specifically, call a diffeomorphism of Minkowski space M *causal* if for any pair of points $x, y \in M$, it preserves the sign $(+, 0, -)$ of the Minkowskian "distance" $(x-y)^2 := (x-y)_\mu (x-y)^\mu$ (summation convention, with the Minkowski metric). In $(1+1)$-dimensional spacetime, an infinite-dimensional group of causal diffeomorphisms exists, defined by independent actions on each of the two light cone coordinates. But in Minklowski spacetimes having more than one space dimension, the causal diffeomorphisms are limited to the finite-dimensional group of Poincaré transformations together with uniform dilations. In Galileian spacetime, on the other hand, diffeomorphisms that act on spatial coordinates only (possibly in a time-dependent way) are among those respecting the causal structure. Thus in the Galilean case we have an infinite-dimensional group.

This incompatibility of general spacetime diffeomorphisms with special relativity, together with the value of the diffeomorphism group approach in describing general quantum kinematics, leads us to the idea of describing measurements of particles or other entities (field quanta) in a *fixed* inertial frame – the frame in which the actual measurements occur. Then we need not worry about covariance; the measurement operators can be noncovariant and nonlocal in the spacetime (although local in space at a fixed time). Of course, the corresponding operators for measurements taken in a different inertial reference frame will be different, and not obtainable directly from the first set of operators until after covariant fields (or some other way of encoding the spacetime symmetry) have been introduced. We thus defer the construction of fields covariant under the spacetime symmetry until later.

Our idea is natural if we focus on describing *spatial configurations* in a relativistic theory (with Poincaré symmetry). One must grapple with the fact that the shapes of spacetime regions, the particular choices of spacelike surfaces such as hyperplanes in Minkowskian spacetime, and the shapes of regions within those surfaces, change with the reference frame of the observer. Therefore, since we are *forced* at some stage to deal with noncovariant objects, it makes sense to begin with them. This is why we specify a frame of reference *before* beginning the constructions that lead to particle configuration spaces, and *without* having yet identified the spacetime symmetry.

5.2. Anticipated steps

Thus we envision the following steps in the program.

1. Choose an inertial frame of reference F (the frame of the observer). We have not yet built in how observations in one inertial frame are related to those in another.

2. Call the spacetime as observed from F by the name M_F. Introduce a coordinate system for M_F in which the coordinate $\mathbf{x}$ refers to space, and t to time. A "spacelike" surface Σ_F, coordinatized by $\mathbf{x}$, is obtained by setting $t = 0$ in M_F. We postulate a Euclidean metric on Σ_F, which is to play the role of the manifold Σ in the earlier discussion. Evidently this approach is general enough to include spatial manifolds Σ_F having nontrivial topology, as well as higher-dimensional spacetimes (e.g., 10- or 26-dimensional) with some of the spatial dimensions compactified.

It is natural to regard the different spacetimes M_F as fibers in a bundle over a base space of inertial frames. Each fiber carries a copy of the theory. The spacetime symmetry should eventually establish the isomorphism of the theories (diffeomorphism group representations, etc.) in different fibers.

3. Define the group $G = \mathcal{D} \times \mathcal{K}$ with respect to Σ_F and consider its continuous unitary representations as discussed above. We remark that even in relativistic QFT, we need to describe (spatial) configurations based on observable locations and motions of entities (particles, excitations). Unitary representations of $G(\Sigma_F)$ are natural because one-parameter groups of diffeomorphisms (i.e., flows) describe possible smooth motions of configurations in physical space. The infinitesimal generators of such flows are local currents.

4. Identify one or more *hierarchies* of representations, describing configurations consisting of entities of the same type (to be interpreted as quanta of the same field). Introduce creation and annihilation fields as operators intertwining the unitary representations in the hierarchy.

At this stage in the general development, we have a full description of the field quanta, but we do not yet have the field – nor do we have a description of the dynamics, which is to be provided as usual by a Hamiltonian operator.

5. The next step is to introduce a (covariant) relativistic field, defined making use of the creation and annihilation fields intertwining the hierarchy of diffeomorphism group representations in inertial frame F. This is where the particular choice of spacetime symmetry (relating different reference frames) is introduced. The configuration-space entities in the hierarchy of diffeomorphism group representations are interpreted as quanta of this field as observable in the reference frame F.

6. Finally, write the Hamiltonian H describing the (relativistic) dynamics. While H can be expressed in terms of the relativistic quantum field, the preceding construction means that it can also be expressed explicitly in terms of the local currents (the infinitesimal generators in the diffeomorphism group representations with which we started), together with the (non-relativistic) intertwining fields. At the end of the construction, the physics described by the relativistic field and Hamiltonian should not depend on the particular reference frame F with which we began.

5.3. Constructing the relativistic free neutral scalar field

Carrying out first four steps in $(3+1)$-dimensional Minkowski space, for the hierarchy of N-particle Bose representations of $G = \mathcal{D} \times \mathcal{K}$, we obtain the creation and annihilation fields ϕ_1^*, ϕ in Fock space satisfying Eq. (10).

To construct the relativistic field $\phi(x)$, we use the three-dimensional inverse transforms of $\phi_1(\mathbf{x}, 0)$ and $\phi_1^*(\mathbf{x}, 0)$, corresponding to Eq. (9); thus,

$$\frac{1}{(2\pi)^{3/2}} \int d^3x \, \phi_1(\mathbf{x}, 0) e^{i\mathbf{k}\cdot\mathbf{x}} = a_{\mathbf{k}} / \omega_{\mathbf{k}}^{1/2} \,, \tag{11}$$

and correspondingly for the adjoint. This is precisely the point where relativistic invariance has been put in "by hand" – the definitions of $a^*_{\mathbf{k}}$ and $a_{\mathbf{k}}$ from the intertwining fields ϕ_1^* and ϕ_1 are such as to satisfy a relativistic bracket.

Then the relativistic field is, as usual,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}\sqrt{2}} \int_{k_0>0} \frac{d^3k}{k_0} (a_{\mathbf{k}} e^{-ikx} + a^*_{\mathbf{k}} e^{ikx}) \,. \tag{12}$$

We stress that both ϕ_1 and ϕ are operator-valued distributions in four dimensions in the same Hilbert space, for any value of c – including the Galilean limit $c \to \infty$.

5.4. The free relativistic Hamiltonian in terms of local currents

We conclude by expressing the free relativistic Hamiltonian in terms of diffeomorphism group generators (i.e., local currents), in this representation. In a calculation beginning with

$$H = \int \frac{d^3k}{k_0} \hbar \, \omega_{\mathbf{k}} \, a^*_{\mathbf{k}} a_{\mathbf{k}} = \frac{1}{2} \int d^3x : \pi(x)^2 + \nabla\phi(x) \cdot \nabla\phi(x) + m^2 \phi(x)^2 : \tag{13}$$

where the colons : : denote normal ordering, we obtain the singular-looking expression,

$$H = \iint d^3x \, d^3y \, F(\mathbf{x} - \mathbf{y}) \, \rho(\mathbf{x}) \, \exp \int_{\mathbf{x}}^{\mathbf{y}} \frac{1}{2\rho(\mathbf{x}')} \mathbf{K}(\mathbf{x}') d\mathbf{x}' \,, \tag{14}$$

where $\mathbf{K}(\mathbf{x}) = \nabla\rho(\mathbf{x}) + 2i\mathbf{J}(\mathbf{x})$, and $F(\mathbf{x} - \mathbf{y}) = \int d^3k \, \omega_{\mathbf{k}} \exp[i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})]$. One can obtain the Galilean free Hamiltonian explicitly as a limiting case of the above relativistic Hamiltonian.

We remark that there are ways to make mathematical sense of the apparently singular expressions in Eq. (14) involving products and quotients of the (spatially) local density and current operators.

6. Discussion and concluding remarks

The example of the free relativistic boson field serves as a model for more general quantum field theories built up from hierarchies of diffeomorphism group representations.

The relativistic fermion case is not quite as straightforward as the boson field. Had we used the hierarchy of fermionic N-particle representations of the group, we would have obtained fields satisfying equal-time canonical anticommutation relations acting in the Fock space of antisymmetric wave functions. But we must take account of particle spin; in the spinless case, after introducing the Hamiltonian, one cannot satisfy the important property of local causality.

Note, however, that ruling out spin 0 fermions does not mean that antisymmetric N-particle representations of diffeomorphism groups are irrelevant. If we consider spin 1 bosons, for example, we need to include both symmetric and antisymmetric spatial wave functions in order to allow for all of the possible spin symmetries under particle exchange.

One natural direction in which to extend these ideas is to non-Fock representations of G describing infinitely many particles. A second direction is toward representations of G describing configurations of extended objects. Intertwining such representations are fields that create vortex loops, strings, or more general embedded manifolds. Fock-like representations of relativistic fields whose quanta have spatial extent would be of great interest from this point of view.

We also anticipate the value of studying interacting relativistic quantum fields constructed from diffeomorphism group representations.

Acknowledgment

GG thanks Los Alamos National Laboratory for hospitality during the course of this research, and Rutgers University for travel support. This work was supported by the U.S. Department of Energy. The document has been assigned report number LA-UR-17-3193 and approved for public release by the Los Alamos National Laboratory.

References

[1] A.S. Wightman and S. Schweber, Configuration space methods in relativistic quantum field theory. *Phys. Rev.* **98**, 812 (1955).

[2] S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory.* Evanston, IL: Row, Peterson and Co. (1961).

[3] G.A. Goldin, R. Menikoff, and D.H. Sharp, Diffeomorphism groups, gauge groups, and quantum theory. *Phys. Rev. Lett.* **51**, 2246–2249 (1983).

[4] G.A. Goldin, Lectures on diffeomorphism groups in quantum physics. In J. Govaerts, M.N. Hounkonnou and A.Z. Msezane (eds.), *Contemporary Problems in Mathematical Physics: Procs. of the Third Int'l. Conference, Cotonou, Benin.* Singapore: World Scientific, 3–93 (2004).

[5] J.M. Leinaas and J. Myrheim, On the theory of identical particles. *Nuovo Cimento* **37B** 1 (1977).

[6] G.A. Goldin R. Menikoff, and D.H. Sharp, Representations of a local current algebra in a non-simply connected space and the Aharonov–Bohm effect. *J. Math. Phys.* **22**, 1664–1668 (1981).

[7] F. Wilczek, Magnetic flux, angular momentum, and statistics. *Phys. Rev. Lett.* **48** 1144 (1982).

[8] F. Wilczek, Quantum mechanics of fractional-spin particles. *Phys. Rev. Lett.* **49** 957 (1982).

[9] G.A. Goldin and D.H. Sharp, The diffeomorphism group approach to anyons. In F. Wilczek (ed.), *Fractional Statistics in Action* (*special issue*)*, Int. J. Mod. Physics B* **5**, 2625–2640 (1991).

[10] G.A. Goldin, R. Menikoff, and D.H. Sharp, Comments on "General theory for quantum statistics in two dimensions" *Phys. Rev. Lett.* **54**, 603 (1985).

[11] G.A. Goldin and R. Menikoff, Quantum-mechanical representations of the group of diffeomorphisms and local current algebra describing tightly bound composite particles. *J. Math. Phys.* **25**, 1880–1884 (1985).

[12] G.A. Goldin, J. Grodnik, R.T. Powers and D.H. Sharp, Nonrelativistic current algebra in the 'N/V' limit. *J. Math Phys.* **15**, 88–100 (1974).

[13] S. Albeverio, Y.G. Kondratiev, and M. Röckner, Diffeomorphism goups and current algebras: Configuration space analysis in quantum theory. *Rev. Math. Phys.* **11**, 1–23 (1999).

[14] Y.G. Kondratiev and T. Kuna, Harmonic analysis on configuration space I: General theory. *Infinite Dimensional Analysis* **5**, 201–233 (2002).

[15] G.A. Goldin and U. Moschella, Quantum phase transitions from a new class of representations of a diffeomorphism group. *J. Physics A: Math. Gen.* **28**, L475–L481 (1995).

[16] G.A. Goldin, U. Moschella, and T. Sakuraba, Self-similar random processes and infinite-dimensional configuration spaces. *Physics of Atomic Nuclei* **68**, 1675–1684 (2005).

[17] G.A. Goldin, R. Menikoff, and D.H. Sharp, Diffeomorphism groups and quantized vortex filaments. *Phys. Rev. Lett.* **58**, 2162–2164 (1987).

[18] G.A. Goldin, R. Menikoff, and D.H. Sharp, Quantum vortex configurations in three dimensions. *Phys. Rev. Lett.* **67**, 3499–3452 (1991).

[19] G.A. Goldin, R. Owczarek, and D.H. Sharp, Quantum kinematics of bosonic vortex loops. *J. Math. Phys.* **46**, 042102 (2005).

[20] G.A. Goldin, Quantum configuration spaces of extended objects, diffeomorphism group representations, and exotic statistics. In P. Kielanowski, S.T. Ali, A. Odzijewicz, M. Schlichenmaier, and T. Voronov (eds.), *Geometric Methods in Physics: XXX Workshop, Białowieża, Poland, June* 26 *to July* 2*,* 2011 (pp. 239–251). Basel: Springer (Birkhäuser).

[21] G.A. Goldin and D.H. Sharp, Diffeomorphism groups, anyon fields, and q-commutators. *Phys. Rev. Lett.* **76**, 1183–1187 (1996).

[22] G.A. Goldin and S. Majid, On the Fock space for nonrelativistic anyon fields and braided tensor products. J. Math. Phys. 45, 3770–3787 (2004).

Gerald A. Goldin
Rutgers University
10 Seminary Place
New Brunswick, NJ 08901, USA
e-mail: geraldgoldin@dimacs.rutgers.edu

David H. Sharp
Theory, Simulation and Computation Directorate
Los Alamos National Laboratory
Los Alamos, NM 87545, USA
e-mail: dcso@lanl.gov

Part II

Noncommutative Geometry

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 59–67

Skew Derivations on Down-up Algebras

Munerah Almulhem and Tomasz Brzeziński

Abstract. A class of skew derivations on complex Noetherian generalized down-up algebras $L = L(f, r, s, \gamma)$ is constructed.

Mathematics Subject Classification (2010). Primary 16S38; Secondary 16W25; 58B32.

Keywords. Generalized down-up algebra; generalized Weyl algebra; skew derivation.

1. Motivation and introduction

Skew derivations, i.e., linear endomorphisms of an algebra that satisfy the Leibniz rule twisted by algebra automorphisms or, more generally, endomorphisms play an increasingly important role in classical ring theory (e.g., of Noetherian rings) and in studies of classes of rings of particular interest in group and quantum group theories; see, e.g., [19, 25]. Many of such rings have the form of *Ore extensions* or *skew-polynomial rings* $S = R[x; \sigma, \delta]$ where σ is an endomorphism of R and δ is a skew σ-derivation on R. In fact, S is often obtained through an iterated skew polynomial construction $\Bbbk[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$ over a base field $\Bbbk$; here each δ_i is a σ_i-skew derivation of the preceding iterated extension (such an algebra is sometimes termed an *Ore algebra*). A great deal of work has been done concerning two "unmixed" cases, in which either $\sigma = 1$ or $\delta = 0$, which provides one with a thorough understanding of many of the classical iterated skew polynomial rings such as Weyl algebras, enveloping algebras of solvable Lie algebras, group algebras of polycyclic groups, and the enveloping algebra $U(sl_2(\Bbbk))$. Since 1980's, the emergence of quantum groups and quantized algebras has brought renewed interest in Ore extensions, where σ and δ are nontrivial, since many of these quantized algebras can be expressed in terms of iterated skew-polynomial rings. Basic examples include the q-Weyl algebra $A_q(\Bbbk)$ and the quantized enveloping algebra $U_q(sl_2(\Bbbk))$; see, e.g., [26] and [18]. Of special interest here are *q-skew derivations.*

In classical ring theory, for instance Kharchenko [20] studied the algebraic dependence problem on skew derivations of (semi-)prime rings and the skew differential identities with automorphisms of (semi-)prime rings. In [12] and [13], Chang considered the fixed power central skew derivations of prime rings and the skew derivations with nilpotent values on semiprime rings. In [14] Chuang and Lee investigated polynomial identities with skew derivations. The work of Bergen and Grzeszczuk [5] connects existence of locally nilpotent derivations with the Ore extension nature of rings.

With the emergence of noncommutative geometry or NCG [15] it has been realized at least as far back as the seminal work of Woronowicz [31] that skew derivations play the role of vector fields and can be used to construct differential structures on noncommutative algebras; see, e.g., [24, Section 4.4]. The insistence on preservation of the classical Leibniz rule for the exterior derivation (so that the coordinate algebra of a noncommutative variety is a subalgebra of the differential graded algebra of forms) forces one to abandon this rule for vector fields and replace it by a weaker condition such as the skew derivation property.

Determining which classes of noncommutative algebras correspond to smooth noncommutative varieties or manifolds is one of the outstanding problems of NCG. In a recent proposal [9], which attempts to explore ideas of both algebraic geometry, such as the *homological smoothness* of Van den Bergh [29] and differential geometry, such as Connes' *spectral triples*, it is argued that a possibility of constructing a suitable graded differential algebra or a differential structure should determine smoothness of a noncommutative variety. This kind of smoothness is referred to as *differential*. The proposal involves a strict Poincaré type duality between differential and integral forms (the reader might like to consult [7] for a concise explanation of terms involved) and is *constructive* in nature. Despite some recent progress in uncovering functorial ways of checking differential smoothness [8] one needs to study algebras on a case by case basis and construct suitable differential and integral complexes. In contrast to other approaches such as [16] or, more recently using the deformation theory, [17, 21, 22], which are based on usual derivations, in the background for such complexes one often finds skew derivations.

Motivated by the role skew derivations play in NCG, in particular in constructing smooth structures, in [1] we undertook a detailed study of skew derivations on generalized Weyl algebras [2]. Despite their almost naïve simplicity, generalized Weyl algebras include an astounding number of examples that have appeared and still appear in NCG, and in other algebraic contexts. In this note we concentrate on one such example in the latter class.

Generalized down-up algebras were introduced by Cassidy and Shelton in [11] as a generalization of the down-up algebras of Benkart and Roby [4]. Significant, classical examples of generalized down-up algebras include the enveloping algebra of sl_2, and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. More recent examples are various algebras similar to the enveloping algebra of sl_2 such as those introduced by Smith [28], Witten [30], Le Bruyn [23] and Rueda [27]. Generalized down-algebras are affine algebras of Gelfand–Kirillov dimension three, i.e., they can be interpreted as coordinate algebras of three-dimensional noncommutative varieties, and in the case in which they are Noetherian domains (on which we focus in this note), it is natural to ask whether they are differentially smooth. In this note we do not attempt to answer this question, but rather provide the first paving stones for a path that might lead to an answer by constructing a class of skew derivations. It is also perhaps worth noting in passing that generalized down-up algebras can themselves be understood as iterated Ore extensions.

2. Generalized down-up and Weyl algebras

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Fix scalars $r, s, \gamma \in \Bbbk$ and a polynomial $f \in \Bbbk[X]$. A *generalized down-up algebra* $L := L(f, r, s, \gamma)$ [11] is a unital associative $\Bbbk$-algebra generated by d, u and h, subject to relations:

$$dh - rhd + \gamma d = 0, \quad hu - ruh + \gamma u = 0, \quad du - sud + f(h) = 0.$$

When f has degree one, we retrieve all down-up algebras of [4]. It was explained in [11] that L has Gelfand–Kirillov dimension three and it is Noetherian if and only if it is a domain, if and only if $rs \neq 0$. From now on, we always assume $rs \neq 0$, and we also set $\Bbbk = \mathbb{C}$.

L is said to be *conformal* if there exists a polynomial $g \in \mathbb{C}[X]$ such that $f(X) = sg(X) - g(rX - \gamma)$, e.g., $L(0, r, s, \gamma)$ is conformal. By [11, Lemma 2.8], L is conformal if $s \neq r^i$ for all $0 \leq i \leq \deg(f)$ (this is a sufficient, but not necessary condition for conformality). If $r \neq 1$, then, by [10, Propostion 1.7], any generalized down-up algebra is isomorphic to one with $\gamma = 0$. A generalized down-up algebra $L(f, r, s, 0)$ is conformal if and only if $r \neq s^i$ for all i in the support of f (i.e., for those $i \in \mathbb{N}$, for which X^i has a non-zero coefficient in the expansion of $f(X)$). In this case the support of g can be assumed to be equal to that of f, and then g is uniquely determined by f; in particular $\deg(f) = \deg(g)$.

Given an algebra R, an automorphism φ of R and a central element $a \in R$, the *generalized Weyl algebra* $R(a, \varphi)$ is a ring extension of R generated by x and y, subject to the relations:

$$xy = \varphi(a), \quad yx = a, \quad xr = \varphi(r)x, \quad yr = \varphi^{-1}(r)y. \tag{1}$$

Generalized Weyl algebras were introduced and studied by Bavula in [2]. Any $R(a, \varphi)$ is a $\mathbb{Z}$-graded algebra with R contained in the degree-zero part and $\deg(x) = 1$, $\deg(y) = -1$. If R is a Noetherian algebra which is a domain and $a \neq 0$ then $R(a, \varphi)$ is a Noetherian domain. Noetherian generalized down-up algebras can be presented as generalized Weyl algebras as follows. Set $a = ud$, let R be the commutative polynomial algebra $\mathbb{C}[h, a]$ and define the automorphism φ by the rules $\varphi(h) = rh - \gamma$ and $\varphi(a) = sa - f(h)$. Then

$$\mathbb{C}[h, a]\,(a, \varphi) \cong L(f, r, s, \gamma).$$

Henceforth we assume that r is not a root of unity, $rs \neq 0$ and $s \neq r^i$, for all $i \in \mathbb{N}$. Thus L is Noetherian and conformal, and with no loss of generality (up to isomorphism) we can also assume that $\gamma = 0$. In this case there is a more convenient presentation of L as a generalized Weyl algebra. Let g be the unique polynomial (of the same degree and with the same support as f) such that $f(X) = sg(X) - g(rX)$. Let $a = ud$ and $k = a - g(h)$. Then $\mathbb{C}[a,k] = \mathbb{C}[h,k]$ and the automorphism φ acts on k by

$$\varphi(k) = \varphi(a - g(h)) = sa - f(h) - g(rh) = sa - sg(h) = sk.$$

Therefore, L is presented as the generalized Weyl algebra $R(\varphi, k + g(h))$, where $R = \mathbb{C}[h,k]$ and φ is the automorphism of R defined by

$$\varphi(h) = rh, \qquad \varphi(k) = sk. \tag{2}$$

The relations are thus:

$$xy = sk + g(rh), \qquad\qquad yx = k + g(h), \tag{3a}$$

$$xp(h,k) = p(rh, sk)x, \qquad yp(h,k) = p(r^{-1}h, s^{-1}k)y, \tag{3b}$$

for all $p(h,k) \in \mathbb{C}[h,k]$. $L(f,r,s,0)$ is recovered from $R(k + g(h), \varphi)$ by the isomorphism $h \mapsto h$, $k \mapsto ud - g(h)$, $x \mapsto d$, $y \mapsto u$.

3. Skew derivations on generalized down-up algebras

For any algebra A, a (*right*) *skew derivation* or a *σ-derivation* is a pair (∂, σ) consisting of an algebra endomorphism $\sigma : A \to A$ and a linear map $\partial : A \to A$ that satisfies the σ-twisted Leibniz rule, for all $a, b \in A$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b). \tag{4}$$

A skew-derivation is said to be *inner* if it is given by a twisted commutator with an element of A, i.e., for all $a \in A$, $\partial(a) = b\sigma(a) - ab$.

In [1, Theorem 3.1] we constructed a large class of skew derivations on arbitrary generalized Weyl algebras. The aim of this section is apply this construction to Noetherian conformal generalized down-up algebras.

Let $A = R(a, \varphi)$ be a generalized Weyl algebra. Any algebra automorphism σ of R such that

$$\sigma \circ \varphi = \varphi \circ \sigma \quad \text{and} \quad \sigma(a) = a, \tag{5}$$

can be extended to an automorphism σ_μ of A by setting,

$$\sigma_\mu \mid_R = \sigma, \qquad \sigma_\mu(x) = \mu^{-1}x, \qquad \sigma_\mu(y) = \mu y, \tag{6}$$

where $\mu \in \mathbb{C}^\times$. The automorphism σ_μ is called a *degree-counting extension* of σ of *coarseness* μ.

In [1], in addition to general inner skew derivations, two classes of *elementary* σ_μ-skew derivation have been identified:

(1) *c*-type derivations: To any $w \in \mathbb{Z}$ and $c_w \in R$, such that

$$bc_w = c_w\varphi^w(\sigma(b)), \qquad \text{for all } b \in R, \tag{7}$$

one can associate a (unique if R is a domain) derivation ∂^c_w on A of $\mathbb{Z}$-degree or *weight* w and such that $\partial^c_w\,(R) = 0$, as follows:

$$\partial_w(x) = \begin{cases} c_w\, x^{w+1}, & w > 0 \\ c_0\, x, & w = 0\ , \\ -\mu^{-1}\varphi(ac_w)\, y^{-w-1} & w < 0 \end{cases} \tag{8a}$$

$$\partial_w(y) = \begin{cases} -\mu\varphi^{-1}(c_w)a\, x^{w-1}, & w > 0 \\ \tilde{c}_0\, y, & w = 0\ , \\ c_w\, y^{-w+1} & w < 0 \end{cases} \tag{8b}$$

where $\tilde{c}_0$ is a solution to the equation $(\tilde{c}_0 + \mu\varphi^{-1}(c_0))a = 0$ that satisfy the twisted-centrality condition (7) with $w = 0$ (this is the content of [1, Lemma 3.3]).

(2) $\boldsymbol{\alpha}$-type derivations: For any $w \in \mathbb{Z}\setminus\{0\}$, any $\varphi^w \circ \sigma$-skew derivation α_w of R such that

$$\alpha_w \circ \varphi = \mu\,\varphi \circ \alpha_w, \tag{9}$$

can be (uniquely) extended to a σ_μ-skew derivation ∂^α_w of A of $\mathbb{Z}$-degree w such that $\partial^\alpha_w(x) = 0$ if $w > 0$ and $\partial^\alpha_w(y) = 0$ if $w < 0$. For all $r \in R$, the formulae are, respectively:

$$\partial^\alpha_w(r) = \alpha_w(r)\, x^w, \qquad \partial^\alpha_w(y) = \mu\, \alpha_w(a)\, x^{w-1}, \qquad w > 0, \tag{10a}$$

$$\partial^\alpha_w(r) = \alpha_w(r)\, y^{-w}, \quad \partial^\alpha_w(x) = \varphi\left(\alpha_w(a)\right)\, y^{-w-1}, \qquad w < 0. \tag{10b}$$

Also a σ-skew derivation α_0 satisfying (9) and such that $\alpha_0(a) = ca$, for some $c \in R$ such that $bc = c\sigma(b)$, for all $b \in R$, can be extended to a σ_μ-skew derivation of A of the $\mathbb{Z}$-degree or weight zero through

$$\partial^\alpha_0(r) = \alpha_0(r), \qquad \partial^\alpha_0(x) = 0, \qquad \partial^\alpha_0(y) = \mu\,\varphi^{-1}(c)\, y, \tag{11}$$

for all $r \in R$ (this is the content of [1, Lemma 3.2]).

All non-inner σ_μ-skew derivations on $R(a, \varphi)$ which vanish either on R or on x, or on y are necessarily linear combinations of elementary skew derivations listed in (1) and (2).

We now specify to conformal Noetherian generalized down-up algebras $L = L(f, r, s, 0)$ viewed as generalized Weyl algebras $\mathbb{C}[h, k](\varphi, k + g(h))$. Our standing assumptions are that the parameters $r, s \in \mathbb{C}$ are neither zero nor roots of unity and $s \neq r^i$, for all $i \in \mathbb{N}$.

The identity map is the only automorphism of $\mathbb{C}[h, k]$ that satisfies (5). Its extension to the whole of L is thus given by:

$$\sigma_\mu(h) = h, \qquad \sigma_\mu(k) = k, \qquad \sigma_\mu(x) = \mu^{-1}x, \qquad \sigma_\mu(y) = \mu y. \tag{12}$$

This automorphism is non-trivial if and only if $\mu \neq 1$, which we assume from now on.

Since σ is the identity map, the w-fold self-composition of φ (2), φ^w, sends a polynomial $p(h, k)$ to $p(r^w h, s^w k)$, and since $\mathbb{C}[h, k]$ is commutative, the equations

(7) can only be satisfied non-trivially if $w = 0$. Consequently, there are only weight zero σ_μ-skew c-type derivations of L:

$$\partial_0^c(x) = c_0(h,k)x, \qquad \partial_0^c(y) = -\mu c_0(r^{-1}h, s^{-1}k)y, \tag{13}$$

for any $c_0(h,k) \in \mathbb{C}[h,k]$; see [1, Lemma 3.3]. These derivations are inner, provided there exist $p(h,k) \in \mathbb{C}[h,k]$ such that

$$c_0(h,k) = \mu^{-1}p(h,k) - p(rh, sk). \tag{14}$$

By comparing the coefficients at powers of h and k, one easily finds that (14) can be solved if, and only if,

$$\mu^{-1} \neq r^\beta s^\gamma, \qquad \text{for all } (\beta,\gamma) \in \operatorname{supp}(c_0(h,k)),$$

where, for any polynomial $\pi \in \mathbb{C}[h,k]$, $\operatorname{supp}(\pi)$ denotes its *support*, defined as the set of all pairs $(m,n) \in \mathbb{N}$, for which π has a non-trivial $h^m k^n$-term. In view of the above, this proves the following

Proposition 3.1. *The Noetherian generalized down-up algebra $L(f,r,s,0) = \mathbb{C}[h,k](\varphi, k+g(h))$ admits a non-inner skew derivation twisted by σ_μ (12) and vanishing on $\mathbb{C}[h,k]$ if, and only if, $\mu = r^{-\beta}s^{-\gamma}$ for some $\beta,\gamma \in \mathbb{N}$. This derivation is of the form (13), where $c_0(h,k) \sim h^\beta k^\gamma$.*

To construct α-type derivations we need to consider skew derivations α_w of $\mathbb{C}[h,k]$ twisted by φ^w. Any such skew derivation is fully determined by its values on generators of $\mathbb{C}[h,k]$, i.e., there exists a unique φ^w-skew derivation α_w which on h and k is equal to any chosen pair of elements of $\mathbb{C}[h,k]$, say

$$\alpha_w(h) = \sum_{i,j} \alpha^w_{h,i\,j}\ h^i k^j, \qquad \alpha_w(k) = \sum_{m,n} \alpha^w_{k,m\,n}\ h^m k^n. \tag{15}$$

The value of α_w on any polynomial is then obtained by applying the φ^w-twisted Leibniz rule. The resulting skew derivation α_w satisfies condition (9) if and only if this condition is satisfied for α_w evaluated on generators of $\mathbb{C}[h,k]$. This in turn is equivalent to equations,

$$r\sum_{i,j} \alpha^w_{h,i\,j}\ h^i k^j = \mu \sum_{i,j} r^i s^j \alpha^w_{h,i\,j}\ h^i k^j, \tag{16a}$$

$$s\sum_{m,n} \alpha^w_{k,m\,n}\ h^m k^n = \mu \sum_{m,n} r^m s^n \alpha^w_{k,m\,n}\ h^m k^n. \tag{16b}$$

Equations (16) yield the following constraints:

$$r^i s^j = \frac{r}{\mu},\ (i,j) \in \operatorname{supp}(\alpha_w(h)); \quad r^m s^n = \frac{s}{\mu},\ (m,n) \in \operatorname{supp}(\alpha_w(k)). \tag{17}$$

Constraints (17) restrict the supports of $\alpha_w(h)$ and $\alpha_w(k)$ as well as possible values of μ. Set

$$r = s^{b_1}, \qquad \mu^{-1} = s^{b_2}, \qquad \text{where } b_1 \in \mathbb{C}^\times \setminus \{\tfrac{1}{q} \mid q \in \mathbb{N}^\times\},\ b_2 \in \mathbb{C}^\times. \tag{18}$$

The removal of zeros and the fractions $1/q$ ensures that s is not a natural power of r, which is one of our standing assumptions (needed for L to be both conformal and

Noetherian), and that $\mu \neq 1$. Note that the vector $\mathbf{b} := (b_1, b_2)$ is not determined uniquely by r, s and μ. Solving constraints (17) we obtain the following forms of $\alpha_w(h)$ and $\alpha_w(k)$:

$$\alpha_w(h) = \sum_{i\in I} \alpha^w_{h,i} h^i k^{b_2+(1-i)b_1}, \quad \alpha_w(k) = \sum_{m\in J} \alpha^w_{k,m} h^m k^{b_2-mb_1+1}, \tag{19}$$

where I and J are any finite subsets of

$$I_{\mathbf{b}} := \{t \in \mathbb{N} \mid b_2 + (1-t)b_1 \in \mathbb{N}\}, \quad J_{\mathbf{b}} := \{t \in \mathbb{N} \mid b_2 - tb_1 + 1 \in \mathbb{N}\}, \tag{20}$$

respectively. In view of the above, these considerations lead to:

Proposition 3.2. *Under standing assumptions on the Noetherian generalized down-up algebra L and with $\mathbf{b}$ given by* (18)*, the following formulae define σ_μ-skew derivations of L of non-zero weight:*

$$\partial^\alpha_w(p(h,k)) = \begin{cases} \alpha_w(p(h,k))x^w & w > 0, \\ \alpha_w(p(h,k))y^{-w} & w < 0, \end{cases} \tag{21a}$$

$$\partial^\alpha_w(x) = \begin{cases} 0 & w > 0, \\ \mu^{-1}\alpha_w(sk + g(rh))y^{-w-1} & w < 0, \end{cases} \tag{21b}$$

$$\partial^\alpha_w(y) = \begin{cases} \mu\alpha_w(k + g(h))x^{w-1} & w > 0, \\ 0 & w < 0, \end{cases} \tag{21c}$$

for all $p(h,k) \in \mathbb{C}[h,k]$, where α_w are given by (19) *with I and J being finite subsets of respectively $I_{\mathbf{b}}$ and $J_{\mathbf{b}}$ given in* (20)*. The σ_μ-skew derivations* (21) *are all* (*possibly non-inner*) *skew-derivations of L of positive weight that vanish on x or of negative weight that vanish on y.*

What $I_{\mathbf{b}}$ and $J_{\mathbf{b}}$ are depends on $\mathbf{b} = (b_1, b_2)$. For example if b_1 is a negative integer and b_2 is a positive integer, then $J_{\mathbf{b}} = \mathbb{N}$, while $I_{\mathbf{b}} = \mathbb{N}$ if $b_2 \geq -b_1$ and $I_{\mathbf{b}} = \mathbb{N} \setminus \{0\}$ otherwise. If b_1 is a positive integer or if b_2 is negative, then the indexing sets $I_{\mathbf{b}}$ or $J_{\mathbf{b}}$ could become finite or even empty. For example, if both b_1 and b_2 are positive, then

$$I_{\mathbf{b}} = \begin{cases} \{0,1\} & b_1 > b_2, \\ \{0,1,2\} & b_1 = b_2, \\ \{0,\ldots,q+1\} & b_1 < b_2, \end{cases} \quad J_{\mathbf{b}} = \begin{cases} \{0\} & b_1 > b_2 + 1, \\ \{0,1\} & b_1 = b_2, b_2+1, \\ \{0,\ldots,q+\delta_{\rho,b_1-1}\} & b_1 < b_2, \end{cases}$$

where q is the quotient and ρ the remainder of the division of b_2 by b_1.

If b_1 is positive and b_2 is negative such that $b_1 < |b_2|$, then $I_{\mathbf{b}} = \emptyset$, hence there are no solutions to (17).

The sets $I_{\mathbf{b}}$ and $J_{\mathbf{b}}$ may be non-empty even if the components of $\mathbf{b}$ are not integral. For example, if $b_1 = b_2$, then $2 \in I_{\mathbf{b}}$ and $1 \in J_{\mathbf{b}}$.

In the weight-zero case in addition to (17), one also needs to require that $\alpha_0(k + g(h))$ contains factor $k + g(h)$. In contrast to the non-zero weight case the values of α_0 on h and k depend heavily on the choice of $g(h)$, and we take this as an excuse for not including them here.

Acknowledgment

The research of the second author is partially supported by the Polish National Science Centre grant 2016/21/B/ST1/02438.

References

[1] M. Almulhem & T. Brzeziński, *Skew derivations on generalized Weyl algebras*, J. Algebra **493** (2018), 194–235.

[2] V.V. Bavula, *Generalized Weyl algebras and their representations*, Algebra i Analiz **4** (1992), 75–97; English transl. in St Petersburg Math. J. **4** (1993), 71–92.

[3] V.V. Bavula & D.A. Jordan, *Isomorphism problems and groups of automorphisms for generalized Weyl algebras*, Trans. Amer. Math. Soc. **353** (2001), 769–794.

[4] G. Benkart & T. Roby, *Down-up algebras*, J. Algebra **209** (1998), 305–344.

[5] J. Bergen & P. Grzeszczuk, On rings with locally nilpotent skew derivations, Comm. Algebra **39** (2011), 3698–3708.

[6] T. Brzeziński, *Non-commutative differential geometry of generalized Weyl algebras*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), 059.

[7] T. Brzeziński, *Differential and integral forms on non-commutative algebras*, [In:] Geometric Methods in Physics XXXV, Workshop 2016, Birkhäuser, Basel, (2018), pp. 249–253.

[8] T. Brzeziński & C. Lomp, *Differential smoothness of skew polynomial rings*, J. Pure Appl. Algebra **222** (2018), 2413–2426.

[9] T. Brzeziński & A. Sitarz, *Smooth geometry of the noncommutative pillow, cones and lens spaces*, J. Noncommut. Geom. **11** (2017), 413–449.

[10] P.A.A.B. Carvalho & S.A. Lopes, *Automorphisms of generalized down-up algebras*, Comm. Algebra **37** (2009), 1622–1646.

[11] T. Cassidy & B. Shelton, *Basic properties of generalized down-up algebras*, J. Algebra **279** (2004), 402–421.

[12] J.C. Chang, *On fixed power central* (α, β)*-derivations*, Bull. Inst. Math. Acad. Sinica, **15** (1987), 163–178.

[13] J.C. Chang & J.S. Lin, (α, β)*-derivations with nilpotent values*, Chinese J. Math. **22** (1994), 349–355.

[14] C.L. Chuang & T.K. Lee, *Identities with a single skew derivation*, J. Algebra **288** (2005), 59–77.

[15] A. Connes, *Noncommutative Geometry*, Academic Press, New York 1994.

[16] M. Dubois-Violette, *Dérivations et calcul différentiel non commutatif*, C. R. Acad. Sci. Paris, Ser. I 307 (1988), 403–408.

[17] E. Eriksen, O.A. Laudal & A. Siqveland, *Noncommutative Deformation Theory*, CRC Press, Boca Raton, FL, 2017.

[18] K.R. Goodearl, *Prime ideals in skew polynomial rings and quantized Weyl algebras*, J. Algebra **150** (1992), 324–377.

[19] K.R. Goodearl & R.B. Wakefield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge, 2004.

[20] V.K. Kharchenko, *Automorphisms and Derivations of Associative Rings*, Kluwer, 1991.

[21] O.A. Laudal, *Phase spaces and deformation theory*, J. Gen. Lie Theory Appl. **5** (2011), Art. ID G110104, 18 pp.

[22] O.A. Laudal, *Geometry of Time-Spaces. Non-commutative Algebraic Geometry, Applied to Quantum Theory*, World Scientific Publishing, Hackensack, NJ, 2011.

[23] L. Le Bruyn, *Conformal sl_2-enveloping algebras*, Comm. Algebra **23** (1995), 1325–1362.

[24] J. Madore, *An Introduction to Noncommutative Geometry and Its Physical Applications*, 2nd ed., Cambridge University Press, Cambridge 1999.

[25] J.C. McConnell & J.C. Robson, *Noncommutative Noetherian Rings*, American Mathematical Society, Providence R.I., 2001.

[26] S. Montgomery & S.P. Smith, *Skew derivations and $U_q(sl(2))$*, Israel J. Math **72** (1990), 158–166.

[27] S. Rueda, *Some algebras similar to the enveloping algebra of $sl(2)$*, Comm. Algebra **33** (2002), 1127–1152.

[28] S.P. Smith, *A class of algebras similar to the enveloping algebra of $sl(2)$*, Trans. Amer. Math. Soc. **322** (1990), 285–314.

[29] M. Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), 1345–1348; Erratum: Proc. Amer. Math. Soc. **130** (2002), 2809–2810.

[30] E. Witten, *Gauge theories, vertex models, and quantum groups*, Nucl. Phys. B **330** (1990), 285–346.

[31] S.L. Woronowicz, *Differential calculus on compact matrix pseudogroups* (*quantum groups*), Comm. Math. Phys. **122** (1989), 125–170.

Munerah Almulhem
Department of Mathematics
Swansea University
Swansea SA2 8PP, UK

and

Department of Mathematics
Imam Abdulrahman bin Faisal University
Dammam 34212, K.S.A.
e-mail: 844404@swansea.ac.uk

Tomasz Brzeziński
Department of Mathematics
Swansea University
Swansea SA2 8PP, UK

and

Department of Mathematics
University of Białystok
K. Ciołkowskiego 1M
PL-15-245 Białystok, Poland
e-mail: T.Brzezinski@swansea.ac.uk

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 69–82

On Noncommutative Geometry of the Standard Model: Fermion Multiplet as Internal Forms

Ludwik Dąbrowski

Abstract. We unveil the geometric nature of the multiplet of fundamental fermions in the Standard Model of fundamental particles as a noncommutative analogue of de Rham forms on the internal finite quantum space.

Mathematics Subject Classification (2010). Primary 58B34; Secondary 46L87, 81T13.

Keywords. Spinors, differential forms, noncommutative geometry; spectral triples; Standard Model.

1. Introduction

From the conceptual point of view the Standard Model (S.M.) of fundamental particles and their interactions is a particular model of $U(1) \times SU(2) \times SU(3)$ gauge fields (bosons) minimally coupled to matter fields (fermions), plus a Higgs field (boson). After the second quantization with gauge fixing, spontaneous symmetry breaking mechanism, regularization and perturbative renormalization it extremely well concords with the experimental data. Even so (unreasonably) successful it however does not explain (though somewhat constrains) the list of particles, in particular the existence of 3 families, contains several parameters and does not include the fourth known interaction: gravitation, with its own fundamental symmetry: general relativity or diffeomorphisms. There have been various attempts to settle some of the above shortcomings: GUT based on a simple group $SU(5)$ or $SO(10)$, modern variants of old Kaluza–Klein model with 'compactified' internal dimensions, and others more recent and fashionable, that are still under extensive massive study.

This work is part of the project Quantum Dynamics sponsored by EU-grant RISE 691246 and Polish Government grant 317281, and was partially supported by by the Polish Ministry of Science and Higher Education 2015–2019 matching fund 3542/H2020/2016/2.

Of our interest in this note is another distinct approach to the S.M. in the framework of noncommutative geometry by A. Connes et al., see, e.g., [6], which is not so widely known among physicists. It interprets the multiplet of fundamental fermions as a field on a finite quantum space, on which the would be "coordinates", as well as the algebra of "functions" fail to commute. In this note we focus on its deeper geometric structure, aiming to shed more light on the nature of this internal quantum space. We shall review the key results of the two recent papers [7, 8], explaining in more detail the classical (commutative) motivation behind them. For that some well-known material in differential geometry will be presented from the viewpoint of the so-called spectral triples; with the only new contribution in the last part of Subsection 2.2 regarding their KO-dimension.

2. Introduction

The noncommutative formulation νS.M. of the Standard Model takes its cue from its geometry which in mathematical terminology corresponds to a connection (locally a multiplet of vector fields) whose structure group is $U(1)\times SU(2)\times SU(3)$ on (a multiplet) of spinors, together with a doublet of scalar fields. Although it does not renounce of *groups*, νS.M. is however based primarily on *algebras.* Moreover to the 75 years-old Gelfand–Naimark (anti)equivalence:

$$\boxed{\text{topological spaces} \longleftrightarrow \text{commutative } C^*\text{-algebras}}$$

and to the Serre–Swan equivalence:

$$\boxed{\text{vector bundles} \longleftrightarrow \text{modules}}$$

it adjoins two other ingredients to encode such structures as *smoothness*, *calculus* and (Riemannian) *metric* on a space M. The first one is a Hilbert space H that carries a unitary representation of a (possibly noncommutative) $*$-algebra A, and so obviously also of its norm completed C^*-algebra. The second one is an analogue of the Dirac operator on H. Together with a $*$-algebra A they satisfy certain analytic conditions: D is selfadjoint, $[D,a]$ are bounded $\forall a \in A$ and $(D-z)^{-1}$ are compact for $z \in \mathbb{C}\setminus\mathbb{R}$, so that they form the so-called *spectral triple* (S.T.) [5]

$$(A, H, D).$$

Such a S.T. is *even* if there is a $\mathbb{Z}_2$-grading χ of H, $\chi^2 = 1, \chi^\dagger = \chi$, with which all $a \in A$ commute and D anticommutes. Furthermore it is *real* if there is a $\mathbb{C}$-antilinear isometric operator J on H, such that denoting B' the commutant of $B \subset \mathcal{B}(H)$,

$$JAJ^{-1} \subset A', \tag{1}$$

which is often termed *order* 0 *condition.* We say that a real S.T. satisfies the *order* 1 *condition* if

$$JAJ^{-1} \subset [D,A]' \tag{2}$$

and the *order* 2 *condition*[1] if

$$J[D,A]J^{-1} \subset [D,A]'. \tag{3}$$

The A-bimodule spanned by $[D,A]$ is often called space of 1-forms, and the algebra generated by A and $[D,A]$ the space of all forms for the Dirac calculus (with the exterior derivative given by the commutator with D). Motivated by the classical examples (cf. the next subsections) slightly abusing the terminology we will call, quite as in [10], *Cliford algebra* the complex $*$-algebra $\mathcal{C}\ell_D(A)$ generated by A and $[D,A]$.

Note that for noncommutative A a priori there is no right action of A on H, but given J there is one:

$$ha := Ja^*J^{-1}h,$$

that commutes with the left action due to the order 0 condition and so H becomes an A-A bimodule. Furthermore, if the order 1 condition holds H becomes a $\mathcal{C}\ell_D(A)$-A bimodule, and if the 2nd-order condition holds H becomes even a $\mathcal{C}\ell_D(A)$-$\mathcal{C}\ell_D(A)$ bimodule.

Connes formulated few other important properties of real spectral triples. One of them requires that the following identities are satisfied

$$J^2 = \epsilon\, \mathsf{id}, \tag{4}$$

$$DJ = \epsilon' JD, \tag{5}$$

and in even case

$$\chi J = \epsilon'' J\chi, \tag{6}$$

where the three signs $\epsilon, \epsilon', \epsilon'' \in \{+,-\}$ specify the so-called KO-dimension modulo 8:

n	0	1	2	3	4	5	6	7
ϵ	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ'	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ''	$+$		$-$		$+$		$-$	

(7)

(If dimension n is even one can alternatively use χJ as a new real structure, which changes the parameter ϵ' to $-\epsilon'$, and ϵ to $\epsilon\epsilon''$.)

2.1. Canonical spectral triple

A prototype example is the *canonical* S.T. on a closed oriented <u>spin</u> manifold M of dimension n equipped with a Riemannian metric g:

$$(C^\infty(M), L^2(S), \not{D}). \tag{8}$$

Here $C^\infty(M)$ is the algebra of smooth complex functions on M, S is the $\mathrm{rank}_{\mathbb{C}} = 2^{[n/2]}$ bundle of Dirac spinors on M, whose sections carry a faithful irreducible representation

$$\gamma : \Gamma(\mathcal{C}\ell(M)) \xrightarrow{\approx} \mathrm{End}_{C^\infty(M)}\Gamma(S) \approx \Gamma(S) \otimes_{C^\infty(M)} \Gamma(S)^*, \tag{9}$$

[1] While the order 1 condition means classically that D is order 1 differential operator, order 0 and 2 conditions don't have such interpretation.

of the algebra of sections of the (simple part of) complex Clifford bundle $\mathcal{C}\ell(M)$, generated by $v \in TM$ with relation $v^2 + g(v,v) = 0$. Furthermore $D\!\!\!/$ is the usual Dirac operator

$$D\!\!\!/ = \gamma \circ \tilde{\nabla} = \sum_j^n \gamma(e_j)\tilde{\nabla}_{e_j}, \tag{10}$$

where $\tilde{\nabla}$ is a lift to S of the Levi-Civita connection on M and e_j, $j = 1, \dots, n$, is a local oriented orthonormal basis of TM.

One has

$$[D\!\!\!/, f] = \gamma(\mathrm{d}f), \quad f \in C^\infty(M),$$

or, what is the same, the symbol of $D\!\!\!/$ is

$$\sigma_{D\!\!\!/}(\xi) = -i\gamma(\xi), \quad \xi \in T^*M,$$

where we have identified $TM \approx T^*M$, and so ∇f with df, using the metric g. Note that the operators of that form together with functions generate an isomorphic copy of the Clifford algebra $\Gamma(\mathcal{C}\ell(M))$.

If $\dim M$ is even there is also a $\mathbb{Z}_2$-grading χ_S of $L^2(S)$, with which all $a \in A$ commute and $D\!\!\!/$ anticommutes. It should be mentioned that $D\!\!\!/$ is an elliptic operator and its index, or more precisely the Fredholm index of $D\!\!\!/|_{\Gamma(S^+)} : \Gamma(S^+) \to \Gamma(S^-)$, where $\Gamma(S^\pm)$ are ± 1 eigenspaces of χ_S, plays an important role in geometry and applications to physics. It can be expressed in terms of the characteristic class called $\hat{A}$-genus, a topological invariant of M.

Furthermore there is a real structure (known as *charge conjugation*) J_S on $L^2(S)$, that satisfies the order 0 condition (1) and the order 1 condition (2), but <u>not</u> the order 2 condition (3). Indeed, J_S and (8) obey a stronger version of (1) and (2) which excludes (3). Namely the norm closure of $C^\infty(M)$, that is the algebra $C(M)$ of continuous functions on M, is the maximal commutant in $B(L^2(S))$ of the norm closure of $\mathcal{C}\ell_{D\!\!\!/}(C^\infty(M)) = C^\infty(M)[D\!\!\!/, C^\infty(M)]$, which is just the algebra of continuous sections $\Gamma(\mathcal{C}\ell(M))$ of the (complexified) Clifford bundle $\mathcal{C}\ell(M)$ on M in the Dirac representation. We can thus say that "the Dirac spinor fields provide a Morita equivalence $C(M) - \Gamma(\mathcal{C}\ell(M))$ bimodule".

As a matter of fact for the latter property it suffices that M is spin$_c$, that can be defined by any of the following three equivalent statements:

i) there exist a principal $Spin_c(n)$-bundle, such that the vector bundle associated with the representation $\rho \times \Box$ is isomorphic to the tangent bundle $T(M)$;
ii) $SO(n)$-bundle of oriented orthogonal frames lifts to $Spin_c(n)$;
iii) the second Stiefel–Whitney class $w_2(M)$ is a modulo 2 reduction of a class in $H^2(M, \mathbb{Z})$.

Here $Spin_c(n)$ is the quotient group of $Spin(n) \times U(1)$ by the subgroup $\mathbb{Z}_2^{\mathrm{diag}} = \{(1,1), (-1,-1)\}$, $\rho : Spin(n) \to SO(n)$ is the nontrivial double covering and $\Box : U(1) \to U(1)$ is the square map.

Importantly however the property that an oriented Riemannian manifold M is spin$_c$ is actually tantamount [12] to

$$\boxed{\exists \text{ a Morita equivalence } \mathcal{C}\ell(M){-}C(M) \text{ bimodule } \Sigma.} \tag{11}$$

Indeed when (11) holds then automatically $\Sigma \approx \Gamma(S)$, where S is the $\mathbb{C}$-vector bundle of Dirac spinors on M.

Therefore spin$_c$ manifolds lend themselves to noncommutative generalization via the algebraic property (11) by taking advantage of the definition of Clifford algebra $\mathcal{C}\ell_D(A)$. Next, the algebraic characterization of *spin* manifolds also admits a noncommutative generalization as the condition (11) plus a real structure (charge conjugation) J that implements it.

We remark that the canonical S.T. fully encodes the geometric data on M, that can be indeed reconstructed [4] from a commutative S.T. with certain few additional properties. One of these properties requires that KO-dimension, defined by (4), (5), (6), is equal for the operators $\not{D}, \chi_S, J_S$ to the dimension of M modulo 8.

2.2. Hodge–de Rham spectral triple

The canonical S.T. (8) is not the only natural S.T. On any oriented closed Riemannian manifold M there is also

$$(C^\infty(M), L^2(\Omega(M)), d + d^*), \tag{12}$$

where $\Omega(M)$ is the graded space of complex de Rham differential forms on M, d is the exterior differential and d^* is its adjoint.

The operator $d + d^*$ is actually Dirac-type since

$$d + d^* = \lambda \circ \nabla, \tag{13}$$

where

$$\lambda : \Gamma(\mathcal{C}\ell(M)) \to \mathrm{End}_{C^\infty(M)}\, \Omega(M),\ \ \lambda(v) = v \wedge - v \lrcorner\,,\ \ v \in T^*M \approx TM, \tag{14}$$

is the (reducible) faithful complex representation on $\Omega(M)$ of the algebra of sections $\Gamma(\mathcal{C}\ell(M))$ of the (complexified) Clifford bundle $\mathcal{C}\ell(M)$ over M. The formula (13) means that

$$[d + d^*, f] = \lambda(\mathrm{d}f)\,, \quad f \in C^\infty(M), \tag{15}$$

what is also the same as the symbol of $d + d^*$ being

$$\sigma_{d+d^*}(\xi) = -i\lambda(\xi), \quad \xi \in T^*M.$$

Note that as for the canonical $\not{D}$ the operators of the form (13) together with functions generate $\Gamma(\mathcal{C}\ell(M))$, and thus indeed $\mathcal{C}\ell_{d+d^*}(C^\infty(M)) \approx \Gamma(\mathcal{C}\ell(M))$.

The representation λ is equivalent to the left regular self-representation of $\Gamma(\mathcal{C}\ell(M))$, via the isomorphism of vector bundles $\mathcal{C}\ell(M) \approx \Omega(M)$. There is also an anti-representation

$$\rho : \Gamma(\mathcal{C}\ell(M)) \to \mathrm{End}_{C^\infty(M)}\, \Omega(M),\ \ \lambda(v) = (v \wedge + v \lrcorner)\, \chi_\Omega\,,\ \ v \in T^*M \approx TM, \tag{16}$$

where

$$\chi_\Omega = \pm 1 \tag{17}$$

on even forms $\Omega(M)^{\mathrm{even}}$, respectively odd forms $\Omega(M)^{\mathrm{odd}}$. It is equivalent to the right regular self-antirepresentation of $\Gamma(\mathcal{C}\ell(M))$.

Furthermore, since the endomorphisms $\lambda(v)$ and $\rho(v')$ commute, $\Omega(M)$ is a $\Gamma(\mathcal{C}\ell(M))$-$\Gamma(\mathcal{C}\ell(M))$ bimodule, which is equivalent to $\Gamma(\mathcal{C}\ell(M))$. Thus, $\Omega(M)$ is actually a self-Morita equivalence $\Gamma(\mathcal{C}\ell(M))$-$\Gamma(\mathcal{C}\ell(M))$ bimodule, the property which in fact provides its unambiguous characterization up to a tensor product with sections of a complex line bundle.

The operator χ_Ω (17) always defines a $\mathbb{Z}_2$-grading of $L^2(\Omega(M))$ according to the parity of forms. If $\dim M = n = 2m$ (even) there is also another grading χ'_Ω given by the normalized Hodge star operator, defined in terms of a local orthonormal oriented basis e^j, $j = 1, \dots, n$, $n = 2m$ of T^*M by

$$\chi'_\Omega(e^{j_1} \wedge \cdots \wedge e^{j_k}) = i^{k(k-1)+m} e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}, \quad 0 \le k \le n, \tag{18}$$

where $j_1, \dots, j_n$ is an even permutation of $1, \dots, n$.

Both the gradings χ_Ω and χ'_Ω commute with $a \in C^\infty(M)$ and anticommute with $d + d^*$. As is well known, they play an important role for the index of the elliptic operator $d + d^*$. More precisely the Fredholm index of

$$(d + d^*)|_{\Omega(M)^{\mathrm{even}}} : \Omega(M)^{\mathrm{even}} \to \Omega(M)^{\mathrm{odd}}$$

computes the Euler character of M, while the index of

$$(d + d^*)|_{\Omega(M)^s} : \Omega(M)^s \to \Omega(M)^a,$$

where $\Omega(M)^s$ and $\Omega(M)^a$ are respectively the ± 1 eigenspaces of χ'_Ω, computes the signature of M.

Furthermore there is also a real structure J_Ω on $L^2(\Omega(M))$ given just by the complex conjugation of forms. It satisfies the conditions (1) and (2) but definitely <u>not</u> (3) and therefore cannot implement the $\Gamma(\mathcal{C}\ell(M))$-$\Gamma(\mathcal{C}\ell(M))$ self-Morita equivalence as above.

In order to implement this equivalence, we need another real structure J'_Ω on $\Omega(M)$, that interchanges the actions λ and ρ. It turns out that it can be defined as

$$J'_\Omega(e^{j_1} \wedge \cdots \wedge e^{j_k}) = e^{j_k} \wedge \cdots \wedge ej_1, \quad 0 \le k \le n, \tag{19}$$

which corresponds to the main anti-involution on $\Gamma(\mathcal{C}\ell(M))$ and can be simply written on $\Omega^k(M)$ as

$$J'_\Omega = (-)^{k(k-1)/2} \circ c.c. \tag{20}$$

This real structure satisfies all the order conditions (1), (2) and (3) and does implement the $\Gamma(\mathcal{C}\ell(M))$-$\Gamma(\mathcal{C}\ell(M))$ self-Morita equivalence as above.

We mention that for the operators $d+d^*, \chi_\Omega, J_\Omega$ one gets the signs $\epsilon = 1, \epsilon' = 1, \epsilon'' = 1$ and so the KO-dimension is 0. Instead for the operators $d + d^*, \chi'_\Omega, J_\Omega$ the signs are $\epsilon = 1, \epsilon' = 1, \epsilon'' = (-1)^m$ and so the KO-dimension is 0 if n=0 mod 4, and 6 if n=2 mod 4 [A. Rubin, MSc Thesis].

As far as the operators $d + d^*, \chi_\Omega$ and J'_Ω are concerned we get the signs $\epsilon = 1, \epsilon'' = 1$ but on $\Omega^k(M)$ d and d^* have different (anti) commutation relations with J'_Ω. Instead for the operators $d+d^*, \chi'_\Omega$ and J'_Ω we get the sign $\epsilon = 1$, but again ϵ' is not determined, while on $\Omega^k(M)$ we obtain

$$J'_\Omega \chi'_\Omega = (-1)^k \chi'_\Omega J'_\Omega \tag{21}$$

so the sign ϵ'' depends on k. These features elude the usual notion of KO-dimension which as known was tailored for the canonical spectral triple.

Closing this section we remark that it is not clear whether, and with which additional conditions, this S.T. equipped with any combination of the gradings and real structures as above may faithfully encode the geometric data on M, that can be then reconstructed.

3. Noncommutative formulation of the Standard Model: νS.M.

Concerning the underlying arena of νS.M., see, e.g., [6], it is

$$(\text{ordinary (spin) manifold } M) \times (\text{finite quantum space } F),$$

described by the algebra $C^\infty(M) \otimes A_F \approx C^\infty(M, A_F)$, where

$$\boxed{A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).}$$

Here $\mathbb{H}$ is the (real) algebra of matrices of the form

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}, \qquad \alpha, \beta \in \mathbb{C},$$

which is isomorphic to the algebra of quaternions.

The Hilbert space is

$$L^2(S) \otimes H_F,$$

where

$$\boxed{H_F = \mathbb{C}^{96} =: H_f \otimes \mathbb{C}^3,}$$

with $\mathbb{C}^3$ corresponding to 3 generations, and

$$H_f = \mathbb{C}^{32} \simeq M_{8\times 4}(\mathbb{C}).$$

The orthonormal basis of H_f is labeled by particles and antiparticles, that we arrange as a 8×4 matrix

$$\begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \\ \bar{\nu}_R & \bar{e}_R & \bar{\nu}_L & \bar{e}_L \\ \bar{u}_R^1 & \bar{d}_R^1 & \bar{u}_L^1 & \bar{d}_L^1 \\ \bar{u}_R^2 & \bar{d}_R^2 & \bar{u}_L^2 & \bar{d}_L^2 \\ \bar{u}_R^3 & \bar{d}_R^3 & \bar{u}_L^3 & \bar{d}_L^3 \end{bmatrix},$$

where the indices 1, 2, 3 correspond to the color quantum number.

The representation π_F of A_F is diagonal in generations and $\pi_F(\lambda, q, m)$ is given on H_f by left multiplication by the matrix:

$$\begin{bmatrix} \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{q}} \\ 0 & 0 & & \end{array}\right] & 0_4 \\ 0_4 & \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array}\right] \end{bmatrix}. \tag{22}$$

Note that $\pi_F(A_F)$ is a real $*$-algebra of operators, and to get its complexification $\mathbb{A}_F$ just replace $\bar{\lambda}$ by an independent $\lambda' \in \mathbb{C}$, and take $q \in M_2(\mathbb{C})$.

The grading (the chirality operator) is

$$\chi_M \otimes \chi_F,$$

where χ_F is diagonal in generations and on H_f reads:

$$\chi_F = \begin{bmatrix} 1_2 & & \\ & -1_2 & \\ & & 0_4 \end{bmatrix} \otimes 1_4 + \begin{bmatrix} 0_4 & \\ & -1_4 \end{bmatrix} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix}. \tag{23}$$

The real conjugation is

$$J_M \otimes J_F,$$

where J_F is also diagonal in generations and on H_f reads:

$$J_F \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2^* \\ v_1^* \end{bmatrix}, \tag{24}$$

that satisfies $J_F^2 = 1$, the order 0 condition:

$$[a, J_F b J_F^{-1}] = 0 \qquad \forall\, a, b \in A_F,$$

and the order 1 condition:

$$[[D, a], J_F b J_F^{-1}] = 0 \qquad \forall\, a, b \in A_F \tag{25}$$

(as in the classical case).

Finally, the Dirac operator is

$$D = \not{D}_M \otimes \mathsf{id} + \chi_M \otimes D_F,$$

where D_F employed by Chamseddine–Connes' reads on H_F

$$D_F = \left[\begin{array}{cccc|cccc} & & \Upsilon_\nu^* & & \Upsilon_R^* & & & \\ & & & \Upsilon_e^* & & & & \\ \Upsilon_\nu & & & & & & & \\ & \Upsilon_e & & & & & & \\ \hline \Upsilon_R & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array}\right] \otimes e_{11} + \left[\begin{array}{cccc|cccc} & & \Upsilon_u^* & & & & & \\ & & & \Upsilon_d^* & & & & \\ \Upsilon_u & & & & & & & \\ & \Upsilon_d & & & & & & \\ \hline & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array}\right] \otimes e_{11}^\perp$$

$$+ e_{55} \otimes \left[\begin{array}{cc|cc} 0 & 0 & \Upsilon_\nu^* & 0 \\ 0 & 0 & 0 & \Upsilon_e^* \\ \hline \Upsilon_\nu & 0 & 0 & 0 \\ 0 & \Upsilon_e & 0 & 0 \end{array}\right] + (e_{66} + e_{77} + e_{88}) \otimes \left[\begin{array}{cc|cc} 0 & 0 & \Upsilon_u^* & 0 \\ 0 & 0 & 0 & \Upsilon_d^* \\ \hline \Upsilon_u & 0 & 0 & 0 \\ 0 & \Upsilon_d & 0 & 0 \end{array}\right]. \qquad (26)$$

Here the first tensor factor acts by the left matrix multiplication and the second one by the right matrix multiplication, e_{jk} are the usual matrix units, the empty spaces stand for 0, and Υ's are in $\mathrm{Mat}(N, \mathbb{C})$ with N equal to the number of generations ($N = 3$ on the current experimental basis).

Concerning the data D_F, χ_F and J_F given above the KO-dimension comes as 6.

3.1. Properties of νS.M.

With the ingredients as listed in the previous section one gets:

- the group $\mathcal{G} := \{U = uJ_FuJ_F^{-1} \,|\, u \in A_F, \det U = 1\}$ turns out to be isomorphic (up to a finite center) with the gauge group $U(1)\times SU(2)\times SU(3)$ of the S.M. (also as functions on M);
- all the fundamental fermions in H have the correct S.M. charges with respect to $\mathcal{G}$ broken to $U(1)_{em} \times SU(3)$;
- the 1-forms $a[D_F, b]$, $a, b \in C^\infty(M, A_F)$ yield the gauge fields A_μ, $W^\pm$, Z, G_μ of the S.M. (from the part D_M of D), plus the weak doublet complex scalar Higgs field (from the part D_F of D).

The merits of the noncommutative formulation are:

- it treats discrete and continuous spaces (or variables) on the same footing;
- both the gauge and the Higgs field arise as parts of a connection;
- the appearance of solely fundamental representations of $\mathcal{G}$ in the S.M. gets an explanation as the fact that they are the only irreducible representations of simple algebras;
- there is an elegant spectral action $\mathrm{Tr} f(D/\Lambda)$, that reproduces the bosonic part of $\mathcal{L}_{S.M.}$ as the lowest terms of asymptotic expansion in Λ, and the matter action $< \phi, D\phi >$ for the (Wick-rotated) fermionic part;
- it couples in a natural way to gravity on M;
- is claimed [2, 3] to predict new relations among the parameters of S.M.

Some of the shortcomings still present are as in the usual S.M.:
the 3 generations (families) put by hand, several free parameters, though most of

them incorporated into a single geometric quantity: D_F. Others are the unimodularity condition to be posed on the gauge group $\mathcal{G}$ and a special treatment needed for the two kinds of fermion doublings due to the presence of chirality ± 1 and particles/antiparticles both in $L^2(S)$ and H_F.

3.2. The geometric nature of H_f

The above "almost commutative" geometry is described by a S.T.

$$(C^\infty(M), L^2(S), D\!\!\!/) \times (A_F, H_f, D_F),$$

that is mathematically a product of the "external" canonical S.T. on spin manifold M with the "internal" *finite* S.T.

Few quite natural questions are in order concerning the geometric interpretation of the internal S.T. (A_F, H_f, D_F):

Does it also correspond to a (noncommutative) spin manifold?
Are the elements of H_f "spinors" in some sense?
In particular "Dirac spinors"?
Or does it correspond rather to de Rham forms?
Or else?

To answer these questions, motivated by the classical case as in Section 2.1, the following definition has been formulated for a general unital S.T.:

Definition ([7]). A real spectral triple (A, H, D, J) is called *spin* (and elements of H are *quantum Dirac spinors*) if H is a Morita equivalence $\mathcal{C}\ell_D(A)$-JAJ^{-1} bimodule (i.e., after norm-completion the algebras $\mathcal{C}\ell_D(A)$ and JAJ^{-1} are maximal one with respect to the other).

Is then the internal S.T. of νS.M. *spin*, like the external one that is given by the canonical S.T. on M?

Building on and extending the classifications of [9] and [11] the answer in [7] is negative. In fact therein after a systematic search an element

$$X = e_{55} \otimes (1 - e_{11})$$

has been found, such that $X \in \mathcal{C}\ell_D(A)'$ but $X \notin JAJ$. A possible way to circumvent this "no go" has been suggested by employing a different grading and adding two extra non-zero matrix elements of D_F, the status of which however requires a further scrutiny (since though desirable for the correct renormalized Higgs mass, they would have unobserved couplings to fermions).

But then, without such additions, may be the internal S.T. of νS.M. is rather an analogue of the other natural classical spectral triple, namely de Rham forms?

To answer this question we have to formulate also these notions noncommutatively using the algebraic description of the Hodge–de Rham spectral triple with the grading χ_Ω and real structure J'_Ω as in Section 2.2.

Definition (cf. [8]). A spectral triple (A, H, D) is called complex Hodge (and vectors in H *complex quantum de Rham forms*) if H is a Morita equivalence $\mathcal{C}\ell_D(A)-$

$\mathcal{C}\ell_D(A)$ bimodule (i.e., after norm-completion these algebras are maximal one with respect to the other).
A complex Hodge spectral triple (A, H, D) with real structure J is called Hodge if J satisfies the order 2 condition and implements the right $\mathcal{C}\ell_D(A)$-action.

The following theorem provides the answer in the case of one generation and thus for $\Upsilon' s \in \mathbb{C}$ in (26).

Theorem ([8]). *For the internal spectral triple of the νS.M. with one generation the Hodge property holds whenever $\Upsilon_x \neq 0$, $\forall\, x \in \{\nu, e, u, d\}$ and*

$$|\Upsilon_\nu| \neq |\Upsilon_u| \qquad or \qquad |\Upsilon_e| \neq |\Upsilon_d|\ . \tag{27}$$

In the rest of this section we will sketch the steps of the proof.

First by direct computation we find that the commutant of A_F in $M_8(\mathbb{C})$ is the algebra C_F with elements of the form

$$\begin{bmatrix} q_{11} & & & q_{12} & \\ & \alpha & & & \\ & & \beta 1_2 & & \\ q_{21} & & & q_{22} & \\ & & & & \delta 1_3 \end{bmatrix}, \tag{28}$$

where $\alpha, \beta, \delta \in \mathbb{C}$, $q = (q_{ij}) \in M_2(\mathbb{C})$. Consequently the commutant of A_F in $\mathrm{End}_{\mathbb{C}}(H)$ is $A'_F = C_F \otimes M_4(\mathbb{C}) \simeq M_4(\mathbb{C})^{\oplus 3} \oplus M_8(\mathbb{C})$ of complex dimension 112.

Next $J_F A_F J_F \subset \mathrm{End}_{\mathbb{C}}(H_F)$ consists of elements of the form:

$$\begin{bmatrix} 1_4 & 0_4 \\ 0_4 & 0_4 \end{bmatrix} \otimes \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array}\right] + \begin{bmatrix} 0_4 & 0_4 \\ 0_4 & 1_4 \end{bmatrix} \otimes \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & q \end{array}\right],$$

where the first factors of the tensor product acts by left matrix multiplication and the second factor by the right matrix multiplication.

Note that A and $A_{\mathbb{C}}$ have the same commutant in $\mathrm{End}_{\mathbb{C}}(H_F)$. The map $a \mapsto J_F \bar{a} J_F$ gives an isomorphism $A_F \to J_F A_F J_F$ and of their complexifications, and also the map $x \mapsto J_F \bar{x} J_F$ is an isomorphism between A'_F and $(J_F A_F J_F)'$.

Therefore the commutant $(J_F A_F J_F)'$ of $J_F A_F J_F$ has elements

$$a \otimes e_{11} + \begin{bmatrix} b & \\ & c \end{bmatrix} \otimes e_{22} + \begin{bmatrix} b & \\ & d \end{bmatrix} \otimes (e_{33} + e_{44}) \tag{29}$$

with $a \in M_8(\mathbb{C})$, $b, c, d \in M_4(\mathbb{C})$.

Furthermore $A'_F \cap (J_F A_F J_F)' \simeq \mathbb{C}^{\oplus 10} \oplus M_2(\mathbb{C})$. It follows that the complex dimension of $A'_F + (J_F A_F J_F)'$ is $2 \cdot 112 - 14 = 210$. The (real) subspace of Hermitian matrices has real dimension 210.

Now we recall that any unital complex $*$-subalgebra of $\mathrm{End}_{\mathbb{C}}(H)$, where $\dim H < \infty$, is a finite direct sum of matrix algebras: $B \simeq \bigoplus_{i=1}^{s} M_{m_i}(\mathbb{C})$. The units P_i, $1 \leq i \leq s$ of $M_{m_i}(\mathbb{C})$ are orthogonal projections and H decomposes as $H \simeq \bigoplus_{i=1}^{s} H_i$, with

$$H_i = P_i \cdot H \simeq \mathbb{C}^{m_i} \otimes \mathbb{C}^{k_i} \ , \tag{30}$$

where k_i is multiplicity of the (unique) irreducible representation $\mathbb{C}^{m_i}$ of $M_{m_i}(\mathbb{C})$ in H_i, and $M_{m_i}(\mathbb{C})$ acts on the 1st factor of $\mathbb{C}^{m_i} \otimes \mathbb{C}^{k_i}$ by matrix product. Then one has the following lemma:

Lemma (A). *The commutant of B in $\mathrm{End}_{\mathbb{C}}(H)$ is $B' \simeq \bigoplus_{i=1}^{s} M_{k_i}(\mathbb{C})$ and the action of B' on $H_i \simeq \mathbb{C}^{m_i} \otimes \mathbb{C}^{k_i}$ is given by matrix multiplication by $M_{k_i}(\mathbb{C})$ of the second factor in the tensor product.*

We will also need:

Lemma (B). *Let (A, H, D, J) be a finite-dimensional real spectral triple. Assume that $B \subseteq \mathrm{End}_{\mathbb{C}}(H)$ is a unital complex $*$-algebra satisfying:*

$$\mathcal{C}\ell_D(A) \subseteq B \qquad \textit{and} \qquad B' = JBJ^{-1} \ .$$

The following are equivalent:

(a) $\mathcal{C}\ell_D(A)' = J\mathcal{C}\ell_D(A)J^{-1}$ (*the Hodge property*)
(b) $\mathcal{C}\ell_D(A)' \subseteq JBJ^{-1}$
(c) $\mathcal{C}\ell_D(A) = B$.

Proof of Lemma (B).

(a $\Rightarrow$ b). The hypothesis $\mathcal{C}\ell_D(A) \subseteq B$ implies $J\mathcal{C}\ell_D(A)J^{-1} \subseteq JBJ^{-1}$; and thus from (a) follows (b).

(b $\Rightarrow$ c). $\mathcal{C}\ell_D(A)' \subseteq JBJ^{-1} = B'$ implies $B \subseteq \mathcal{C}\ell_D(A)$, and so using our assumptions: $\mathcal{C}\ell_D(A) = B$.

(c $\Rightarrow$ a). If (c) holds then $B' = JBJ^{-1}$ translates to $\mathcal{C}\ell_D(A)' = J\mathcal{C}\ell_D(A)J^{-1}$. □

Now, proceeding with the proof of the theorem, we take

$$B := \mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \tag{31}$$

with $(\lambda, m, a, b) \in B$ represented on H_F as

$$\begin{bmatrix} \lambda & 0 \\ 0 & m \end{bmatrix} \otimes e_{22} \otimes 1 + a \otimes e_{11} \otimes e_{11} + b \otimes e_{11} \otimes (1 - e_{11}) \ . \tag{32}$$

Next we:

- check that $\mathcal{C}\ell_{D_F}(A_F) \subset B$;
- check that B and $J_F B J_F^{-1}$ commute, and so $J_F B J_F^{-1} \subseteq B'$;
- note that (32) is equivalent to the representation of B on:

$$(\mathbb{C} \otimes \mathbb{C}^4) \oplus (\mathbb{C}^3 \otimes \mathbb{C}^4) \oplus (\mathbb{C}^4 \otimes \mathbb{C}) \oplus (\mathbb{C}^4 \otimes \mathbb{C}^3)$$

given by matrix multiplication on the first factors;

- use Lemma (A) to infer that
$$B' \simeq M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \simeq B$$
and so we have $J_F B J_F^{-1} = B'$;
- find that $\mathcal{C}\ell_{D_F}(A_F)' \subseteq J_F B J_F^{-1}$;
- finish the proof by Lemma (B).

4. Conclusions

The Connes–Chamseddine noncommutative formulation of the Standard Model interprets the geometry of the S.M. as gravity on the product of a spin manifold M with a finite noncommutative 'internal' space F. The multiplet of fundamental fermions (each one a Dirac spinor on M) defines fields on F that constitute H_F.

We show that the geometric nature of the latter one is not a noncommutative analogue of Dirac spinors on F (unless > 2 new parameters are introduced in the matrix D_F, so fields on M with physical status under scrutiny) but rather of de Rham forms on F if the conditions (27) are satisfied (for one generation). Conversely, the geometric qualification of the internal spectral triple as being Hodge constrains somewhat the parameters Υ occurring in the matrix D_F.

What happens for 3 generations of particles and so 96×96 matrices?
It can be seen (not easily) that then the spin property also does NOT hold, and that as adverted in [1], indeed the order 2 condition $\mathcal{C}\ell_D(A)' \supset J\mathcal{C}\ell_D(A)J$ holds. Whether the Hodge property is satisfied, or H_F corresponds rather to three copies of quantum de Rham forms on F, is currently under investigation.

References

[1] L. Boyle and S. Farnsworth, *Non-Commutative Geometry, Non-Associative Geometry and the Standard Model of Particle Physics*, New J. Phys. 16 (2014) 123027.

[2] A.H. Chamseddine and A. Connes, *Space-Time from the spectral point of view*, in the Proceedings of the 12th Marcel Grossmann meeting, World Sci. Singapore, 2012.

[3] A.H. Chamseddine and A. Connes, *Resilience of the Spectral Standard Model*, JHEP 1209 (2012) 104.

[4] A. Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom. 7 (2013) 1–82.

[5] A. Connes, *Gravity coupled with matter and foundation of non-commutative geometry*, Commun. Math. Phys. 182 (1996) 155–176.

[6] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*, Colloquium Publications, vol. 55, AMS, 2008.

[7] F. D'Andrea and L. Dąbrowski, *The Standard Model in Noncommutative Geometry and Morita equivalence*, J. Noncommut. Geom. 10 (2016) 551–578.

[8] L. Dąbrowski, F. D'Andrea and A Sitarz, *The Standard Model in noncommutative geometry: fundamental fermions as internal forms*, Lett. Math. Phys. 108 (2018) 1323, arXiv:1703.05279

[9] T. Krajewski, *Classification of Finite Spectral Triples*, J. Geom. Phys. 28 (1998) 1–30.

[10] S. Lord, A. Rennie and J.C. Várilly, *Riemannian manifolds in noncommutative geometry*, J. Geom. Phys. 62 (2012) 1611–1638.

[11] M. Paschke and A. Sitarz, *Discrete spectral triples and their symmetries*, J. Math. Phys. 39 (1996) 6191–6205.

[12] R.J. Plymen, *Strong Morita equivalence, spinors and symplectic spinors*, J. Operator Theory 16 (1986), 305–324.

Ludwik Dąbrowski
Scuola Internazionale Superiore di Studi Avanzati (SISSA)
via Bonomea 265
I-34136 Trieste, Italy
e-mail: dabrow@sissa.it

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 83–93

Recursion Operator in a Noncommutative Minkowski Phase Space

Mahouton Norbert Hounkonnou, Mahougnon Justin Landalidji and Ezinvi Baloïtcha

Abstract. A recursion operator for a geodesic flow, in a noncommutative (NC) phase space endowed with a Minkowski metric, is constructed and discussed in this work. A NC Hamiltonian function $\mathcal{H}_{\text{nc}}$ describing the dynamics of a free particle system in such a phase space, equipped with a noncommutative symplectic form ω_{nc} is defined. A related NC Poisson bracket is obtained. This permits to construct the NC Hamiltonian vector field, also called NC geodesic flow. Further, using a canonical transformation induced by a generating function from the Hamilton–Jacobi equation, we obtain a relationship between old and new coordinates, and their conjugate momenta. These new coordinates are used to re-write the NC recursion operator in a simpler form, and to deduce the corresponding constants of motion. Finally, all obtained physical quantities are re-expressed and analyzed in the initial NC canonical coordinates.

Mathematics Subject Classification (2010). 37C10; 37J35.

Keywords. Noncommutative Minkowski phase space, recursion operator, geodesic flow, Nijenhuis torsion, constants of motion.

1. Introduction

In the last few decades there was a renewed interest in completely integrable Hamiltonian systems, the concept of which goes back to Liouville in 1897 [14] and Poincaré in 1899 [18]. They are dynamical systems admitting a Hamiltonian description and possessing sufficiently many constants of motion, so that they can be integrated by quadratures. Some qualitative features of these systems remain true in some special classes of infinite-dimensional Hamiltonian systems expressed by nonlinear evolution equations as, for instance, Korteweg–de Vries and sine-Gordon [25].

A relevant progress in the study of these systems with an infinite-dimensional phase manifold $\mathcal{M}$ was the introduction of the Lax Representation [13]. It played an important role in formulating the Inverse Scattering Method [1], one of the most remarkable result of theoretical physics in last decades. This method allows the integration of nonlinear dynamics, both with finitely or infinitely many degrees of freedom, for which a Lax representation can be given [8], this being both of physical and mathematical relevance [5].

Another progress, in the analysis of the integrability, was the important remark that many of these systems are Hamiltonian dynamics with respect to two compatible symplectic structures [9, 15, 25], this leading to a geometrical interpretation of the so-called recursion operator [13]. For more details, see [20] and references therein. A description of integrability working both for systems with finitely many degrees of freedom and for field theory can be given in terms of invariant, diagonalizable mixed $(1,1)$-tensor field, having bidimensional eigenspaces and vanishing Nijenhuis torsion. A natural approach to integrability is to try to find sufficient conditions for the eigenvalues of the recursion operator to be in involution. Thereby, a new characterization of integrable Hamiltonian systems is given by De Filippo *et al* through the following Theorem [6]:

Theorem 1. *Let X be a dynamical vector field on a $2n$-dimensional manifold $\mathcal{M}$. If the vector field X admits a diagonalizable mixed $(1,1)$-tensor field T which is invariant under X, has a vanishing Nijenhuis torsion and has doubly degenerate eigenvalues with nowhere vanishing differentials, then there exist a symplectic structure and a Hamiltonian function H such that the vector field X is separable, Hamiltonian vector field of H, and H is completely integrable with respect to the symplectic structure.*

Such a $(1,1)$-tensor field T is called a recursion operator of X.

In a particular case of $\mathbb{R}^{2n}$, a recursion operator can be constructed as follows [22]:

Lemma 2. *Let us consider vector fields*

$$X_l = -\frac{\partial}{\partial x_{n+l}}, \quad l = 1, \dots, n$$

on $\mathbb{R}^{2n}$ and let T be a $(1,1)$-tensor field on $\mathbb{R}^{2n}$ given by

$$T = \sum_{i=1}^{n} x_i \left(\frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right).$$

Then, we have that the Nijenhuis torsion $\mathcal{N}_T$ and the Lie derivative $\mathcal{L}_{X_l}$ of T are vanishing, i.e.,

$$(\mathcal{N}_T)^h_{ij} := T^k_i \frac{\partial T^h_j}{\partial x^k} - T^k_j \frac{\partial T^h_i}{\partial x^k} + T^h_k \frac{\partial T^k_i}{\partial x^j} - T^h_k \frac{\partial T^k_j}{\partial x^i} = 0, \quad \mathcal{L}_{X_l} T = 0. \tag{1}$$

That is the $(1,1)$-tensor field T is a recursion operator of $X_l, (l = 1, \dots, n)$.

On the other hand, this $(1,1)$-tensor field T is used as an operator which generate enough constants of motion [12]. Based on Theorem 1, a series of investigations was done (see, e.g., [6, 7, 10, 12, 17, 20–23, 26] and references therein).

One of powerful methods of describing completely integrable Hamiltonian systems with involutive Hamiltonian functions or constants of motion uses the recursion operator admitting a vanishing Nijenhuis torsion.

Recently, in 2015, Takeuchi constructed recursion operators of Hamiltonian vector fields of geodesic flows for some Riemannian and Minkowski metrics [21], and obtained related constants of motion. Further, he used five particular solutions of the Einstein equation in the Schwarzschild, Reissner–Nordstrøm, Kerr, Kerr–Newman, and FLRW metrics, and showed that the Hamiltonian functions of the associated corresponding geodesic flows form a system of variables separation equations. Then, he constructed recursion operators inducing the complete integrability of the Hamiltonian functions. In the present work, we investigate the same problem in a deformed Minkowski phase space.

This paper is organized as follows. In Section 2, we consider a noncommutative Minkowski phase space, and define the NC Hamiltonian function and symplectic form, as well as the corresponding NC Poisson bracket. In Section 3, we construct the associated NC recursion operator for the NC Hamiltonian vector field of the geodesic flow, and obtain related constants of motion. In Section 4, we end with some concluding remarks.

2. Noncommutative Minkowski phase space

The noncommutativity between space-time coordinates was first introduced by Snyder [19]. Later, Alain Connes developed the noncommutative geometry [3] and applied it to various physical situations [4]. Since then, the noncommutative geometry remained a very active research subject in several domains of theoretical physics and mathematics.

Noncommutativity between phase space variables is here understood by replacing the usual product with the β-star product, also known as the Moyal product law between two arbitrary functions of position and momentum, as follows [11, 16, 24]:

$$(f *_\beta g)(q,p) = f(q_i,p_i)\exp\left(\frac{1}{2}\beta^{ab}\overleftarrow{\partial}_a\overrightarrow{\partial_b}\right)g(q_j,p_j)\Big|_{(q_i,p_i)=(q_j,p_j)}, \tag{2}$$

where

$$\beta_{ab} = \begin{pmatrix} \alpha_{ij} & \delta_{ij}+\gamma_{ij} \\ -\delta_{ij}-\gamma_{ij} & \lambda_{ij} \end{pmatrix}, \tag{3}$$

α and λ are antisymmetric $n\times n$ matrices which represent the noncommutativity in coordinates and momenta, respectively; γ is some combination of α and λ. The $*_\beta$ deformed Poisson bracket can be written as

$$\{f,g\}_\beta = f *_\beta g - g *_\beta f. \tag{4}$$

So, we can show that:

$$\{q_i,q_j\}_\beta = \alpha_{ij},\ \{q_i,p_j\}_\beta = \delta_{ij}+\gamma_{ij},\ \{p_i,q_j\}_\beta = -\delta_{ij}-\gamma_{ij},\ \{p_i,p_j\}_\beta = \lambda_{ij}. \tag{5}$$

Now, consider the following transformations:

$$q'_i = q_i - \frac{1}{2}\sum_{j=1}^{n}\alpha_{ij}p_j, \quad p'_i = p_i + \frac{1}{2}\sum_{j=1}^{n}\lambda_{ij}q_j, \tag{6}$$

where q'_i and p'_j obey the same commutation relations as in (5), but with respect to the usual Poisson bracket:

$$\{q'_i, q'_j\} = \alpha_{ij},\ \{q'_i, p'_j\} = \delta_{ij} + \gamma_{ij},\ \{p'_i, q'_j\} = -\delta_{ij} - \gamma_{ij},\ \{p'_i, p'_j\} = \lambda_{ij}, \tag{7}$$

with q_i and p_j satisfying the following commutation relations:

$$\{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0. \tag{8}$$

In our framework, we consider the noncommutative Minkowski phase space with the metric defined by

$$ds'^2 = -dq'^2_1 + dq'^2_2 + dq'^2_3 + dq'^2_4, \tag{9}$$

where q'_1 is time coordinate; q'_2, q'_3, q'_4 are space coordinates. The tensor metric is given by

$$g'_{ij} = g'^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

and the equation of geodesic is

$$\frac{d^2 q'^{\mu}}{dt^2} + \Gamma'^{\mu}_{\nu\lambda}\frac{dq'^{\nu}}{dt}\frac{dq'^{\lambda}}{dt} = \frac{d^2 q'^{\mu}}{dt^2} = 0, \quad (\mu = 1, 2, 3, 4), \tag{11}$$

with the Christoffel symbols $\Gamma'^{\mu}_{\nu\lambda} = 0$.
Set:

$$q'_i = q_i - \frac{1}{2}\sum_{j=1}^{4}\alpha_{ij}p_j, \quad p'_i = p_i + \frac{1}{2}\sum_{j=1}^{4}\lambda_{ij}q_j,\ \lambda_{1j} = \alpha_{1j} = 0,\ p_1 > 0. \tag{12}$$

Then, the commutation relations (7) become:

$$\{q'_i, q'_j\} = \alpha_{ij},\ \{q'_i, p'_j\} = \delta_{ij} + \gamma_{ij},\ \{p'_i, q'_j\} = -\delta_{ij} - \gamma_{ij},\ \{p'_i, p'_j\} = \lambda_{ij}. \tag{13}$$

2.1. NC Hamilton function and NC symplectic form

The NC Hamiltonian function $\mathcal{H}_{\mathrm{nc}}$ describing the dynamics of a free particle system, in the considered NC Minkowski phase space is defined as follows:

$$\mathcal{H}_{\mathrm{nc}} := \frac{1}{2}\bigg(- p'^2_1 + \sum_{k=2}^{4} p'^2_k \bigg).$$

Using equations (12), we get

$$\mathcal{H}_{\mathrm{nc}} = \frac{1}{2}\Bigg[- p_1^2 + \sum_{k=2}^{4}\bigg(p_k + \frac{1}{2}\sum_{j=2}^{4}\lambda_{kj}q_j\bigg)^2\Bigg]. \tag{14}$$

Proposition 3. *The exterior derivative of the Hamiltonian function $\mathcal{H}_{nc}$ is given by*

$$d\mathcal{H}_{nc} = -p_1 dp_1 + \sum_{k=2}^{4} \varpi_k dp_k + \frac{1}{2} \sum_{k,i=2}^{4} \lambda_{ik} \Omega_i dq_k, \tag{15}$$

where

$$\varpi_k = \left(p_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)$$

and

$$\Omega_i = \left(p_i + \frac{1}{2} \sum_{j=2}^{4} \lambda_{ij} q_j \right).$$

The NC symplectic form is now defined by

$$\omega_{\mathrm{nc}} := \sum_{i=1}^{4} dp_i' \wedge dq_i' = dp_1' \wedge dq_1' + \sum_{k=2}^{4} dp_k' \wedge dq_k'. \tag{16}$$

Proposition 4. *Considering the NC Minkowski phase space, the symplectic form associated with the Hamiltonian function $\mathcal{H}_{nc}$ is given by*

$$\omega_{nc} = \sum_{\nu=1}^{4} \theta_\nu dp_\nu \wedge dq_\nu, \tag{17}$$

where

$$\theta_\nu = \sum_{\mu=1}^{4} \left(\delta_{\mu\nu} + \frac{1}{4} \lambda_{\mu\nu} \alpha_{\mu\nu} \right) \neq 0, \quad \delta_{\mu\nu} = \begin{cases} 0, & \text{if } \mu \neq \nu \\ 1, & \text{if } \mu = \nu. \end{cases} \tag{18}$$

2.2. NC Poisson bracket and NC Hamiltonian vector field

Proposition 5. *The bracket given by*

$$\{f, g\}_{nc} = \sum_{\nu=1}^{4} \theta_\nu^{-1} \left(\frac{\partial f}{\partial p_\nu} \frac{\partial g}{\partial q_\nu} - \frac{\partial f}{\partial q_\nu} \frac{\partial g}{\partial p_\nu} \right) \tag{19}$$

is a Poisson bracket which respects the symplectic form ω_{nc}, where f and g are arbitrary differentiable coordinate functions on the NC Minkowski phase space.

Proposition 6. *In the NC Minkowski phase space, the Hamiltonian vector field is given by*

$$X_{\mathcal{H}_{nc}} = -p_1 \frac{\partial}{\partial q_1} + \sum_{k=2}^{4} \theta_k^{-1} \left(\varpi_k \frac{\partial}{\partial q_k} - \frac{1}{2} \sum_{i=2}^{4} \lambda_{ik} \Omega_i \frac{\partial}{\partial p_k} \right), \tag{20}$$

where

$$\varpi_k = \left(p_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right) \qquad \text{and} \qquad \Omega_i = \left(p_i + \frac{1}{2} \sum_{j=2}^{4} \lambda_{ij} q_j \right).$$

3. NC Recursion operator

In this section, we construct the recursion operator for the geodesic flow in the NC Minkowski phase space, and derive the constants of motion. We consider the Hamilton–Jacobi equation [2] for the Hamiltonian function (14), and introduce a generating function W^{nc} satisfying the following relations:

$$p = \frac{\partial W^{\mathrm{nc}}}{\partial q} \quad \text{and} \quad P = -\frac{\partial W^{\mathrm{nc}}}{\partial Q}. \tag{21}$$

This allows us to obtain a relationship between the (p, q) and (P, Q) coordinates. Then, using Lemma 2, we build a recursion operator for the NC Hamiltonian vector field $X_{\mathcal{H}_{\mathrm{nc}}}$.

3.1. NC Hamilton–Jacobi equation and generating function

The NC Hamilton–Jacobi equation is a nonlinear equation given by

$$E_{\mathrm{nc}} = \mathcal{H}_{\mathrm{nc}}\left(q, \frac{\partial W^{\mathrm{nc}}}{\partial q}\right). \tag{22}$$

Thus,

$$E_{\mathrm{nc}} = \frac{1}{2}\left\{-\left(\frac{\partial W^{\mathrm{nc}}}{\partial q_1}\right)^2 + \sum_{k=2}^{4}\left(\frac{\partial W^{\mathrm{nc}}}{\partial q_k} + \frac{1}{2}\sum_{j=2}^{4}\lambda_{kj}q_j\right)^2\right\},$$

where E_{nc} is a constant. Setting $W^{\mathrm{nc}} = \sum_{i=1}^{4} W_i^{\mathrm{nc}}(q_i)$, where $W_i^{\mathrm{nc}}(q_i) = a_i q_i$ and a_i $(i = 1, 2, 3, 4)$ are constants, not depending on q_i, leads to $a_i = \frac{\partial W_i^{\mathrm{nc}}}{\partial q_i}$, and (22) becomes

$$2E_{\mathrm{nc}} = -a_1^2 + \sum_{k=2}^{4}\left(a_k + \frac{1}{2}\sum_{j=2}^{4}\lambda_{kj}q_j\right)^2.$$

Assume now $\left[\sum_{k=2}^{4}\left(a_k + \frac{1}{2}\sum_{j=2}^{4}\lambda_{kj}q_j\right)^2 - 2E_{\mathrm{nc}}\right] > 0$. Then,

$$a_1 = \pm\sqrt{\sum_{k=2}^{4}\left(a_k + \frac{1}{2}\sum_{j=2}^{4}\lambda_{kj}q_j\right)^2 - 2E_{\mathrm{nc}}}.$$

Considering the future domain yields

$$a_1 = \sqrt{\sum_{k=2}^{4}\left(a_k + \frac{1}{2}\sum_{j=2}^{4}\lambda_{kj}q_j\right)^2 - 2E_{\mathrm{nc}}}, \tag{23}$$

and

$$W^{\mathrm{nc}} = \left(\sqrt{\sum_{k=2}^{4}\left(a_k + \frac{1}{2}\sum_{j=2}^{4}\lambda_{kj}q_j\right)^2 - 2E_{\mathrm{nc}}}\right)q_1 + \sum_{k=2}^{4}a_k q_k. \tag{24}$$

Now, we introduce a generating function by using the above solution such that $W^{\rm nc} = W^{\rm nc}(q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4)$ becomes

$$W^{\rm nc} = \left(\sqrt{\sum_{k=2}^{4} Q_k^2 - 2Q_1}\right) q_1 + \sum_{k=2}^{4} \left(Q_k - \frac{1}{2}\sum_{j=2}^{4} \lambda_{kj} q_j\right) q_k, \tag{25}$$

where

$$\left(\sum_{k=2}^{4} Q_k^2 - 2Q_1\right) > 0,\ Q_1 = E_{\rm nc}, \tag{26}$$

and

$$Q_k = a_k + \frac{1}{2}\sum_{j=2}^{4} \lambda_{kj} q_j, (k = 2, 3, 4). \tag{27}$$

Thanks to the canonical transformations (21), we obtain the following relationship between the canonical coordinate system (P, Q) and the original coordinate system (p, q) :

$$\begin{cases} p_1 = \sqrt{\displaystyle\sum_{k=2}^{4} Q_k^2 - 2Q_1} \\ p_k = Q_k \\ q_1 = P_1 \sqrt{\displaystyle\sum_{k=2}^{4} Q_k^2 - 2Q_1} \\ q_k = -P_k - Q_k P_1 \end{cases} ; \quad \begin{cases} P_1 = \dfrac{q_1}{p_1} \\ P_k = -\dfrac{p_k q_1}{p_1} - q_k \\ Q_1 = \mathcal{H}_{\rm nc} \\ Q_k = p_k, \end{cases} , \tag{28}$$

where $k = 2, 3, 4$.

3.2. $(1,1)$-tensor field T as recursion operator

In the (P, Q) coordinate systems, the Hamiltonian vector field is defined by

$$X_{\mathcal{H}_{\rm nc}} := \{\mathcal{H}_{\rm nc}, .\}_{\rm nc} = \sum_{\nu=1}^{4} \theta_\nu^{-1} \left(\frac{\partial \mathcal{H}_{\rm nc}}{\partial P_\nu}\frac{\partial}{\partial Q_\nu} - \frac{\partial \mathcal{H}_{\rm nc}}{\partial Q_\nu}\frac{\partial}{\partial P_\nu}\right).$$

Setting $\mathcal{H}_{\rm nc} = Q_1$ and $\theta_1 = 1$ transforms the NC Hamiltonian vector field $X_{\mathcal{H}_{\rm nc}}$ and symplectic form $\omega_{\rm nc}$ into the forms

$$X_{\mathcal{H}_{\rm nc}} = -\frac{\partial \mathcal{H}_{\rm nc}}{\partial Q_1}\frac{\partial}{\partial P_1} = -\frac{\partial}{\partial P_1}, \tag{29}$$

and

$$\omega_{\rm nc} = \sum_{\nu=1}^{4} \theta_\nu dP_\nu \wedge dQ_\nu, \tag{30}$$

respectively. A tensor field T of $(1,1)$-type can then be expressed as:

$$T = \sum_{\nu=1}^{4} Q_\nu \left(\frac{\partial}{\partial P_\nu} \otimes dP_\nu + \frac{\partial}{\partial Q_\nu} \otimes dQ_\nu\right). \tag{31}$$

Letting $x_\nu = Q_\nu$ and $x_{\nu+4} = P_\nu$, where $\nu = 1, 2, 3, 4$, affords the tensor field

$$T = \sum_{i,j=1}^{8} T^i_j \frac{\partial}{\partial x^i} \otimes dx^j,$$

with $x \equiv (Q_1, \ldots, Q_4, P_1, \ldots, P_4)$. The matrix (T^i_j) is given by

$$(T^i_j) = \begin{pmatrix} {}^tA & O \\ O & A \end{pmatrix}, \quad (A^i_j) = \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & Q_4 \end{pmatrix}.$$

Then, by Lemma 2, T satisfies $\mathcal{L}_{X_{\mathcal{H}_{\mathrm{nc}}}} T = 0$, the Nijenhuis torsion $\mathcal{N}_T$ of T is vanishing, i.e., $\mathcal{N}_T = 0$, and $degQ_i = 2$. Hence, T is a recursion operator of the Hamiltonian vector field $X_{\mathcal{H}_{\mathrm{nc}}}$. The constants of motion $Tr(T^l)$, $(l = 1, 2, 3, 4)$, of the geodesic flow are:

$$Tr(T^l) = 2(Q_1^l + Q_2^l + Q_3^l + Q_4^l), \quad l = 1, 2, 3, 4.$$

Proposition 7. *Assume:*

(1) $\lambda_{1\mu} = \alpha_{1\mu} = 0, \quad \mu = 1,\ 2,\ 3,\ 4;$
(2) $\lambda_{\nu\mu}\theta_\nu = \lambda_{\mu\nu}\theta_\mu, \quad$ *for every* $\nu, \mu = 2,\ 3,\ 4.$

Then, the geodesic flow has a recursion operator T given by

$$T = \sum_{\mu,\nu=1}^{4} \left(M^\mu_\nu \frac{\partial}{\partial q_\nu} \otimes dq_\mu + N^\mu_\nu \frac{\partial}{\partial p_\nu} \otimes dp_\mu + L^\mu_\nu \frac{\partial}{\partial q_\nu} \otimes dp_\mu + R^\mu_\nu \frac{\partial}{\partial p_\nu} \otimes dq_\mu \right), \quad (32)$$

where

$$M = \begin{pmatrix} \mathcal{H}_{nc} & \frac{p_2}{p_1}\left(p_2 - \mathcal{H}_{nc}\right) & \frac{p_3}{p_1}\left(p_3 - \mathcal{H}_{nc}\right) & \frac{p_4}{p_1}\left(p_4 - \mathcal{H}_{nc}\right) \\ \frac{q_1\mathcal{H}_{nc}}{p_1^2} S_2 & p_2 & 0 & 0 \\ \frac{q_1\mathcal{H}_{nc}}{p_1^2} S_3 & 0 & p_3 & 0 \\ \frac{q_1\mathcal{H}_{nc}}{p_1^2} S_4 & 0 & 0 & p_4 \end{pmatrix},$$

$$N = \begin{pmatrix} \mathcal{H}_{nc} & 0 & 0 & 0 \\ \frac{p_2}{p_1} V_2 & p_2 & 0 & 0 \\ \frac{p_3}{p_1} V_3 & 0 & p_3 & 0 \\ \frac{p_4}{p_1} V_4 & 0 & 0 & p_4 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & \frac{q_1 p_2}{p_1^2}\left[\frac{p_2}{p_1}\left(\mathcal{H}_{nc} - p_2\right)\right] & \frac{q_1}{p_1}\left[\frac{p_3}{p_1}\left(\mathcal{H}_{nc} - p_3\right)\right] & \frac{q_1}{p_1}\left[\frac{p_4}{p_1}\left(\mathcal{H}_{nc} - p_4\right)\right] \\ \frac{q_1 p_2}{p_1^2} V_2 & 0 & 0 & 0 \\ \frac{q_1 p_2}{p_1^2} V_3 & 0 & 0 & 0 \\ \frac{q_1 p_2}{p_1^2} V_4 & 0 & 0 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\mathcal{H}_{nc}}{p_1} S_2 & 0 & 0 & 0 \\ \frac{\mathcal{H}_{nc}}{p_1} S_3 & 0 & 0 & 0 \\ \frac{\mathcal{H}_{nc}}{p_1} S_4 & 0 & 0 & 0 \end{pmatrix},$$

with $V_k = p_k - \mathcal{H}_{nc} - \frac{\mathcal{H}_{nc}}{2p_k}\sum_{j=2}^{4}\lambda_{kj}q_j$ *and* $S_k = -\frac{1}{2}\sum_{i=2}^{4}\lambda_{ik}\Omega_i, (k = 2,\ 3,\ 4)$.

The constants of motion in the original coordinate system (p, q) *are* $Tr(T^l)$, $(l = 1,\ 2,\ 3,\ 4)$*:*

$$Tr(T^l) = 2\mathcal{H}_{nc}^l + 2(p_2^l + p_3^l + p_4^l) = \frac{1}{2^{l-1}}\left(-p_1^2 + \sum_{k=2}^{4}\varpi_k^2\right)^l + 2(p_2^l + p_3^l + p_4^l). \quad (33)$$

4. Concluding remarks

In this paper, we have constructed a recursion operator of a Hamiltonian vector field for a geodesic flow in a noncommutative Minkowski phase space, and computed the associated constants of motion. For the vanishing deformation parameter β, the NC Minkowski phase space turns to be the usual one, and all the results displayed in this work are reduced to the particular cases examined in [21] and [22].

References

[1] M.J. Ablowitz, H. Segur, *Solitons and the Inverse Scattering Transform.* 1st Edition, Society for Industrial and Applied Mathematics, 1981.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics.* 2nd Edition, Springer-Verlag, 1978.

[3] A. Connes, *Cyclic Cohomology and Noncommutative Differential Geometry.* Proceedings of the International Congress of Mathematicians, 1986.

[4] A. Connes, *Geometrie noncommutative.* 1st Edition, Academic Press, 1994.

[5] L.D. Faddeev, L.A. Takhtajan *Hamiltonian Methods in the Theory of Solitons.* 1st Edition, Springer-Verlag, 1986.

[6] S. De Filippo, G. Marmo, M. Salerno and G. Vilasi, *A New Characterization of Completely Integrable Systems.* Nuovo Cimento B **83** (1984), 97–112.

[7] S. De Filippo, G. Marmo and G. Vilasi, *A Geometrical Setting for the Lax Representation.* Phys. Lett. B **117** (1982), 418–422.

[8] C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura, *Method for solving the Korteweg–de Vries equation.* Phys. Rev. Lett. **19** (1967), 1095–1097.

[9] I.M. Gelfand, I.Ya. Dorfman, *The Schouten Bracket and Hamiltonian Operators.* Funct. Anal. Appl. **14** (1980), 71–74.

[10] K. Hosokawa, T. Takeuchi, *A Construction for the Concrete Example of a Recursion.* Operator, JP Journal of Geometry and Topology **14** (2013), 99–183.

[11] N. Khosravi, S. Jalalzadeh, H.R. Sepangi, *Non-commutative multi-dimensional cosmology.* JHEP **01** (2006) 134, 1–10

[12] G. Landi, G. Marmo and G. Vilasi, *Recursion Operators: Meaning and Existence for Completely Integrable Systems.* J. Math. Phys. **35** (1994), 808–815.

[13] P.D. Lax *Integrals of Nonlinear Equations of Evolution and Solitary Waves.* Commun. Pure Appl. Math. **21** (1968), 467–490.

[14] R. Liouville, *Sur le mouvement d'un corps solide pesant suspendu par l'un de ses points.* Acta Math. **20** (1897), 239–284.

[15] F. Magri, *A simple model of the integrable Hamiltonian equation.* J. Math. Phys. **19** (1978), 1156–1162.

[16] B. Malekolkalami, K. Atazadeh and B. Vakili, *Late time acceleration in a noncommutative model of modified cosmology.* Phys. Lett. B **739** (2014), 400–404.

[17] G. Marmo and G. Vilasi, *When Do Recursion Operators Generate New Conservation Laws?* Phys. Lett. B **277** (1992), 137–140.

[18] H. Poincaré, *Sur les quadratures mécaniques.* Acta Math. **13** (1899), 382–387.

[19] H. Snyder, *Quantized Space-Time.* Phys. Rev. (1947), 38–71.

[20] G. Sparano and G. Vilasi, *Noncommutative Integrability and Recursion Operators.* J. Geom. Phys. **36** (2000), 270–284.

[21] T. Takeuchi, *On the construction of recursion operators for some geodesic flows.* Departement of Mathematics, PhD thesis, Tokyo University of Science, 2015.

[22] T. Takeuchi, *A Construction of a Recursion Operator for Some Solutions of Einstein Field Equations.* Proceedings of the Fifteenth International Conference on Geometry, Integrability and Quantization **15** (2014), 249–258.

[23] T. Takeuchi, *On the Construction of Recursion Operators for the Kerr–Newman and FLRW Metrics*. J. Geom. Phys. **37** (2015), 85–96.

[24] B. Vakili, P. Pedram and S. Jalalzadeh, *Late time acceleration in a deformed phase space model of dilaton cosmology*. Phys. Lett. B **687** (2010), 119–123.

[25] G. Vilasi,*On the Hamiltonian Structures of the Korteweg–de Vries and Sine-Gordon Theories*. Phys. Lett. B **94** (1980), 195–198.

[26] G. Vilasi, *Hamiltonian Dynamics*. 1st Edition, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

Mahouton Norbert Hounkonnou, Mahougnon Justin Landalidji and Ezinvi Baloïtcha
International Chair of Mathematical Physics and Applications
(ICMPA-UNESCO Chair)
University of Abomey-Calavi
072 B.P. 50
Cotonou, Republic of Benin
e-mail: norbert.hounkonnou@cipma.uac.bj
with copy to
hounkonnou@yahoo.fr
landalidjijustin@yahoo.fr
ezinvi.baloitcha@cipma.uac.bj

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 95–101

Decompactifying Spectral Triples

Andrzej Sitarz

Abstract. We show that one can approximate different geometries, including the locally compact ones using the approximation of their compactifications with a suitably chosen conformal rescaling. We illustrate the idea showing the family of Dirac operators on the "fuzzy circle" that approximate the flat Dirac operator on the line.

Mathematics Subject Classification (2010). Primary 58B34; Secondary 46L85, 46L87, 81T75.

Keywords. Connes' noncommutative geometry, spectral triples.

1. Introduction

One of the appealing features of noncommutative geometry is the possibility to approximate the algebra of functions on a space by the finite-dimensional algebras, which are not necessarily commutative. In contrast to the world of triangulations of manifolds or approximation by lattices, in the noncommutative approximations some of the symmetries might be preserved, as is the case for the fuzzy sphere [6].

Yet the algebras provide only part of the information about the space and it is entirely wrong to interpret the algebra alone as the noncommutative space and see it as already equipped with geometry, while in reality it corresponds only to the topological space. For the same reason a space becomes indeed a sphere only when equipped with the appropriate metric and compatible differential calculus.

Still, a difference between two topological spaces might be very small, as, for example, happens for the infinite (hyper) planes and compact spheres. The respective $\mathbb{C}^*$ algebras of continuous functions differ only by a unit, which suggests that a suitable variations of the metric might allow to cast some noncompact models (like a plane) in the formalism of compact models (like a sphere).

Motivated by few classical examples we shall briefly present the idea how to compactify some known noncompact noncommutative geometries as well as how to decompactify some compact commutative and noncommutative geometries using

Supported by NCN grant 2016/21/B/ST1/02438.

the tools of Connes' noncommutative differential geometry [2] and spectral triple approach [3, 4]. We shall illustrate the idea with a very simple object: the circle.

2. Compact spectral triples and the metric

Let us recall the simplest definition of a spectral triple.

Definition 1. The data for a compact spectral triple consist of a unital algebra $\mathcal{A}$, represented faithfully on a Hilbert space $\mathcal{H}$ as bounded operators $\pi(a), a \in \mathcal{A}$ and a densely defined selfadjoint operator D satisfying the following conditions, here $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$ in $B(\mathcal{H})$):

- $\forall a \in \mathcal{A}$: $[D, \pi(a)] \in B(\mathcal{H})$,
- for even spectral triples: $\exists \gamma \in \mathcal{A}' : \gamma^2 = 1, \gamma = \gamma^\dagger, \gamma D + D\gamma = 0$,
- $(D+i)^{-1}$ is compact.

The idea of a spectral triple comes from the spin geometry, where the algebra are smooth functions over a compact spin manifold, the Hilbert space are all L^2 section of the spinor bundle on which functions act by pointwise multiplication and D is the usual spin Dirac operator.

There are some more conditions, which restrict possible constructions, like the existence of reality structure and order-one condition.

2.1. Conformally rescaled spectral triples

Once we have a spectral triple with a Dirac operator D, there exist two possibilities to find a family of possible Dirac operators. The first way is to modify D by bounded operators, which are one-forms in the differential calculus generated by the commutators with D. Taking $A = \sum_i a_i[D, b_i]$ and $A = A^*$, we can consider a family of operators $D_A = D + A$, which satisfy again conditions of a spectral triple. Such modifications (fluctuations) correspond in the classical case to gauge connections and in principle do not change the metric.

Another possibility is to modify the Dirac operator by a conformal rescaling, assuming that the commutant $\mathcal{A}'$ is big enough. Technically, taking $h = h^* \in \mathcal{A}'$ we can define:

$$D_h = h\,D\,h,$$

which, again, will be an admissible Dirac operator. In the classical case, where $\mathcal{A} \subset \mathcal{A}'$ this corresponds to a conformal rescaling of the metric by h^{-2}. The properties of such conformally modified family of spectral triples were discussed in [8]. A more detailed study of the reality structure and order-one condition for conformally modified geometries was presented in [1].

Using the conformal rescaling one can modify the metric, also to the effect of obtaining from locally compact spectral geometries the spectral triples with a finite volume (compact). Such construction was successfully carried out for the Moyal plane, which after unitization and a suitable choice of the rescaling function was shown to provide a geometry with finite volume and the metric, that was a Moyal counterpart of the Fubini–Study metric for the plane [5].

In this note we want to demonstrate that such process could be also carried in the other direction, thus giving a possibility to study finite approximations of locally compact geometries like a hyperbolic plane, for instance. We test the ideas on the simplest possible commutative example, given by the spectral triple over a circle.

3. The circle

Let us consider the circle parametrized by $z = e^{2\pi i s}$, the algebra of smooth functions $C^\infty(S^1)$ and the standard Dirac operator

$$D = -i\frac{\partial}{\partial s},$$

so that $(C^\infty(S^1), L^2(S^1), D)$ is a spectral triple. It could be easily shown that this Dirac operator corresponds to the circle of length 1 (so that its radius is $\frac{1}{2\pi}$).

3.1. How to decompactify a circle

If $h \in C^\infty(S^1)$ then $D_h = hDh$ is the Dirac operator that, although conformally rescaled, has the same spectrum, with the eigenfunction to the eigenvalue λ:

$$\phi_\lambda(s) = C \exp\left(\int_0^s \frac{i\lambda - h(\xi)h'(\xi)}{h^2(\xi)} d\xi\right).$$

If h is a smooth function on the circle, the continuity condition fixes λ to be:

$$\lambda\left(\int_0^1 \frac{1}{h^2(\xi)} d\xi\right) \in (2\pi)\mathbb{Z},$$

and therefore, up to the global rescaling by the volume, the spectrum of D_h is the same as the spectrum of D.

Consider now, a particular choice of such rescaling function, with a family parametrized by $0 \leq r < 1$:

$$h_r(s) = \sqrt{1 - r\cos(2\pi s)}.$$

For $r = 0$ we have $D_r = D$, whereas in the limiting case $r = 1$ we have the Dirac operator, which is unitarily equivalent to the Dirac operator on the circle with the metric

$$(1 - r\cos(2\pi s))^{-2} ds^2,$$

and this, in turn corresponds to the constant metric on the real line using the usual projection from the circle onto $\mathbb{R}$. The eigenvalues of this Dirac operator depend only on r and we have:

$$\lambda \in (2\pi\sqrt{1 - r^2})\,\mathbb{Z},$$

so they are the same as for the equivariant Dirac operator on the circle with a volume $(1 - r^2)^{-1}$. What changes, however, is the distance, as one can easily see (Fig. 1). Contrary to the case of rescaled radius, the distances do not change with the same proportion but the ratio of growth depends on the position of the interval on the circle.

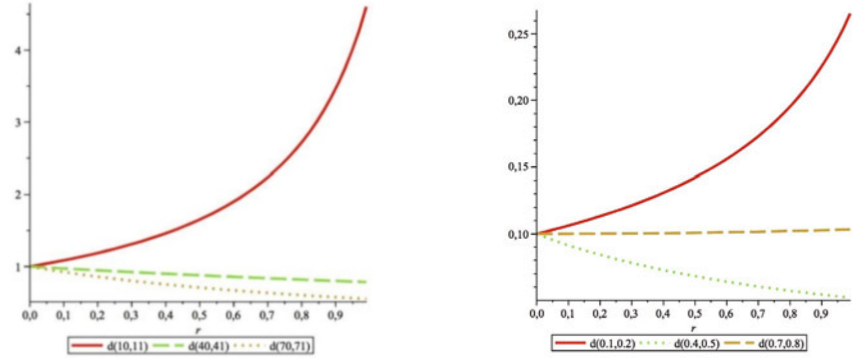


FIGURE 1. Distances between $d(n, n+1)$ as compared to distances for interval on the circle for the families of conformally rescaled Dirac operators D_r.

3.2. The "fuzzy" circle

Although the notion of a fuzzy circle is not as well established as that of the fuzzy sphere, we can consider the following approximation based on the idea of discretized version of the circle.

Let us fix N and take an algebra of functions on the set of N points, labeled $0, \ldots, N-1$ which we can visualize as sitting on the circle (that is point n has neighbours $n-1$ mod N and $n+1$ mod N). Consider a shift operator T, which acts on the functions as:

$$Tf(x) = f(x+1),$$

where the operation of addition is taken mod N. We propose to study the following selfadjoint operator as the candidate for the Dirac,

$$D = \frac{1}{2i}(T - T^*).$$

To argue that the above operator could be naturally taken as the Dirac operator let us check the distance, which comes from the Connes distance formula.

Lemma 2. *For the above operator D on the algebra of functions on N points the Connes distance between neigbouring points is* 1.

Proof. We have that the distance:

$$d(n, n+1) = \sup_{\|[D,f]\| \leq 1} |f(n) - f(n+1)|.$$

First of all, let us compute the commutator $[D, f]$ for an arbitrary f as an operator acting on a function Ψ

$$([D, f]\Psi)(n) = \frac{1}{2i}\big((f(n+1) - f(n))\Psi(n+1) + (f(n) - f(n-1))\Psi(n-1)\big).$$

Now, this operator is a sum of a two components:

$$[D, f]\Psi = \frac{1}{2i}\left((\partial f)T + T^*(\partial f)\right)\Psi,$$

where $(\partial f)(n) = f(n+1) - f(n)$. Since T is unitary then it is easy to see:

$$\|[D, f]\| \leq \sup_n (\partial f)(n),$$

and therefore:

$$d(n, n+1) \geq 1.$$

On the other hand, if $|f(n+1) - f(n)| > 1$ for sume function f then the maximum norm of the matrix $[D, f]$ is bigger than 1 and consequently, $\|[D, f]\| > 1$, hence such function f cannot be in the domain of functions that are taken to measure the distance. □

Since independently of N the length of each smallest interval is 1, to have a correct rescaling of the circle (independently of N we need to assume that it is the same circle of radius 1) we must take as the Dirac operator not D alone but:

$$D_N = \frac{N}{2\pi} D.$$

The spectrum of such Dirac operator is:

$$\operatorname{spec} D_N = \left\{ \frac{N}{2\pi} \sin\left(\frac{2k}{N}\pi\right) \right\}, \qquad k = 0, 1, \ldots, N-1,$$

and it is easy to see that in the limit $N \to \infty$ we have that the lowest eigenvalues grow linearly with k as in the case of the usual Dirac operator on the circle.

3.3. Conformally rescaled fuzzy circle

Now, we shall use the same procedure as in the case of the circle and rescale the Dirac operator on the fuzzy circle by a respective family of functions. We define the function h so that it is an approximation of the rescaling we used,

$$h(n) = \sqrt{1 - r\cos\left(2\pi \frac{n}{N}\right)},$$

and we take

$$D_{r,N} = hD_N h.$$

Now, the distribution of eigenvalues is completely different from the continuous case, yet for small values it still is linear (Fig. 2):

To see that we indeed approximate the noncompact line, let us see the comparison of scaling of distances with r between the adjacent points for the fuzzy circle with similar scaling for the intervals on the conformally rescaled circle.

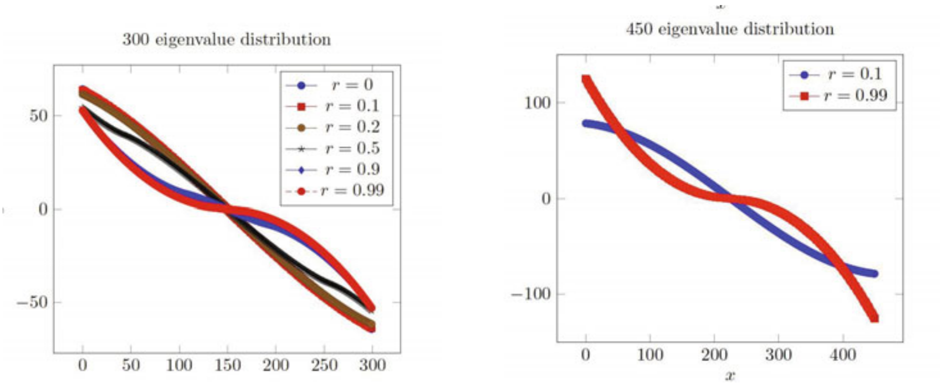


FIGURE 2. Eigenvalue distribution of $D_{r,N}$, for $N = 300$ and $N = 450$ case.

3.4. Conclusions and outlook

We have shown that using the conformal rescaling of the Dirac operator it is possible not only to compactify some of the locally compact commutative and noncomutative geometries but also do the inverse. In particular, all finite approximations of geometries by functions on lattices or matrix algebras could be used to introduce families of Dirac operators that in the limit give locally compact geometries.

The presented toy model of a circle and its decompatified fuzzy version is just a test-ground for further studies. One of the most used noncommutative geometries are fuzzy spheres [6]. Since the classical sphere could be modeled as a one point compactification of a plane with the Fubini–Study metric, we could reverse the process and describe the plane as a sphere without one point with a certain degenerate metric.

If the fuzzy sphere is generated by three matrices X, Y, Z, such that $X^2 + Y^2 + Z^2 = \text{const}$, then we suggest that the following family of operators:

$$D_r = (\sqrt{1 - rZ})D(\sqrt{1 - rZ}),$$

approximates the flat Dirac operator on the plane (which is obtained in the $r \to 1$ limit).

We conjecture that similarly as the fuzzy sphere converges in the Gromov–Rieffel–Hausdorff distance to the classical sphere [7], the conformally modified fuzzy spheres will converge to the respective conformal modification of the sphere and in the $r = 1$ limit to the flat plane.

References

[1] T. Brzeziński, N. Ciccola, L. Dąbrowski, A. Sitarz, *Twisted reality condition for Dirac operators*, Math Phys Anal Geom **19** (2016), 16

[2] A. Connes, *Noncommutative geometry and physics*, Academic Press, Inc, San Diego (1994)

[3] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. **36** (1995), 6194–6231.

[4] A. Connes, *Gravity coupled with matter and foundation of non-commutative geometry*, Commun. Math. Phys. **182** (1996), 155–176.

[5] M. Eckstein, A. Sitarz, R. Wulkenhaar, *The Moyal Sphere*, Journal of Mathematical Physics **57**, 112301 (2016)

[6] J. Madore, *The fuzzy sphere*, Class. Quantum Grav. **9**, (1992), 69–87

[7] M. Rieffel, *Gromov–Hausdorff distance for quantum metric spaces. Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance*, Mem. Am. Math. Soc. *168*, (796) (2004)

[8] A. Sitarz, *Conformally rescaled noncommutative geometries*, in: Geometric Methods in Physics, Trends in Mathematics, Springer 2015

Andrzej Sitarz
Institute of Physics
Jagiellonian University
prof. Stanislawa Lojasiewicza 11
PL-30-348 Kraków, Poland

and

Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8
PL-00-950 Warszawa, Poland
e-mail: andrzej.sitarz@uj.edu.pl

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 103–109

Dirac Operator on a Noncommutative Toeplitz Torus

Fredy Díaz García and Elmar Wagner

Abstract. We construct a 1^+-summable regular even spectral triple for a noncommutative torus defined by a C*-subalgebra of the Toeplitz algebra.

Mathematics Subject Classification (2010). Primary 58B34; Secondary 46L87.

Keywords. Dirac operator, noncommutative torus, spectral triple, Toeplitz algebra.

1. Introduction

In noncommutative geometry [1], a noncommutative topological space is presented by a noncommutative C*-algebra. Usually definitions of such C*-algebras are motivated by imitating some features of the classical spaces. For instance, a noncommutative version of any compact two-dimensional surface without boundary can be found in [7], where the corresponding C*-algebras are defined as subalgebras of the Toeplitz algebra.

The metric aspects of a noncommutative space are captured by the notation of a spectral triple [2]. Given a unital C*-algebra A, a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ for A consists of a dense *-subalgebra $\mathcal{A} \subset A$, a Hilbert space $\mathcal{H}$ together with a faithful *-representation $\pi : \mathcal{A} \to B(\mathcal{H})$, and a self-adjoint operator D on $\mathcal{H}$, called *Dirac operator*, such that

$$[D, \pi(a)] \in B(\mathcal{H}) \quad \text{for all} \quad a \in \mathcal{A}, \tag{1}$$

$$(D + \mathrm{i})^{-1} \in K(\mathcal{H}). \tag{2}$$

Here $K(\mathcal{H})$ denotes the set of compact operators on $\mathcal{H}$.

The purpose of the present paper is the construction of a spectral triple for the noncommutative torus from [7]. The noncommutative torus was chosen because the self-adjoint operator D from the spectral triple has a similar structure to the Dirac operator on a classical torus with a flat metric. Our main theorem shows that this spectral triple is even, regular, and 1^+-summable.

For the convenience of the reader, we recall the definitions of the just mentioned properties of a spectral triple (see [4]). By a slight abuse of notation, we will not distinguish between a densely defined closable operator and its closure. A spectral triple $(\mathcal{A},\mathcal{H},D)$ is said to be *even*, if there exists a grading operator $\gamma\in B(\mathcal{H})$ satisfying

$$\gamma^*=\gamma,\quad \gamma^2=1,\quad \gamma D=-D\gamma,\quad \gamma\pi(a)=\pi(a)\gamma \text{ for all } a\in\mathcal{A}. \tag{3}$$

We call $(\mathcal{A},\mathcal{H},D)$ *regular*, if $\delta^k(a)\in B(\mathcal{H})$ and $\delta^k([D,a])\in B(\mathcal{H})$ for all $a\in\mathcal{A}$ and $k\in\mathbb{N}$, where $\delta(x):=[|D|,x]$ for $x\in B(\mathcal{H})$. The term 1^+-*summable* means that $(1+|D|)^{-(1+\epsilon)}$ is a trace class operator for all $\epsilon>0$ but $(1+|D|)^{-1}$ is not a trace class operator.

Consider the polar decomposition $D=F|D|$ of the Dirac operator. The grading operator γ gives rise to a decomposition $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ such that $\gamma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $F=\begin{pmatrix}0&F_{+-}\\F_{-+}&0\end{pmatrix}$. If the spectral triple satisfies the properties of the previous paragraph, then F_{+-} and F_{-+} are Fredholm operators and one defines $\operatorname{ind}(D):=\operatorname{ind}(F_{+-})$. The operator F is called the fundamental class of D and it is said to be non-trivial if $\operatorname{ind}(D)\neq 0$.

2. Noncommutative Toeplitz torus

Let $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ be the open unit disc and $\bar{\mathbb{D}}:=\{z\in\mathbb{C}:|z|\le 1\}$ its closure in $\mathbb{C}$. Consider the Hilbert space $L_2(\mathbb{D})$ with respect to the standard Lebesgue measure and its closed subspace $A_2(\mathbb{D})$ consisting of all L_2-functions which are holomorphic in $\mathbb{D}$. We denote by P the orthogonal projection from $L_2(\mathbb{D})$ onto $A_2(\mathbb{D})$. For all $f\in C(\bar{\mathbb{D}})$, the Toeplitz operator $T_f\in B(A_2(\mathbb{D}))$ is defined by

$$T_f(\psi):=P(f\psi),\qquad \psi\in A_2(\mathbb{D})\subset L_2(\mathbb{D}),$$

and the Toeplitz algebra $\mathcal{T}$ is the C*-algebra generated by all T_f in $B(A_2(\mathbb{D}))$.

It is well known (see, e.g., [6]) that the compact operators $K(A_2(\mathbb{D}))$ belong to $\mathcal{T}$ and that the quotient $\mathcal{T}/K(A_2(\mathbb{D}))\cong C(\mathbb{S}^1)$ gives rise to the C*-algebra extension

$$0\longrightarrow K(A_2(\mathbb{D}))\longrightarrow\mathcal{T}\xrightarrow{\sigma}C(\mathbb{S}^1)\longrightarrow 0, \tag{4}$$

where $\sigma:\mathcal{T}\longrightarrow C(\mathbb{S}^1)$ is given by $\sigma(T_f)=f\restriction_{\mathbb{S}^1}$ for all $f\in C(\bar{\mathbb{D}})$.

There are alternative descriptions for the Toeplitz algebra. For instance, consider the Hilbert space $L_2(\mathbb{S}^1)$ with respect to the Lebesgue measure on $\mathbb{S}^1$ and the orthonormal basis $\{\frac{1}{\sqrt{2\pi}}u^k:k\in\mathbb{Z}\}$, where $u\in C(\mathbb{S}^1)\subset L_2(\mathbb{S}^1)$ is the unitary function given by $u(\zeta)=\zeta,\ \zeta\in\mathbb{S}^1$. Let P_+ denote the orthogonal projection from $L_2(\mathbb{S}^1)$ onto $\overline{\operatorname{span}}\{u^n:n\in\mathbb{N}\}\cong\ell_2(\mathbb{N})$. For all $f\in C(\mathbb{S}^1)$, define $\hat{T}_f\in B(\ell_2(\mathbb{N}))$ by

$$\hat{T}_f(\phi):=P_+(f\phi),\qquad \phi\in\overline{\operatorname{span}}\{u^n:n\in\mathbb{N}\}\subset L_2(\mathbb{S}^1). \tag{5}$$

Then $\mathcal{T}$ is isomorphic to the C*-subalgebra of $B(\ell_2(\mathbb{N}))$ generated by the set $\{\hat{T}_f : f \in C(\mathbb{S}^1)\}$, and the C*-algebra extension (4) becomes

$$0 \longrightarrow K(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\ \sigma\ } C(\mathbb{S}^1) \longrightarrow 0 \tag{6}$$

with $\sigma(\hat{T}_f) = f$.

Let us also mention that $\mathcal{T}$ may be considered as a deformation of the C*-algebra of continuous functions on the closed unit disc $\bar{\mathbb{D}}$ (see [5]). From this point of view, the equivalent C*-algebra extensions (4) and (6) correspond to the exact sequence

$$0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C(\bar{\mathbb{D}}) \xrightarrow{\ \tau\ } C(\mathbb{S}^1) \longrightarrow 0\,, \tag{7}$$

where $\tau(f) = f\!\restriction_{\mathbb{S}^1}$.

Recall that the torus $\mathbb{T}^2$ can be constructed as a topological manifold by dividing the boundary $\mathbb{S}^1 = \partial\bar{\mathbb{D}}$ into four quadrants and gluing opposite edges together. Then the C*-algebra of continuous functions on $\mathbb{T}^2$ is isomorphic to

$$C(\mathbb{T}^2) := \{f \in C(\bar{\mathbb{D}}) : f(\mathrm{e}^{\mathrm{i}t}) = f(-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}),\ f(\mathrm{e}^{-\mathrm{i}t}) = f(\mathrm{i}\mathrm{e}^{\mathrm{i}t}),\ t \in [0, \tfrac{\pi}{2}]\}. \tag{8}$$

Motivated by (8) and the analogy between (7) and (4) (or (6)), we state the following definition of the noncommutative Toeplitz torus:

Definition 1. The C*-algebra of the noncommutative Toeplitz torus is defined by

$$C(\mathbb{T}^2_q) := \{a \in \mathcal{T} : \sigma(a)(\mathrm{e}^{\mathrm{i}t}) = \sigma(a)(-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}),\ \sigma(a)(\mathrm{e}^{-\mathrm{i}t}) = \sigma(a)(\mathrm{i}\mathrm{e}^{\mathrm{i}t}),\ t \in [0, \tfrac{\pi}{2}]\}.$$

That $C(\mathbb{T}^2_q)$ is a C*-subalgebra of $\mathcal{T}$ follows from the fact that σ is a C*-algebra homomorphism. Note that gluing the point $\mathrm{e}^{\mathrm{i}t} \in \mathbb{S}^1$ to $-\mathrm{i}\mathrm{e}^{-\mathrm{i}t} \in \mathbb{S}^1$ and the point $\mathrm{e}^{-\mathrm{i}t} \in \mathbb{S}^1$ to $\mathrm{i}\mathrm{e}^{\mathrm{i}t} \in \mathbb{S}^1$ for all $t \in [0, \frac{\pi}{2}]$ yields a topological space homeomorphic to the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^1$ of two pointed circles. Setting

$$C(\mathbb{S}^1 \vee \mathbb{S}^1) := \{f \in C(\mathbb{S}^1) : f(\mathrm{e}^{\mathrm{i}t}) = f(-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}),\ f(\mathrm{e}^{-\mathrm{i}t}) = f(\mathrm{i}\mathrm{e}^{\mathrm{i}t}),\ t \in [0, \tfrac{\pi}{2}]\}, \tag{9}$$

we can write

$$C(\mathbb{T}^2_q) = \{a \in \mathcal{T} : \sigma(a) \in C(\mathbb{S}^1 \vee \mathbb{S}^1)\}. \tag{10}$$

Moreover, (6) and (10) yield the C*-algebra extension

$$0 \longrightarrow K(\ell_2(\mathbb{N})) \longrightarrow C(\mathbb{T}^2_q) \xrightarrow{\ \sigma\ } C(\mathbb{S}^1 \vee \mathbb{S}^1) \longrightarrow 0\,.$$

3. Spectral triple on the noncommutative Toeplitz torus

The Dirac operator on a local chart in two dimensions with the flat metric, see [3], up to constant and change of orientation is given by

$$D = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ -\frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix}, \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big). \tag{11}$$

Since $\frac{\partial}{\partial z}$ acts on $A_2(\mathbb{D})$ in the obvious way, we want to use the same structure to define a spectral triple for the noncommutative Toeplitz torus. Clearly, one can construct a noncommutative version of any (orientable) compact surface without boundary by choosing appropriate boundary conditions in Definition 1 (see [7]). However, by the Gauss–Bonnet theorem, only the (classical) torus admits a dense local chart with a flat metric, therefore we restrict here our discussion to the quantum analogue of the torus.

Our principal aim is to find a dense *-subalgebra $\mathcal{A} \subset C(\mathbb{T}^2_q)$ and an operator ∂_z, which should be closely related to $\frac{\partial}{\partial z}$ from (11), such that $[\partial_z, a]$ is bounded for all $a \in \mathcal{A}$. Recall that an orthonormal basis for $A_2(\mathbb{D})$ is given by $\{\varphi_n : n \in \mathbb{N}\}$, where $\varphi_n := \frac{\sqrt{n+1}}{\sqrt{\pi}} z^n$ [6]. Complex differentiation yields $\frac{\partial}{\partial z}(\varphi_n) = \sqrt{n(n+1)}\,\varphi_{n-1}$. If we define an operator ∂_z on $A_2(\mathbb{D})$ by $\partial_z(\varphi_n) := n\varphi_{n-1}$, then $\frac{\partial}{\partial z} - \partial_z$ extends to a bounded operator on $A_2(\mathbb{D})$ since the coefficients $\sqrt{n(n+1)} - n$ are uniformly bounded. As a consequence, the commutators $[\frac{\partial}{\partial z}, a]$ are bounded for all $a \in \mathcal{A}$ if and only if the commutators with ∂_z are bounded.

In order to simplify the notation, we will use the description of the Toeplitz algebra on $\ell_2(\mathbb{N}) \cong \overline{\mathrm{span}}\{u^n : n \in \mathbb{N}\} \subset L_2(\mathbb{S}^1)$. For $m \in \mathbb{Z}$, set $e_m := \frac{1}{\sqrt{2\pi}} u^m$ and let ∂_z be defined by

$$\partial_z(e_n) := n e_{n-1} \ \text{ on } \ \mathrm{dom}(\partial_z) := \Big\{ \sum_{n\in\mathbb{N}} \alpha_n e_n \in \ell_2(\mathbb{N}) : \sum_{n\in\mathbb{N}} n^2 |\alpha_n|^2 < \infty \Big\}. \tag{12}$$

Moreover, consider the number operator N on $\ell_2(\mathbb{N})$ determined by

$$N(e_n) := n e_n \quad \text{on} \quad \mathrm{dom}(N) := \mathrm{dom}(\partial_z). \tag{13}$$

Let S be the unilateral shift operator on $\ell_2(\mathbb{N})$ so that we have

$$S(e_n) = e_{n+1}, \quad n \in \mathbb{N}, \quad S^*(e_n) = e_{n-1}, \quad n > 1, \quad S^*(e_0) = 0. \tag{14}$$

Since N is a self-adjoint positive operator on $\mathrm{dom}(N) = \mathrm{dom}(\partial_z)$ and since S^* is a partial isometry such that $\ker(S^*) = \mathrm{Ran}(N)^\perp$, it follows that $\partial_z = S^* N$ is the polar decomposition of the closed operator ∂_z. Clearly, $\partial_z^* = NS$, so

$$\partial_z^*(e_n) = (n+1)\, e_{n+1} \quad \text{and} \quad \mathrm{dom}(\partial_z^*) = \mathrm{dom}(N). \tag{15}$$

Under the unitary isomorphism $A_2(\mathbb{D}) \cong \ell_2(\mathbb{N})$ given by $\varphi_n \mapsto e_n$ on the bases described above, the operator ∂_z on $\ell_2(\mathbb{N})$ is unitary equivalent to a bounded perturbation of the Cauchy–Riemann operator $\frac{\partial}{\partial z}$ on $A_2(\mathbb{D})$. Therefore we take ∂_z on $\ell_2(\mathbb{N})$ as a replacement for $\frac{\partial}{\partial z}$ on $A_2(\mathbb{D})$.

Note that, in the commutative case and with functions represented by multiplication operators, one has $[\frac{\partial}{\partial z}, f] = \frac{\partial f}{\partial z}$ for all $f \in C^{(1)}(\mathbb{D})$ but clearly not all continuous functions are differentiable. In the following, we will single out a dense *-subalgebra $\mathcal{A} \subset C(\mathbb{T}^2_q) \subset B(\ell_2(\mathbb{N}))$ which can be viewed as an algebra of infinitely differentiable functions. With $C(\mathbb{S}^1 \vee \mathbb{S}^1) \subset C(\mathbb{S}^1)$ defined in (9), set $C^\infty(\mathbb{S}^1 \vee \mathbb{S}^1) := C(\mathbb{S}^1 \vee \mathbb{S}^1) \cap C^\infty(\mathbb{S}^1)$ and let

$$\mathcal{A}_0 := \{\hat{T}_f : f \in C^\infty(\mathbb{S}^1 \vee \mathbb{S}^1)\} \subset C(\mathbb{T}^2_q).$$

Using the obvious embedding $\mathrm{End}(\mathrm{span}\{e_1,\dots,e_n\}) \subset K(\ell_2(\mathbb{N})) \subset C(\mathbb{T}_q^2)$, consider

$$\mathcal{F}_0 := \bigcup_{n\in\mathbb{N}} \mathrm{End}(\mathrm{span}\{e_1,\dots,e_n\}) \subset C(\mathbb{T}_q^2).$$

We will take $\mathcal{A}$ to be the *-subalgebra of $C(\mathbb{T}_q^2)$ generated by the elements of $\mathcal{A}_0$ and $\mathcal{F}_0$, i.e.,

$$\mathcal{A} := \text{*-alg}(\mathcal{A}_0 \cup \mathcal{F}_0) \subset C(\mathbb{T}_q^2). \tag{16}$$

Lemma 2. *The algebra $\mathcal{A}$ defined in* (16) *is dense in $C(\mathbb{T}_q^2)$ and its elements admit bounded commutators with ∂_z and ∂_z^*. Furthermore, $\delta_N^k(a)$, $\delta_N^k([\partial_z,a])$ and $\delta_N^k([\partial_z^*,a])$ are bounded for all $a\in\mathcal{A}$ and $k\in\mathbb{N}$, where $\delta_N(x) := [N,x]$ for $x\in B(\ell_2(\mathbb{N}))$.*

Proof. The set $\mathcal{F}_0$ contains all finite operators on $\mathrm{span}\{e_n : n\in\mathbb{N}\}$, therefore it is dense in $K(\ell_2(\mathbb{N}))$. As a consequence, all compact operators $K(\ell_2(\mathbb{N}))$ belong to the closure of $\mathcal{A}$. From (5), it follows that $\|\hat{T}_f\| \le \|f\|_\infty$. By the Stone–Weierstrass theorem, $C^\infty(\mathbb{S}^1\vee\mathbb{S}^1)$ is dense in $C(\mathbb{S}^1\vee\mathbb{S}^1)$ with respect to the norm $\|\cdot\|_\infty$. Thus each $\hat{T}_g \in C(\mathbb{T}_q^2)$ can be approximated by elements from $\mathcal{A}_0$. Let $a\in C(\mathbb{T}_q^2)$. Writing $a = a - \hat{T}_{\sigma(a)} + \hat{T}_{\sigma(a)}$, where $\hat{T}_{\sigma(a)} \in C(\mathbb{T}_q^2)$ and $a - \hat{T}_{\sigma(a)} \in K(\ell_2(\mathbb{N}))$, we conclude that a lies in the closure of $\mathcal{A}$, so $\mathcal{A}$ is dense in $C(\mathbb{T}_q^2)$.

By the Leibniz rule $[A,BC] = [A,B]C + B[A,C]$ for the commutator $[\cdot,\cdot]$, it suffices to prove the boundedness of the commutators for the elements belonging to the generating set $\mathcal{A}_0\cup\mathcal{F}_0$. From the definitions of $\mathcal{F}_0$ and N, it follows that $Na\in\mathcal{F}_0$ and $aN\in\mathcal{F}_0$ for all $a\in\mathcal{F}_0$. This immediately implies that $\delta_N^k(a)\in B(\ell_2(\mathbb{N}))$ for all $k\in\mathbb{N}$ since each term of the iterated commutators belongs to $\mathcal{F}_0 \subset B(\ell_2(\mathbb{N}))$. Note also that $aS^*\in\mathcal{F}_0$ and $S^*a\in\mathcal{F}_0$ for all $a\in\mathcal{F}_0$, therefore $[\partial_z,a] = S^*(Na) - (aS^*)N \in \mathcal{F}_0$. In particular, $[\partial_z,a]$ and $\delta_N^k([\partial_z,a])$ are bounded for all $k\in\mathbb{N}$.

Next consider $\hat{T}_f\in\mathcal{A}_0$. To determine the action of $\hat{T}_f$ on $\ell_2(\mathbb{N})$, we represent f by its Fourier series $f = \sum_{k\in\mathbb{Z}} \hat{f}(k)\,u^k$, where $\hat{f}(k)\in\mathbb{C}$. Since multiplication by u^k yields $u^k e_m = e_{m+k}$, one obtains from (5)

$$\hat{T}_f(e_m) = P_+\Big(\sum_{k\in\mathbb{Z}} \hat{f}(k)\,u^k\,e_m\Big) = P_+\Big(\sum_{k\in\mathbb{Z}} \hat{f}(k)\,e_{m+k}\Big) = \sum_{n\in\mathbb{N}} \hat{f}(n-m)\,e_n\,. \tag{17}$$

If $f\in C^\infty(\mathbb{S}^1)$, then partial integration shows that $f'\in C(\mathbb{S}^1)$ has the Fourier series $f' = \sum_{k\in\mathbb{Z}} \mathrm{i}k\hat{f}(k)\,u^k$. Therefore, for all $m\in\mathbb{N}$,

$$\begin{aligned}[N,\hat{T}_f](e_m) &= \sum_{n\in\mathbb{N}} n\hat{f}(n-m)\,e_n - \sum_{n\in\mathbb{N}} m\hat{f}(n-m)\,e_n = \sum_{n\in\mathbb{N}} (n-m)\hat{f}(n-m)\,e_n \\ &= -\mathrm{i}\,P_+\Big(\sum_{k\in\mathbb{Z}} \mathrm{i}(k-m)\hat{f}(k-m)\,e_k\Big) = -\mathrm{i}\hat{T}_{f'}(e_m)\end{aligned} \tag{18}$$

by (17) and the Fourier series of f'. Similarly,

$$\begin{aligned}[\partial_z, \hat{T}_f](e_m) &= \sum_{n\in\mathbb{N}} n\hat{f}(n-m)\,e_{n-1} - \sum_{n\in\mathbb{N}} m\hat{f}(n-(m-1))\,e_n \\ &= \sum_{n\in\mathbb{N}} (n-m+1)\hat{f}(n-m+1)\,e_n = -\mathrm{i}\,P_+\Big(\bar{u}\sum_{k\in\mathbb{Z}} \mathrm{i}(k-m)\hat{f}(k-m)\,e_k\Big) \\ &= -\mathrm{i}\hat{T}_{\bar{u}f'}(e_m). \end{aligned} \tag{19}$$

This yields $[\partial_z, \hat{T}_f] = -\mathrm{i}\hat{T}_{\bar{u}f'} \in B(\ell_2(\mathbb{N}))$, $\delta_N^k(\hat{T}_f) = (-\mathrm{i})^k\hat{T}_{f^{(k)}} \in B(\ell_2(\mathbb{N}))$, and $\delta_N^k([\partial_z, \hat{T}_f]) = (-\mathrm{i})^{k+1}\hat{T}_{(\bar{u}f')^{(k)}} \in B(\ell_2(\mathbb{N}))$, the latter because $\bar{u}f'$ is a C^∞-function. The statement for ∂_z^* can be proven analogously or by using $[\partial_z^*, a] = -[\partial_z, a^*]^*$ together with $a^* \in \mathcal{F}_0$ for all $a \in \mathcal{F}_0$ and $\hat{T}_f^* = \hat{T}_{\bar{f}}$ for all $f \in C(\mathbb{S}^1)$. □

Now we are in a position to construct our spectral triple and describe its fundamental properties.

Theorem 3. *Let $\mathcal{A}$ denote the dense *-subalgebra of $C(\mathbb{T}_q^2)$ from Lemma 2. Set $\mathcal{H} := \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$ and define a *-representation $\pi : \mathcal{A} \to B(\ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N}))$ by $\pi(a) := a \oplus a$. Consider the self-adjoint operator*

$$D := \begin{pmatrix} 0 & \partial_z \\ \partial_z^* & 0 \end{pmatrix} \quad \textit{on} \quad \mathrm{dom}(D) := \mathrm{dom}(N) \oplus \mathrm{dom}(N).$$

Then $(\mathcal{A}, \mathcal{H}, D)$ is a 1^+-summable regular even spectral triple for $C(\mathbb{T}_q^2)$ with grading operator $\gamma := \mathrm{id} \oplus (-\mathrm{id})$. The Dirac operator D has discrete spectrum $\mathrm{spec}(D) = \mathbb{Z}$, each eigenvalue $k \in \mathrm{spec}(D)$ has multiplicity 1, and a complete set of eigenvectors $\{b_k : k \in \mathbb{Z}\}$ satisfying $Db_k = k\,b_k$ is given by

$$b_k := \tfrac{1}{\sqrt{2}}(e_{k-1} \oplus e_k), \quad b_{-k} := \tfrac{1}{\sqrt{2}}(-e_{k-1} \oplus e_k), \quad k > 0, \qquad b_0 := 0 \oplus e_0.$$

Its fundamental class $F = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ is non-trivial and $\mathrm{ind}(D) = 1$.

Proof. We have already mentioned that the operator $\partial_z = S^*N$ is closed. Hence D is self-adjoint by its definition. Since $[D, \pi(a)]$ has $[\partial_z, a]$ and $[\partial_z^*, a]$ as its non-zero matrix entries, the boundedness of these commutators for all $a \in \mathcal{A}$ follows from Lemma 2. As $\partial_z = S^*N$ and $\partial_z^* = NS = S(N+1)$, the polar decomposition of D reads as

$$D = F\,|D| = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}\begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix}. \tag{20}$$

In particular, the entries of the commutators with $|D|$ are given by commutators with N, thus the regularity can easily be deduced from Lemma 2. Clearly, γ, D and $\pi(a)$ satisfy (3), so the spectral triple is even. From (12) and (15), it follows immediately that $D(b_k) = k\,b_k$ for all $k \in \mathbb{Z}$. Since $\{b_k : k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}$, we have $\mathrm{spec}(D) = \mathbb{Z}$ and each eigenvalue has multiplicity 1. The 1^+-summability follows from the convergence behavior of the infinite sum $\sum_{k\in\mathbb{Z}}(1 + |k|)^{-(1+\epsilon)}$, $\epsilon \geq 0$. Finally, by the polar decomposition given in (20), $\mathrm{ind}(D) = \mathrm{ind}(S^*) = 1$. □

Acknowledgment

This work was supported by CIC-UMSNH and is part of the international project “Quantum Dynamics” supported by EU grant H2020-MSCA-RISE-2015-691246 and co-financed by Polish Government grant W2/H2020/2016 awarded for the years 2016–2019.

References

[1] A. Connes, *Noncommutative geometry.* Academic Press, San Diego, 1994.

[2] A. Connes, *Noncommutative geometry and reality.* J. Math. Phys. **36** (1995), 6194–6231.

[3] T. Friedrich, *Dirac operators in Riemannian geometry.* American Mathematical Society, Providence, RI, 2000.

[4] J.M. Gracia-Bondía, H. Figueroa and J.C. Várilly, *Elements of Noncommutative Geometry.* Birkhäuser, Boston, 2001.

[5] S. Klimek and A. Lesniewski, *A two-parameter quantum deformation of the unit disc.* J. Funct. Anal. **115** (1993), 1–23.

[6] N.L. Vasilevski, *Commutative algebras of Toeplitz operators on the Bergman space.* Birkhäuser Verlag, Basel, 2008.

[7] E. Wagner, *Compact quantum surfaces of any genus*, in preparation.

Fredy Díaz García and Elmar Wagner
Instituto de Física y Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Cd. Universitaria, Edificio C-3
58040 Morelia, Michoacán, México
e-mail: lenonndiaz@gmail.com
elmar@ifm.umich.mx

Part III

Quantization

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 113–118

Field Quantization in the Presence of External Fields

Fardin Kheirandish

Abstract. By quantizing a general field theory in the presence of anisotropic media, a general formula for fluctuation-induced free energy is obtained.

Mathematics Subject Classification (2010). 81S05; 81S40; 81T55; 81T28; 82B10.

Keywords. Fluctuation-induced force, field quantization, anisotropic matter, partition function.

1. Introduction

Since the seminal paper of Casimir [1] on fluctuation induced force between two parallel plates made of perfect conductors due to vacuum fluctuations of electromagnetic field and its generalization to the case of dielectric slabs by Lifshitz [2], an extensive work has been done on fluctuation-induced forces [3–12].

Starting from a Lagrangian, we derive a general formula for fluctuation-induced free energy for two separate anisotropic material objects interacting linearly with a general fluctuating field.

2. Model

Suppose A_1 and A_2 are two separate pieces of anisotropic matter interacting linearly with a general fluctuating field through coupling tensors $g^{(1)}_{ij}$ and $g^{(2)}_{ij}$ $(i,j=1,2,3)$, respectively, see Figure 1. The total Lagrangian can be described by

$$\mathcal{L} = -\frac{1}{2}\mathbf{F}\cdot\hat{\mathbf{o}}\cdot\mathbf{F} - \frac{1}{2}\int_0^\infty d\nu\,\mathbf{X}_\nu\cdot(\partial_t^2+\nu^2)\,\mathbf{X}_\nu + \int_0^\infty d\nu\,\mathbf{F}\cdot\bar{\mathbf{g}}\cdot\partial_t\mathbf{X}_\nu, \tag{1}$$

where $\mathbf{F}(\mathbf{r},t)$ describes the fluctuating field and $\mathbf{X}_\nu(\mathbf{r},t)$ describes anisotropic material. The coupling strength defined by the coupling tensor $\bar{\mathbf{g}}(\nu,\mathbf{r})$, exists only inside

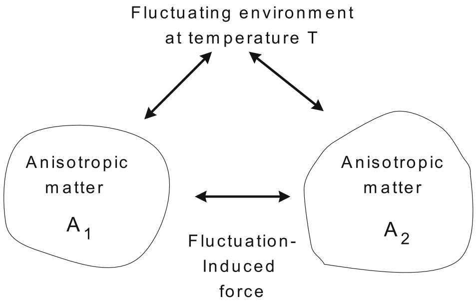


FIGURE 1. Regions A_1 and A_2 contain anisotropic matter interacting linearly with the fluctuating environment described by general field $\mathbf{F}$.

the regions A_1 and A_2, occupied with an homogeneous but anisotropic matter

$$\mathbf{g}_{ij}(\nu, \mathbf{r}) = \begin{cases} g_{ij}^{(1)}(\nu), & \mathbf{r} \in A_1; \\ g_{ij}^{(2)}(\nu), & \mathbf{r} \in A_2; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For example when the fluctuating field is electromagnetic vacuum field, then $\hat{\mathbf{o}} = \partial_t^2 + \nabla \times \nabla \times$, [11, 12]. The fluctuating field can be assumed as a scalar, vector, tensor or a spinor field interacting linearly with material fields and the only modification will be a rearrangement of indices on fields and coupling functions in the total Lagrangian density. Here, the anisotropic matter is modeled by bosonic fields as a continuum of quantum harmonic oscillators [13–18], but it can also be modeled by fermionic fields like metallic objects, the form of the final results does not depend on these details and differences between media are included in response tensors. From Heisenberg equations of motion, we find the following equations for fluctuating and material fields, respectively

$$\hat{\mathbf{o}} \cdot \mathbf{F} = \int_0^\infty d\nu \, \bar{\mathbf{g}}(\nu, \mathbf{r}) \cdot \partial_t \mathbf{X}_\nu, \tag{3}$$

$$(\partial_t^2 + \nu^2) \, \mathbf{X}_\nu = -\bar{\mathbf{g}}(\nu, \mathbf{r}) \cdot \partial_t \mathbf{F}. \tag{4}$$

Equation (4) can be solved formally as

$$\mathbf{X}_\nu = \mathbf{X}_\nu^{(n)} - \int_0^t dt' \, G_\nu(t - t') \bar{\mathbf{g}} \cdot \partial_{t'} \mathbf{F}, \tag{5}$$

where $G_\nu(t - t')$ is the retarded Green function that can be expressed in terms of the Heaviside step function as

$$G_\nu(t - t') = \Theta(t - t') \, \frac{\sin \nu(t - t')}{\nu}, \tag{6}$$

and $\mathbf{X}_\nu^{(n)}$, is the homogeneous solution $(\partial_t^2 + \nu^2) \, \mathbf{X}_\nu^{(n)} = 0$, or material quantum noise field. By inserting the solution (5) into (3), we find the quantum Langevin equation for the fluctuating field in the presence of material fields

$$\hat{\mathbf{o}} \cdot \mathbf{F} + \partial_t \int_0^t dt' \, \bar{\boldsymbol{\chi}}(\mathbf{r}, t - t') \cdot \partial_{t'} \mathbf{F} = \int_0^\infty d\nu \, \bar{\mathbf{g}} \cdot \partial_t \mathbf{X}_\nu^{(n)}, \tag{7}$$

where the response or memory tensor is defined by

$$\bar{\boldsymbol{\chi}}(\mathbf{r}, t - t') = \int_0^\infty d\nu \, \bar{\mathbf{g}} \cdot \bar{\mathbf{g}} \, G_\nu(t - t'). \tag{8}$$

For notational simplicity, in Eq. (8) we have assumed that the coupling tensors are symmetric $\bar{\mathbf{g}} = \bar{\mathbf{g}}^t$ [18], that is the imaginary part of the susceptibility tensor is symmetric, see Eq. (9). One can proceed without this assumption and consider $\bar{\mathbf{g}} \cdot \bar{\mathbf{g}}^t$ instead of $\bar{\mathbf{g}} \cdot \bar{\mathbf{g}}$. Equation (8) is a sine transform and its inverse leads to the following relation between coupling and memory tensor in the frequency space

$$\bar{\mathbf{g}}(\nu, \mathbf{r}) = \sqrt{\frac{2\nu}{\pi} \, \mathrm{Im}[\bar{\boldsymbol{\chi}}(\mathbf{r}, \nu)]}. \tag{9}$$

Therefore, if we are given a definite response tensor, we can adjust the coupling tensor according to Eq. (9).

3. Partition function

To find the partition function, we first switch to the Euclidean Lagrangian $\mathcal{L}_E$, obtained by a Wick rotation on time coordinate $it = \tau, (\partial_t = i\partial_\tau)$, which implies

$$\hat{\mathbf{o}}(\partial_t^2, \partial_i) \to \hat{\mathbf{o}}'(-\partial_\tau^2, \partial_i), \tag{10}$$

and all fields are now functions of $(\mathbf{r}, \tau)$. The total partition function is defined by [11]

$$\begin{aligned} Z = \int \prod_{\nu \geq 0} D[\mathbf{X}_\nu] D[\mathbf{F}] \, & e^{-\frac{1}{2} \int d\mathbf{r} \int_0^\beta d\tau [\mathbf{F} \cdot \hat{\mathbf{o}}' \cdot \mathbf{F} + \mathbf{F} \cdot \mathbf{J}]} \\ & \times e^{-\frac{1}{2} \int d\mathbf{r} \int_0^\beta d\tau \int_0^\infty d\nu \, \mathbf{X}_\nu \cdot (-\partial_\tau^2 + \nu^2) \, \mathbf{X}_\nu}, \end{aligned} \tag{11}$$

where $\beta = 1/k_B T$, k_B is the Boltzmann constant and T is the temperature of the fluctuating medium described by the field $\mathbf{F}$. The source term $\mathbf{J}$ in Eq. (11) is defined by

$$\mathbf{J} = i \int_0^\infty d\nu \, \bar{\mathbf{g}} \cdot \partial_\tau \mathbf{X}_\nu. \tag{12}$$

To find the partition function, periodic boundary conditions on bosonic fields are imposed

$$
\begin{aligned}
\mathbf{F}(\mathbf{r},\tau) = \mathbf{F}(\mathbf{r},\tau+\beta) \;&= \sum_{n=0}^{\infty\,\prime} [\mathbf{F}_n(\mathbf{r}) e^{-i\omega_n \tau} + c.c.], \\
\mathbf{X}_\nu(\mathbf{r},\tau) = \mathbf{X}_\nu(\mathbf{r},\tau+\beta) &= \sum_{n=0}^{\infty\,\prime} [\mathbf{X}_{\nu,n}(\mathbf{r}) e^{-i\omega_n \tau} + c.c.],
\end{aligned} \tag{13}
$$

where $\omega_n = 2\pi n/\beta$ are Matsubara frequencies and the prime over the summation means the term corresponding to $n = 0$, should be given half weight. Note that for fermionic fields antiperiodic boundary conditions should be applied. Inserting Eqs. (13) into Eq. (11), we find

$$
\begin{aligned}
Z = \int \prod_{n,\nu \geq 0} D[\mathbf{X}_{\nu,n}] D[\mathbf{X}^*_{\nu,n}] \prod_{n \geq 0} D[\mathbf{F}_n] D[\mathbf{F}^*_n] \\
\times\, e^{-\frac{1}{2}\int d\mathbf{r} \sum_{n=0}^{\infty\,\prime} (\mathbf{F}_n \cdot \beta \hat{\mathbf{o}}_n \cdot \mathbf{F}^*_n + \mathbf{F}^*_n \cdot \beta \hat{\mathbf{o}}_n \cdot \mathbf{F}_n + \mathbf{F}_n \cdot \mathbf{J}^*_n + \mathbf{F}^*_n \cdot \mathbf{J}_n)} \\
\times\, e^{-\frac{1}{2}\int d\mathbf{r} \int_0^\infty d\nu\, (\mathbf{X}^*_{\nu,n} \cdot \beta(\omega_n^2+\nu^2)\, \mathbf{X}_{\nu,n} + \mathbf{X}_{\nu,n} \cdot \beta(\omega_n^2+\nu^2)\, \mathbf{X}^*_{\nu,n})},
\end{aligned} \tag{14}
$$

where for convenience we have defined $\hat{\mathbf{o}}_n = \hat{\mathbf{o}}'(\omega_n^2, \partial_i)$. By making use of the well-known formula [19]

$$
\int D[\varphi] D[\varphi^*]\, e^{-\int d\mathbf{r}\,(\varphi^* \hat{A} \varphi + \varphi \hat{A} \varphi^* + \rho \varphi^* + \varrho^* \varphi)} = (\det \hat{A})^{-1}\, e^{\int d\mathbf{r}\, \rho^* \hat{A}^{-1} \rho}, \tag{15}
$$

we can integrate over fluctuating field and material degrees of freedom and find the total partition function as

$$
\begin{aligned}
Z = \underbrace{\prod_{n \geq 0}^{\infty\,\prime} (\det[\beta \hat{\mathbf{o}}_n])^{-1}}_{Z_F} \underbrace{\prod_{n \geq 0}^{\infty\,\prime} \prod_{\nu \geq 0}^{\infty} (\det[\beta(\omega_n^2 + \nu^2)])^{-1}}_{Z_m} \\
\times \underbrace{\prod_{n \geq 0}^{\infty\,\prime} \prod_{\nu \geq 0}^{\infty} (\det[1 + \omega_n^2\, G_\nu(\omega_n)\, \bar{\mathbf{g}} \cdot \mathbf{G}_0 \cdot \bar{\mathbf{g}}])^{-1}}_{Z_{\text{eff}}},
\end{aligned} \tag{16}
$$

where $\mathbf{G}_0$ is the dyadic Green function of the fluctuating field in free space $\hat{\mathbf{o}}_n \cdot \mathbf{G}_0 = \mathbb{I}$. In Eq. (16), the first product term is the partition function of the fluctuating field (Z_F), the second product term is the partition function of the material field (Z_m) and the last term which is the relevant term for our purposes, originates from interaction between the fluctuating field and material field (Z_{eff}). Using the identity $\ln[\det \hat{O}] = \mathrm{Tr} \ln[\hat{O}]$, and definition of the relevant or effective free energy $F_{\text{eff}} = -k_B T \ln Z_{\text{eff}}$, we find

$$F_{\text{eff}} = k_B T \sum_{n\geq 0}^{\infty}{}' \operatorname{Tr} \ln[1 + \omega_n^2\, G_\nu(\omega_n)\, \bar{\mathbf{g}} \cdot \mathbf{G}_0 \cdot \bar{\mathbf{g}}]. \tag{17}$$

By making use of the expansion

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}, \tag{18}$$

and Fourier transform of the memory or response tensor, Eq. (8)

$$\chi_{ij}(\mathbf{r}, \omega) = \int_0^\infty d\nu\, \frac{\mathbf{g}_{ik}(\nu, \mathbf{r})\, \mathbf{g}_{kj}(\nu, \mathbf{r})}{\omega^2 + \nu^2}, \tag{19}$$

we find the free energy in terms of the response tensor as

$$F_{\text{eff}} = k_B T \sum_{n=0}^{\infty}{}' \operatorname{Tr}_{|i,\mathbf{r}\rangle}\, \ln[1 + \bar{\boldsymbol{\chi}}(i\omega_n) \cdot \bar{\mathbf{G}}_0(i\omega_n)], \tag{20}$$

where $\operatorname{Tr}_{|i,\mathbf{r}\rangle}$, means taking trace over position and internal degrees of freedom ($i = 1, 2, 3$). Equation (20) is the elegant formula for the fluctuation-induced free energy obtained for a general fluctuating field interacting linearly with anisotropic media in finite temperature. By making use of the expansion (18) for the logarithm, we obtain the following series in the susceptibility tensor

$$F_{\text{eff}} = k_B T \sum_{n=0}^{\infty}{}' \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \operatorname{Tr}_{|i,\mathbf{r}\rangle} \left(\bar{\boldsymbol{\chi}}(i\omega_n) \cdot \bar{\mathbf{G}}_0(i\omega_n)\right)^m, \tag{21}$$

which is a generalization of the result reported in [11, 20] for the case of the electromagnetic field in the presence of isotropic matter.

For the fermionic material fields we find the same formula and the details of the material properties are included in the response tensor $\bar{\boldsymbol{\chi}}$ of the medium. In zero temperature, using the correspondence

$$\int_0^\infty \frac{d\zeta}{2\pi} \leftrightarrow k_B T \sum_{n=0}^{\infty}{}', \tag{22}$$

we find

$$F = \int_0^\infty \frac{d\zeta}{2\pi} \operatorname{Tr}_{|i,\mathbf{r}\rangle} \left[\ln(1 + \bar{\boldsymbol{\chi}}(i\zeta) \cdot \bar{\mathbf{G}}_0(i\zeta))\right]. \tag{23}$$

References

[1] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

[2] E.M. Lifshitz, Sov. Phys. JETP **2**, 73 (1956); I.E. Dzyaloshinskii, E.M. Lifshitz, and L.P. Pitaevskii, Adv. Phys. **10**, 165 (1961).

[3] J.N. Israelachvili, *Intermolecular and Surface Forces* (Academic, London, 1992).

[4] M. Kardar and R. Golestanian, Rev. Mod. Phys. **71**, 1233 (1999).

[5] M. Bordag, U. Mohideen, V.M. Mostepanenko, Phys. Rep. **353**, 1 (2001).

[6] K.A. Milton, The Casimir Effect: Physical Manifestations of Zero-Point Energy (World Scientific, Singapore, 2001).

[7] M. Bordag, G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, Advances in the Casimir Effect (Oxford university press, 2008).

[8] F. Kheirandish, E. Amooghorban and M. Soltani, Phys. Rev. A **83**, 032507 (2011).

[9] F. Kheirandish, M. Soltani, J. Sarabadani, Annals of Physics, **326**, 657–667 (2011).

[10] E. Amooghorban, M. Wubs, A. Mortensen and F. Kheirandish, Phys. Rev. A **84**, 013806 (2011).

[11] F. Kheirandish and S. Salimi, Phys. Rev. A **84**, 062122 (2011).

[12] F. Kheirandish and M. Jafari, Phys. Rev. A **86**, 022503 (2012).

[13] J.J. Hopfield, Phys. Rev. 112, 1555 (1958).

[14] B. Huttner and S.M. Barnett, Phys. Rev. A 46, 4306 (1992); B. Huttner and S. Barnett, Europhys. Lett. 18, 487 (1992); S. Barnett, R. Matloob, and R. Loudon, J. Mod. Opt. 42, 1165 (1995).

[15] F. Kheirandish and M. Amooshahi, Phys. Rev. A **74**, 042102 (2006); M. Amooshahi and F. Kheirandish, ibid. **76**, 062103 (2007).

[16] F. Kheirandish, M. Amooshahi, and M. Soltani, J. Phys. B: Mol. Opt. Phys. 42, 075504 (2009).

[17] F. Kheirandish and M. Soltani, Phys. Rev. A**78**, 012102 (2008).

[18] F. Kheirandish and M. Amooshahi, J. Phys. A: Math. Theor. **41**, 275402 (2008).

[19] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 2nd edn. (Oxford, Oxford University Press, 1993).

[20] R. Golestanian, Phys. Rev. Lett. **95**, 230601 (2005).

Fardin Kheirandish
Department of Physics, Faculty of Science
University of Kurdistan
P.O.Box 66177-15175
Sanandaj, Iran
e-mail: f.kheirandish@uok.ac.ir

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 119–129

Quantization of Mathematical Theory of Non-Smooth Strings

A.G. Sergeev

Abstract. The mathematical problem of quantization of the theory of smooth strings consists of quantization of the space Ω_d of smooth loops taking values in the d-dimensional Minkowski space R^d. The latter problem can be solved in frames of the standard Dirac approach. However, a natural symplectic form on Ω_d may be extended to the Hilbert completion of Ω_d coinciding with the Sobolev space $V_d := H_0^{1/2}(S^1, R^d)$ of half-differentiable loops with values in R^d. So it is reasonable to consider V_d as the phase space of non-smooth string theory and try to quantize it. We explain how to do it using ideas from noncommutative geometry.

Mathematics Subject Classification (2010). Primary 30C62,81T75.

Keywords. Loop spaces, quantization, Sobolev space of half-differentiable functions, quasisymmetric homeomorphisms.

The problem of quantization of the theory of smooth strings mathematically is reduced to the quantization of the space Ω_d of smooth loops taking values in the d-dimensional Minkowski space. Here Ω_d, provided with a natural symplectic form, is treated as the phase space of smooth string theory. The problem of quantization of Ω_d may be solved in frames of the standard Dirac approach (cf., e.g., [6]). We describe the main steps of this construction in the first part of this paper Sections 1, 2).

However, the symplectic form of Ω_d may be extended to the Hilbert completion of Ω_d coinciding with the Sobolev space $V_d := H_0^{1/2}(S^1, R^d)$ of half-differentiable loops with values in R^d. So it is reasonable to consider V_d as the phase space of non-smooth string theory and try to quantize it. There is a natural group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle associated with this Sobolev space. This group acts on V_d by reparameterization and this action preserves the symplectic form. If this action would be smooth we could quantize

While preparing this survey the author was partially supported by RFBR grants 16-01-00117, 16-52-12012, the Program of support of leading scientific schools NSh-9110.2016.1 and the Program of Presidium of RAS "Nonlinear dynamic".

the space V_d along the same lines as in the case of the space Ω_d of smooth loops. However, this action is not smooth implying that we are not able to construct any classical system associated with the phase space V_d provided with the action of the group $\mathrm{QS}(S^1)$. Nevertheless, we can define a quantum system associated with V_d using the ideas from noncommutative geometry as in [8].

1. Classical system associated with the loop space Ω_d

1.1. Phase space Ω_d

The loop space Ω_d is the space $C_0^\infty(S^1, R^d)$ of smooth maps $S^1 \to R^d$ with zero average along the circle. The elements $x \in \Omega_d$ have Fourier decompositions of the form

$$x(\theta) := x(e^{i\theta}) = \sum_{k \neq 0} x_k e^{ik\theta}$$

with coefficients $x_k \in \mathbb{C}^d$ satisfying the relation: $x_k = \bar{x}_{-k}$.

The standard symplectic form on Ω_d is defined in terms of Fourier coefficients by the formula

$$\omega(\xi, \eta) = -i \sum_{k \neq 0} k\langle \xi_k, \eta_{-k}\rangle = 2 \operatorname{Im} \sum_{k>0} k\langle \xi_k, \bar{\eta}_k\rangle$$

where we denote by $\langle \cdot\,, \cdot\rangle$ the standard inner product in R^d.

We can also define a complex structure on Ω_d in terms of Fourier coefficients by the operator

$$\xi(\theta) = \sum_{k \neq 0} \xi_k e^{ik\theta} \longmapsto (J^0\xi)(\theta) = -i \sum_{k>0} \xi_k e^{ik\theta} + i \sum_{k<0} \xi_k e^{ik\theta}.$$

Both introduced structures are compatible with each other, in particular, they determine a Riemannian metric on Ω_d by the formula $g^0(\xi, \eta) := \omega(\xi, J^0\eta)$ or in terms of Fourier coefficients

$$g^0(\xi, \eta) = 2 \operatorname{Re} \sum_{k>0} k\langle \xi_k, \bar{\eta}_k\rangle = \sum_{k \neq 0} |k|\langle \xi_k, \bar{\eta}_k\rangle.$$

In other words, Ω_d has the structure of a Kähler–Fréchet space.

1.2. Algebra of observables $\mathcal{A}_d$

The space Ω_d is provided with the algebra of observables $\mathcal{A}_d$ given by the semi-direct product $\operatorname{heis}(\Omega_d) \rtimes \operatorname{Vect}(S^1)$ of the Heisenberg algebra $\operatorname{heis}(\Omega_d)$ and Lie algebra $\operatorname{Vect}(S^1)$ of tangent vector fields on the circle.

In more detail, the Heisenberg algebra $\operatorname{heis}(\Omega_d)$ is a central extension of the Abelian Lie algebra Ω_d generated by the coordinate functions on Ω_d. In other words, it coincides with the vector space $\operatorname{heis}(\Omega_d) = \Omega_d \oplus \mathbb{R}$ provided with the Lie bracket of the form

$$[(x, s), (y, t)] := (0, \omega(x, y)) \ , \quad x, y \in \Omega,\ s, t, \in \mathbb{R}.$$

The second component of $\mathcal{A}_d$ is given by the Lie algebra of the Lie group $\mathrm{Diff}_+(S^1)$ of orientation-preserving diffeomorphisms of the circle. It coincides with the Lie algebra $\mathrm{Vect}(S^1)$ of smooth tangent vector fields on the circle S^1.

1.3. Definition of quantization

The quantization of the classical system, given by the pair $(\Omega_d, \mathcal{A}_d)$, consisting of the phase space Ω_d and the algebra of observables $\mathcal{A}_d$, is an irreducible linear representation

$$r : \mathcal{A}_d \longrightarrow \mathrm{End}^*(H)$$

of observables $f \in \mathcal{A}_d$ by selfadjoint operators $r(f)$ acting in a complex Hilbert space H called the quantization space.

This representation should satisfy the condition

$$r\left(\{f, g\}\right) = \frac{1}{i}\left(r(f)r(g) - r(g)r(f)\right)$$

for any $f, g \in \mathcal{A}_d$ where $\{f, g\}$ is the Poisson bracket of observables f, g determined by the symplectic structure. It is also assumed that $r(1) = \mathrm{id}$.

For infinite-dimensional algebra of observables $\mathcal{A}_d$ it is more natural to deal with projective representations. Having such representation of $\mathcal{A}_d$, we can construct the quantization of the extended system $(V_d, \widetilde{\mathcal{A}_d})$ where $\widetilde{\mathcal{A}_d}$ is an appropriate central extension of $\mathcal{A}_d$.

1.4. The action of the diffeomorphism group of the circle on Ω_d

The diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$ acts on Ω_d by reparameterization: $f : x \mapsto x \circ f$, $x \in \Omega_d$, $f \in \mathrm{Diff}_+(S^1)$. This action is symplectic, i.e., it preserves symplectic form ω.

However, in general it does not preserve the complex structure J^0. More precisely, a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ transforms the complex structure J^0 into a new complex structure

$$J^f := f_*^{-1} \circ J^0 \circ f_*,$$

where f_* is the tangent map to f. This new complex structure J^f coincides with the original complex structure J^0 if and only if $f \in \text{Möb}(S^1)$ where $\text{Möb}(S^1)$ is the group of fractional-linear automorphisms of the unit disc restricted to S^1.

We shall call the complex structures J^f on Ω_d obtained from J^0 by the action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ admissible.

The space of admissible complex structures on Ω_d is identified with the Kähler–Fréchet manifold

$$\mathcal{S} := \mathrm{Diff}_+(S^1)/\,\text{Möb}(S^1).$$

2. Quantization of the classical system $(\Omega_d, \mathcal{A}_d)$

2.1. Sobolev space V_d

Quantization of the first component of the algebra of observables $\mathcal{A}_d$, given by the Heisenberg algebra heis(Ω_d), is realized in the Fock space associated with the Sobolev space $V_d := H_0^{1/2}(S^1, R^d)$ of half-differentiable loops with values in R^d.

Recall that the Sobolev space $V_d := H_0^{1/2}(S^1, R^d)$ consists of the maps $x \in L^2(S^1, \mathbb{C}^d)$ with zero average along the circle, having Fourier decompositions of the form

$$x(\theta) := x(e^{i\theta}) = \sum_{k\neq 0} x_k e^{ik\theta}$$

with coefficients $x_k \in \mathbb{C}^d$ satisfying the relation: $x_k = \bar{x}_{-k}$, and having the finite Sobolev norm of order 1/2

$$\|x\|_{V_d}^2 = \sum_{k\neq 0} |k| \|x_k\|^2 = 2\sum_{k>0} k\|x_k\|^2 < \infty.$$

This space may be considered as a natural Hilbert completion of the Fréchet space Ω_d with respect to the Sobolev norm. In particular, the complex structure operator J^0 and symplectic form ω, introduced above for the space Ω_d, extend to V_d transforming it into a Kähler–Hilbert space with Riemannian metric

$$g^0(\xi, \eta) = \omega(\xi, J^0\eta) = 2\,\mathrm{Re}\sum_{k>0} k\langle \xi_k, \bar{\eta}_k\rangle = \sum_{k\neq 0} |k|\langle \xi_k, \bar{\eta}_k\rangle.$$

This metric extends to a Hermitian inner product on the compexified Sobolev space $V_d^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C}^d)$ given by the formula

$$\langle \xi, \eta\rangle = \sum_{k\neq 0} |k|\langle \xi_k, \bar{\eta}_k\rangle.$$

The form ω and complex structure operator J^0 extend to $V_d^{\mathbb{C}}$ complex-linearly.

The space $V_d^{\mathbb{C}}$ decomposes into the direct sum

$$V_d^{\mathbb{C}} = W_+ \oplus W_- =: W_0 \oplus \overline{W}_0$$

of the subspaces $W_\pm$ being the $(\mp i)$-eigenspaces of operator J^0 given by

$$W_\pm = \left\{ x \in V_d^{\mathbb{C}} :\ x(\theta) = \sum_{\pm k>0} x_k e^{ik\theta} \right\}.$$

2.2. Fock space associated with V_d

The Fock space $F_0 \equiv F(V_d^{\mathbb{C}}, J^0)$ is the completion of the algebra of symmetric polynomials on W_0 with respect to a natural norm generated by $< \cdot, \cdot >$.

In more detail, denote by $\mathfrak{S}(W_0)$ the algebra of symmetric polynomials in variables $z \in W_0 \equiv W_+$ and introduce the inner product on $\mathfrak{S}(W_0)$ induced by the

Hermitian inner product $\langle\cdot,\cdot\rangle$ on $V_d^{\mathbb{C}}$. This inner product on monomials is given by the formula

$$\langle z_1\otimes\cdots\otimes z_n, z_1'\otimes\cdots\otimes z_n'\rangle = \sum_{\{i_1,\ldots,i_n\}} \langle z_1, z_{i_1}'\rangle\cdot\ \cdots\ \cdot\langle z_n, z_{i_n}'\rangle$$

where the summation is taken over all permutations $\{i_1,\ldots,i_n\}$ of the set $\{1,\ldots,n\}$ (inner product of monomials of different degrees is set to 0 by definition).

Extend this inner product on monomials to the whole algebra $\mathfrak{S}(W_0)$ by linearity. The completion $\widehat{\mathfrak{S}(W_0)}$ of the space $\mathfrak{S}(W_0)$ with respect to the introduced norm is called the Fock space

$$F_0 \equiv F(V_d^{\mathbb{C}}, J^0)$$

over $V_d^{\mathbb{C}}$ with respect to the complex structure J^0.

In an analogous way, any admissible complex structure J on V_d yields the decomposition

$$V_d^{\mathbb{C}} = W\oplus\overline{W}$$

of the complexified Sobolev space $V_d^{\mathbb{C}}$ into the direct sum of $(\mp i)$-eigenspaces of operator J on $V^{\mathbb{C}}$. These subspaces are isotropic with respect to ω and the above decomposition is orthogonal with respect to the Riemannian metric $\langle\cdot\,,\cdot\rangle_J$ on $V_d^{\mathbb{C}}$ determined by J and ω.

Using this decomposition, we define the Fock space

$$F_J \equiv F(V_d^{\mathbb{C}}, J)$$

as the completion of the algebra $\mathfrak{S}(W)$ of symmetric polynomials on W with respect to the norm generated by $\langle\cdot\,,\cdot\rangle_J$.

If $\{w_n\}_{n=1}^{\infty}$ is an orthonormal basis of W one can take for the orthonormal basis of the Fock space F_J the monomials of the form

$$P_K(z) = \frac{1}{\sqrt{k!}}\langle z, w_1\rangle_J^{k_1}\cdot\ \cdots\ \cdot\langle z, w_n\rangle_J^{k_n}, \quad z\in W,$$

where $K=(k_1,\ldots,k_n,0,\ldots)$ is a finite sequence of natural numbers $k_i\in\mathbb{N}$ and $k! = k_1!\cdot\ \cdots\ \cdot k_n!$.

2.3. Heisenberg representation

There is a standard irreducible Heisenberg representation r_J of the Heisenberg algebra $\mathrm{heis}(V_d)$ in the Fock space $F_J = F(V_d^{\mathbb{C}}, J)$ which is constructed in the following way.

The elements of the algebra $\mathfrak{S}(W)$ may be considered as holomorphic functions on the space $\overline{W}$ by identifying $z\in W$ with the holomorphic function

$$\overline{W}\ni\bar{w}\longmapsto\langle z, w\rangle_J \quad \text{on } \overline{W}.$$

Respectively, the space F_J may be considered as a space of functions holomorphic on $\overline{W}$.

The Heisenberg representation r_J of the Heisenberg algebra $\mathrm{heis}(V_d)$ in the Fock space F_J is given by the formula

$$V_d \ni v \longmapsto r_J(v)f(\bar{w}) = \partial_v f(\bar{w}) + \langle v, w\rangle_J f(\bar{w}) \tag{1}$$

where ∂_v is the derivative in direction of vector v. Extending r_J to the complexified algebra $\mathrm{heis}^{\mathbb{C}}(V_d)$ by the same formula (1), we obtain that

$$r_J(\bar{z})f(\bar{w}) = \partial_{\bar{z}} f(\bar{w}) \text{ for } \bar{z} \in \overline{W},$$
$$r_J(z)f(\bar{w}) = \langle z, w\rangle_J f(\bar{w}) \text{ for } z \in W.$$

The Heisenberg representation is conveniently described in terms of creation and annihilation operators on the space F_J given by the formulas

$$a_J^*(v) = \frac{r_J(v) + ir_J(Jv)}{2}, \quad a_J(v) = \frac{r_J(v) - ir_J(Jv)}{2}$$

where $v \in V_d^{\mathbb{C}}$. It implies that

$$a_J^*(z)f(\bar{w}) = \langle z, w\rangle_J f(\bar{w}) \text{ for } z \in W,$$
$$a_J(\bar{z})f(\bar{w}) = \partial_{\bar{z}} f(\bar{w}) \text{ for } \bar{z} \in \overline{W}.$$

2.4. Quantization of the algebra $\mathcal{A}_d$

To quantize the second component of the algebra of observables $\mathcal{A}_d$, represented by the algebra $\mathrm{Vect}(S^1)$ of tangent vector fields on the circle, we should study the action of the group $\mathrm{Diff}_+(S^1)$ of diffeomorphisms of the circle on the Fock spaces F_J.

If $J = J^f$ is an admissible complex structure obtained from J^0 by the action of a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ we denote by $r_{J^f} \equiv r^f$ the corresponding representation of the Heisenberg algebra $\mathrm{heis}(V_d)$ in the Fock space $F_{J^f} \equiv F^f$.

By the Goodman–Wallach theorem [3] the natural action of the group $\mathrm{Diff}_+(S^1)$ on the space $\mathcal{S} := \mathrm{Diff}_+(S^1)/\,\mathrm{Möb}(S^1)$ of admissible complex structures on Ω_d by left translations may be pulled up to a projective unitary action

$$U^f : F_0 \longrightarrow F^f$$

of the group $\mathrm{Diff}_+(S^1)$ on Fock spaces intertwining the representations r_0 and r^f:

$$r^f U^f(v) = U^f r_0(v) \text{ for } v \in F_0.$$

Introduce the Fock bundle

$$\mathcal{F} := \bigcup_{J \in \mathcal{S}} F_J \longrightarrow \mathcal{S} = \mathrm{Diff}_+(S^1)/\,\mathrm{Möb}(S^1).$$

It is a holomorphic Hilbert bundle over the space $\mathcal{S}$ (cf., e.g., [6]).

The infinitesimal version of the action of the group $\mathrm{Diff}_+(S^1)$ on the Fock bundle yields a projective unitary representation ρ of the Lie algebra $\mathrm{Vect}(S^1)$ in the space F_0. The cocycle of this representation was computed by Bowick and Rajeev [1].

In the basis $\{e_n := ie^{in\theta}d/d\theta\}$ of the algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$ it is given by the formula

$$[\rho(e_m), \rho(e_n)] - \rho\left([e_m, e_n]\right) = \tfrac{d}{12}(m^3 - m)\delta_{m,-n}.$$

This cocycle determines a central extension $\widetilde{\mathcal{A}_d}$ of the original algebra of observables $\mathcal{A}_d$ together with the unitary representation of $\widetilde{\mathcal{A}_d}$ in the Fock space F_0 concluding the quantization of the classical system $(\Omega, \widetilde{\mathcal{A}_d})$.

3. Quantization of theory of half-differentiable strings

3.1. Motivation

Having quantized the classical system $(\Omega_d, \mathcal{A}_d)$ we may ask why in our approach we have restricted ourselves to smooth strings?

The only "physical parameter" in the considered theory is the symplectic form ω on Ω_d which extends to the Sobolev space $V_d = H_0^{1/2}(S^1, R^d)$ of half-differentiable loops. In fact, V_d is the largest in the Sobolev scale of spaces $H_0^s(S^1, R^d)$, $s > 0$, on which this form is correctly defined.

So why not to take this Sobolev space, already "chosen by symplectic form itself", for the phase space of string theory? It seems that the only reason for choosing Ω_d as the phase space of this theory is that we prefer to deal with smooth objects.

Motivated by these considerations, we shall assume from now on that the phase space of our theory is the Sobolev space V_d of half-differentiable loops.

We should choose next a natural algebra of observables on this phase space. The first component of the algebra $\mathcal{A}_d$, coinciding with the Heisenberg algebra, describing the pure kinematics, should be included into our algebra of observables anyhow.

As for the second component of $\mathcal{A}_d$, namely the Lie algebra $\mathrm{Vect}(S^1)$, its choice in the smooth case was explained by the fact that the corresponding Lie group $\mathrm{Diff}_+(S^1)$, consisting of diffeomorphisms of the circle, acts on the space Ω_d by reparameterization, i.e., change of variable. Such choice was evidently dictated by the fact that we were dealing with smooth loops only.

Returning to the choice of the second component of the algebra of observables, we can pose the following question: what is the natural group acting on the space V_d by reparameterization? The answer to this question is given by the Nag–Sullivan theorem below. However, before that we need to introduce some notions from the theory of quasiconformal maps.

3.2. Quasisymmetric homeomorphisms

Recall that an orientation-preserving homeomorphism $w : \Delta \to \Delta$ of the unit disc Δ onto itself with locally integrable derivatives is called quasiconformal if there exists a bounded measurable function $\mu \in L^\infty(\Delta, \mathbb{C})$ with the norm $\|\mu\|_\infty =: k < 1$

such that the following Beltrami equation

$$w_{\bar{z}} = \mu w_z$$

holds almost everywhere in Δ. The function μ is called the Beltrami differential.

In the case when $k = 0$, i.e., $\mu = 0$, the Beltrami equation coincides with the Cauchy–Riemann equation so in this case the map w is conformal.

Recall some important properties of quasiconformal maps:

1. Quasiconformal homeomorphisms $w : \Delta \to \Delta$ extend continuously (in fact, Hölder-continuously) to the boundary $S^1 = \partial\Delta$ as homeomorphisms $S^1 \to S^1$.
2. The composition of quasiconformal maps $\Delta \to \Delta$ is again a quasiconformal map. The same is true for the maps inverse to quasiconformal ones. Hence, quasiconformal automorphisms of the disc Δ form a group with respect to the composition.

An orientation-preserving homeomorphism $f : S^1 \to S^1$ is called quasisymmetric if it extends to a quasiconformal homeomorphism $w : \Delta \to \Delta$ of the unit disc Δ onto itself. Since the quasiconformal automorphisms of the disc Δ form a group the same is true also for quasisymmetric homeomorphisms of S^1. The group of quasisymmetric homeomorphisms of S^1 is denoted by $\mathrm{QS}(S^1)$.

3.3. Nag–Sullivan theorem

Let f be an orientation-preserving homeomorphism $S^1 \to S^1$. Associate with it an operator T_f acting on V_d by the formula

$$(T_f h(z) = h(f(z)) - \frac{1}{2\pi}\int_0^{2\pi} h(f(e^{i\theta}))d\theta, \quad z = e^{i\theta},\ h \in V_d.$$

Theorem 1 (Nag–Sullivan [4]). *The operator T_f maps the space V_d into itself if and only if $f \in \mathrm{QS}(S^1)$. The action of the operator $T_f : V_d \to V_d$ with $f \in \mathrm{QS}(S^1)$ on the space V_d preserves the symplectic form ω, i.e., $\omega(T_f\xi, T_f\eta) = \omega(\xi, \eta)$ for any $\xi, \eta \in V_d$.*

This theorem implies that a natural group acting on the Sobolev space V_d is the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle. If this group would be a Lie group, acting smoothly on V_d then we could take for the second component of our algebra of observables the Lie algebra of this group and construct the quantization of the arising classical system in the same way as in the case of smooth loops. However, neither the group $\mathrm{QS}(S^1)$ nor its action on the Sobolev space V_d are smooth. By this reason we cannot construct any classical system with the phase space V_d provided with the action of the group $\mathrm{QS}(S^1)$.

Instead, we shall construct directly a quantum system associated with V_d. In other words, we change our original point of view on quantization and construct first the quantum system, associated with the space V_d and the group $\mathrm{QS}(S^1)$, passing by the stage of construction of the classical system.

To do that we have to replace our original definition of quantization by the one proposed by Connes.

3.4. Connes quantization

In Dirac definition we are quantizing the classical systems $(M, \mathcal{A})$ where M is the phase space and $\mathcal{A}$ is a Lie algebra of observables, consisting of smooth functions on M, provided with the Poisson bracket.

In Connes' approach [2] a classical system is given by the pair $(M, \mathfrak{A})$ where M is again the phase space and the algebra of observables $\mathfrak{A}$ is an associative involutive algebra of smooth functions on M provided with involution and exterior derivative d.

The quantization of such system is given by an irreducible linear representation π of observables from $\mathfrak{A}$ by closed linear operators acting in the quantization space H. The involution in $\mathfrak{A}$ under this representation is transformed into Hermitian conjugation while the derivative d is sent to the commutator with some symmetry operator S where S is a selfadjoint operator on H with square $S^2 = I$. In other words,

$$\pi : df \longmapsto d^q f := [S, \pi(f)], \quad f \in \mathfrak{A}.$$

If all observables are smooth functions on M (as we have assumed up to this point) then there is not much difference between these two approaches to quantization. However, if we allow the algebra of observables $\mathcal{A}$ to contain nonsmooth functions the Dirac definition looses its sense. In Connes' approach the differential of a nonsmooth observable $f \in \mathfrak{A}$ is also not defined in the classical sense. However, it may happen that its quantum analogue

$$d^q f := [S, \pi(f)]$$

is correctly defined.

Before we switch to the construction of the quantum system, associated with the Sobolev space V_d and the group $\mathrm{QS}(S^1)$ consider the following simple example.

Take for the algebra of observables the algebra $\mathfrak{A} = L^\infty(S^1, \mathbb{C})$ of bounded functions on the circle S^1. Any function $f \in \mathfrak{A}$ determines a bounded multiplication operator M_f in the Hilbert space $H = L^2(S^1, \mathbb{C})$ acting by the formula:

$$M_f : h \in H \longmapsto fh \in H.$$

The symmetry operator S on H is given by the Hilbert transform:

$$(Sh)(\phi) = \frac{1}{2\pi} \,\mathrm{P.V.} \int_0^{2\pi} K(\phi, \psi) f(\psi) d\psi, \quad f \in H,$$

where the integral is taken in the principal value sense, i.e.,

$$\mathrm{P.V.} \int_0^{2\pi} K(\phi, \psi) f(\psi) d\psi := \lim_{\epsilon \to 0} \left[\int_0^{\phi - \epsilon} + \int_{\phi + \epsilon}^{2\pi} \right] K(\phi, \psi) f(\psi) d\psi.$$

(Here and in the sequel we identify the functions $f(z)$ on the circle S^1 with the functions $f(\phi) := f(e^{i\phi})$ on the interval $[0, 2\pi]$.) The Hilbert kernel in this formula is given by the expression

$$K(\phi, \psi) = 1 + i \cot \tfrac{\phi - \psi}{2} \,.$$

Note that for $\phi \to \psi$ it behaves like $1 + \frac{2i}{\phi-\psi}$.

The differential of a general observable $f \in \mathfrak{A}$ is not defined in the classical sense but its quantum analogue

$$d^q f := [S, M_f]$$

is correctly defined as an operator in H (this operator is correctly defined even for functions from $\mathrm{BMO}(S^1)$).

Namely, it is an integral operator given by the formula

$$(d^q f)(h)(\phi) = \frac{1}{2\pi}\int_0^{2\pi} k_f(\phi,\psi)h(\psi)d\psi, \quad h \in H,$$

where $k_f(\phi,\psi) = K(\phi,\psi)(f(\phi) - f(\psi))$. For $\phi \to \psi$ the kernel $k_f(\phi,\psi)$ behaves like

$$\mathrm{const}\,\frac{f(\phi)-f(\psi)}{\phi-\psi}\,.$$

The quasiclassical limit of this operator, established by restricting it to smooth functions and taking the trace on the diagonal $\phi = \psi$, coincides with the multiplication operator $h \mapsto f' \cdot h$.

In other words, the quantization procedure in this example essentially reduces to the replacement of the derivative by its finite-difference analogue. Such a quantization, given by the correspondence

$$\mathfrak{A} \ni f \longmapsto d^q f : H \to H,$$

Connes [2] calls the "quantum calculus" by analogy with the finite-difference calculus (cf. also [7]).

3.5. Quantization of the Sobolev space V_d

Returning to the quantization of the Sobolev space V_d, it would be more convenient to switch from S^1 to the real line $\mathbb{R}$. Then V_d will be replaced by the Sobolev space $H^{1/2}(\mathbb{R})$ of half-differentiable vector-functions on the real line (which we continue to denote by V_d) and $\mathrm{QS}(S^1)$ will be substituted by the group $\mathrm{QS}(\mathbb{R})$ of quasisymmetric homeomorphisms of $\mathbb{R}$, extending to quasiconformal homeomorphisms of the upper half-plane. Then for any $h \in V_d$ we introduce the operator $d^q h : V_d^{\mathbb{C}} \to V_d^{\mathbb{C}}$ by the formula

$$(d^q h)(v)(x) = \int_{\mathbb{R}} \frac{h(x+t)-h(t)}{t}\, v(t)dt, \quad v \in V_d^{\mathbb{C}}.$$

According to [5], the tangent space to $\mathrm{QS}(\mathbb{R})$ at the origin coincides with the Zygmund space $\Lambda(\mathbb{R})$ of functions $f(x)$ satisfying the condition

$$|f(x+t) + f(x-t) - 2f(x)| \le C|t|$$

uniformly for $x \in \mathbb{R}$, $t > 0$, and growing not faster than const $\cdot\, x^2$ for $|x| \to \infty$. This motivates the introduction of the following operator $d^q g$ for $g \in \mathrm{QS}(\mathbb{R})$

$$d^q g(v) = \int_{\mathbb{R}} \frac{g(x+t) + g(x-t) - 2g(x)}{t} v(t)dt, \quad v \in V_d^{\mathbb{C}}.$$

We define now the quantized infinitesimal action of $\mathrm{QS}(\mathbb{R})$ on $V_d^{\mathbb{C}}$ as the composition $T_g^q h := d^q h(g) \circ d^q g$ of the introduced operators. The quasiclassical limit of this operator is equal to the operator of multiplication by $h'(g)g'$.

This action extends to the whole Fock space F_0 in the following way. We define it first on the elements of the orthonormal basis of F_0, given by monomials $P_K(z)$, by the Leibniz rule. Then we extend this operator to the whole algebra $\mathfrak{S}(W_+)$ of symmetric polynomials on W_+ by linearity. The closure of the obtained operator yields an operator $T_g^q h$ in the Fock space F_0. In the same way the operator $d^q h$ is extended to a closed operator $d^q h$ in F_0.

The desired quantum algebra of observables $\mathfrak{A}_d^q$ is the Lie algebra generated by the constructed operators $d^q h$ and $T_g^q h$ in F_0 with $g \in \mathrm{QS}(\mathbb{R})$, $h \in V_d$.

References

[1] M.J. Bowick, S.G. Rajeev, *The holomorphic geometry of closed bosonic string theory and Diff S^1/S^1*. Nucl. Phys. **B293** (1987), 348–384.

[2] A. Connes, *Geometrie Non Commutative*. Intereditions, 1990.

[3] R. Goodman, N.R. Wallach, *Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle*. J. Reine Angew. Math. **B347** (1984), 67–133.

[4] S. Nag, D. Sullivan, *Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle*. Osaka J. Math. **32** (1995), 1–34.

[5] M. Reimann, *Ordinary differential equations and quasiconformal mappings*, Invent. Math. **33** (1976), 247–270.

[6] A. Sergeev, *Kähler Geometry of Loop Spaces*. World Scientific, 2011.

[7] A. Sergeev, *Quantum calculus and quasiconformal mappings*. Math. Notes **100** (2016), 123–131.

[8] A. Sergeev, *Quantization of Sobolev space of half-differentiable functions*. Sbornik Mathematics **207** (2016), 96–104.

A.G. Sergeev
Steklov Mathematical Institute
Gubkina 8
Moscow 119991
Russian Federation
e-mail: sergeev@mi.ras.ru

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 131–138

The Reasonable Effectiveness of Mathematical Deformation Theory in Physics

Daniel Sternheimer

Abstract. This is a brief reminder, with extensions, from a different angle and for a less specialized audience, of my presentation at WGMP32 in July 2013, to which I refer for more details on the topics hinted at in the title, mainly deformation theory applied to quantization and symmetries (of elementary particles).

Mathematics Subject Classification (2010). 53D55; 81R50, 17B37, 53Z05, 81S10, 81V25, 83C57.

Keywords. Deformation theory, deformation quantization, symmetries of hadrons, models, Anti de Sitter, singletons, quantum groups at root of unity, "quantum deformations", black holes, dark matter.

1. Presentation

In 1960 Wigner [21] marveled about "the unreasonable effectiveness of mathematics in the natural sciences." In 1963/64 appeared index theorems for pseudodifferential operators. I participated (exposé 23, [9]) in the Cartan–Schwartz seminar that developed the announcement of the first result (a parallel seminar was held in Princeton under Richard Palais, and different proofs and extensions were published by Atiyah et al. a few years later). In 1964 appeared Gerstenhaber's theory of deformations of algebras [17]. [Murray Gerstenhaber is now 90 and still active.] Soon after Gerstenhaber's seminal work it was realized that, from the viewpoint of symmetries, special relativity is a deformation.

The underlying idea is that new fundamental physical theories can, so far a posteriori, be seen as emerging from existing ones via some kind of deformation. The main paradigms are the physics revolutions from the beginning of the twentieth century, special relativity (symmetry deformation from the Galilean to the Poincaré groups) and quantum mechanics (via deformation quantization).

Indeed in the mid-seventies all this converged to explain quantum mechanics as a deformation of classical mechanics [3], what is now called deformation quantization. Quantum groups and noncommutative geometry can be considered as avatars of that framework. In this very short paper (see [20], and references therein, for more details on most of these ideas) I present these notions in general terms, describe some results. and indicate some perspectives, including suggestions to use the framework to put on "non-clay feet" the "colossal" standard model of elementary particles, and maybe explain the "dark universe."

1.1. The problem

"It isn't that they can't see the solution. It is that they can't see the problem." That is a quote from a detective story by G.K. Chesterton (1874–1936) ("The Point of a Pin" in "The Scandal of Father Brown" (1935)). The problem, in my view, is that *the Standard Model of elementary particles could be a colossus with clay feet.* Cf. in the Bible (Daniel 2:41–43), the interpretation by Belteshazzar (a.k.a. prophet Daniel) of Nebuchadnezzar's dream.

The physical consequences of the approach described here might be revolutionary but in any case there are, in the mathematical tools required to jump start the process, potentially important developments to be made.

1.2. Some motivating quotes

Albert Einstein: "The important thing is not to stop questioning. Curiosity has its own reason for existing."

"You can never solve a [fundamental, precision added by DS] problem on the level on which it was created."

Gerard 't Hooft (about Abdus Salam) [18]: "To obtain the Grand Picture of the physical world we inhabit [...] courage is required. Every now and then, one has to take a step backwards, one has to ask silly questions, one must question established wisdom, one must play with ideas like being a child. And one must not be afraid of making dumb mistakes. By his adversaries, Abdus Salam was accused of all these things."

Eugene Paul Wigner [21]: "Mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections. Secondly, just because of this circumstance, and because we do not understand the reasons of their usefulness, we cannot know whether a theory formulated in terms of mathematical concepts is uniquely appropriate."

Sir Michael Atiyah (at ICMP London 2000, [2]): "Mathematics and physics are two communities separated by a common language."

Paul Adrien Maurice Dirac [11]: "Two points of view may be mathematically equivalent [...] But it may be that one point of view may suggest a future development which another point does not suggest [...] Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics."

2. Headlines

A scientist should ask himself three questions: Why, What and How. Of course, work is 99% perspiration and 1% inspiration. Finding how is 99% of the research work, but it is important to know what one is doing and even more why one does such a research.

What we call "physical mathematics" can be defined as mathematics inspired by physics. While in mathematical physics one studies physical problems with mathematical tools and (hopefully) rigor. [Theoretical physics uses mathematical language without caring much about rigor.] In addition, as to "what" and "how" to research there are important differences between mathematicians and physicists. Indeed, even when taking their inspiration from physics (which fortunately is again often the case now, see, e.g., [2]), mathematicians tend to study problems in as general a context as possible, which may be very hard. But when the aim is to tackle specific physical problems, and though generalizations may turn out to have unexpected consequences, it is often enough to develop tools adapted to the desired applications.

That is the spirit which inspired the approaches Moshe Flato and I developed (with coworkers of course) since the mid 60s, and which I am continuing this millennium. In what follows I give a flavor of two main topics we tackled, i.e., original applications both of symmetries to particle physics and of quantization as a deformation, and of the combining program that I am now trying to push forward for the coming generation(s).

2.1. Deformation quantization and avatars

As we said above the two major physical theories of the first half of the twentieth century, relativity and quantization, can now be understood as based on deformations of some algebras. That is the starting point of Moshe Flato's "deformation philosophy". Deformations (in the sense of Gerstenhaber [17]) are classified by cohomologies. The deformation aspect of relativity became obvious in 1964, as soon as deformation theory of algebras (and groups) appeared, since one can deform the Galilean group symmetry of Newtonian mechanics $SO(3) \rtimes \mathbb{R}^3 \rtimes \mathbb{R}^4$ to the Poincaré group $SO(3,1) \rtimes \mathbb{R}^4$.

Though (when Moshe arrived in Paris) I studied in [9] the composition of symbols of elliptic operators, and in spite of the fact that the idea that some passage from classical to quantum mechanics had been "in the back of the mind" of many, it took a dozen more years before quantization also could be mathematically understood as a deformation, with what is now called deformation quantization, often without quoting our founding 1978 papers [3]. (These have nevertheless been cited over 1000 times if one includes the physics literature, and so far 271 times for paper II in the mathematics literature, according to MathSciNet.)

Explaining the process in some detail would be beyond the scope of this short overview, so we refer to [20] and references therein. In a nutshell the idea is that, instead of a complete change in the nature of observables in classical mechanics (from functions on phase space, a symplectic or Poisson manifold, to operators on some Hilbert space in quantum mechanics), the algebra of quantum mechanics observables can be built on the same classical observables but with a deformed composition law (what we called a "star product" since the deformed composition law was denoted $\star$). The main paradigm is the harmonic oscillator. That can be extended to field theory (infinitely many degrees of freedom), and more.

Related representations of Lie groups can then be performed on functions on orbits with deformed products, instead of operators. Similarly, though they arose in quite different contexts in the 80s, based on previous works from the 70s by their initiators (Faddeev's Leningrad school for quantum groups coming from quantum integrable models, and Alain Connes' seminal works on von Neumann algebras for noncommutative geometry), both can be (see, e.g., [10, 12]) considered as avatars of our framework.

In particular, in the "generic case", quantum groups are deformations of an algebra of functions on a Poisson Lie group or of a dual algebra [5, 6]. The idea was extended in the 90s to multiparameter deformations (with commuting parameters), and (unrelatedly) to the case when the parameter is a root of unity – in which case the deformed Hopf algebra is finite-dimensional, with finitely many irreducible representations. That idea has not yet been extended to multiparameter deformations at roots of unity, nor a fortiori to noncommutative parameters (a notion which is not part of Gerstenhaber's approach and is not yet defined).

2.2. Symmetries of elementary particles

A posteriori one can say that the geometric aspect of deformation theory was known in physics since the antiquity, in particular when (in the fifth century B.C.) Pythagoras conjectured that, like other celestial bodies, the earth is not flat; two centuries later Aristotle gave phenomenological indications why this is true, and ca. 240 B.C. Eratosthenes came with an experimental proof of the phenomenon, giving a remarkably precise evaluation of the radius of the "spherical" earth. In mathematics one had to wait for Riemann's surface theory to get an analogue.

In another context, in the latter part of last century arose the standard model of elementary particles, based on empirically guessed symmetries. The untold rationale was that symmetries are important to explain the spectra observed in atomic and crystalline spectroscopy (as shown, e.g., in Moshe Flato's M.Sc. thesis under Racah [19]). There one knows the forces and symmetries make calculations feasible. In nuclear spectroscopy, the subject of Moshe's PhD thesis under Racah (which he never defended because Racah died unexpectedly in Firenze on his way to meet Moshe in Paris, and by then Moshe had already completed a D.Sc. in Paris), symmetries can be used as "spectrum generating".

That is also how symmetries were introduced empirically in particle physics, starting with isospin $SU(2)$ since the 30s, then with a rank 2 compact Lie group (thoroughly studied in the less known [4]) after "strangeness", a new quantum number, was introduced in the 50s (in particular thanks to people like Murray Gell-Mann, who was then daring to tackle unpopular topics, before becoming a kind of "guru" for at least a generation). That was quickly restricted to "flavor" $SU(3)$.

A natural question was then what (if any) is the relation between such "internal" symmetries and the "external" symmetries like the Poincaré group, in particular to explain the mass spectrum inside a multiplet. In 1965 we objected [15] to a "no-go theorem" claiming to show that the connection must be a direct sum, giving counterexamples shortly afterward. One should be careful with no-go theorems in physics, which often rely on unstated hypotheses. [Another issue, which was not so explicitly mentioned, is, e.g., how one can have the 3 octets of the "eightfold way", based on the same adjoint representation of $SU(3)$, associated with bosons of spin 0 and 1, and fermions of spin $\frac{1}{2}$.]

But what (if anything) to do with the basic 3-dimensional representations of $SU(3)$? In 1964 Gell-Mann (and others) came with the suggestion to associate them with "quarks", hypothetical entities of fractional charge which (being "confined" and of spin $\frac{1}{2}$) cannot be directly observed nor coexist, e.g., in a hadron (strongly interacting particle). Initially there were three "flavors" of quarks. Later consequences of the quark hypothesis were observed and we now have 3 generations (6 flavors) of quarks. To make possible their coexistence in a hadron they were given (three) different colors, whence "color $SU(3)$." Soon, on the basis of that empirically guessed symmetry, in a process reverse to what was done in spectroscopy, dynamics were developed, QCD (quantum chromodynamics) with a non-abelian gauge $SU(3)$ on the pattern of QED (quantum electrodynamics, with abelian $U(1)$ gauge). The rest is history but not the end of the story.

Already in 1988 [13], Flato and Fronsdal explained how, if one deforms space-time from Minkowski to Anti de Sitter (also a 4-dimensional space-time but with tiny negative curvature) and as a consequence the Poincaré group to AdS $SO(2,3)$, one can explain the photon as dynamically composite of two Dirac "singletons" (massless particles in 1+2-dimensional space-time) in a way compatible with QED. That was an instance of the AdS/CFT correspondence which we had detailed, e.g., in 1981 [1].

After numerous of papers on singleton physics, in Moshe's last paper [14] we described how the AdS deformation of Poincaré may explain the newly discovered neutrino oscillations, which showed that neutrinos are not massless. Going one step further, shortly afterward Fronsdal [16] explained how, on the pattern of the electroweak model and on the basis of AdS deformation, the leptons (electron, muon, tau, their antiparticles and neutrinos) can be considered as composites of singletons, initially massless, massified by 5 Higgs. (This predicts, e.g., 2 new "W and Z like" bosons.)

2.3. Combining both, and perspectives

In line with our deformation philosophy, the idea is that the question of connection between symmetries could be a false problem: the "internal" symmetries on which the Standard Model is based might "emerge" from the symmetry of relativity, first by "geometric" deformation (to Anti de Sitter, with singleton physics for photons and leptons) followed (for hadrons) by a quantum group deformation quantization.

The program to deform Poincaré to AdS for particle symmetries was developed by Flato and coworkers since the 70s (see, e.g., [1, 13, 14] and references therein), in parallel with deformation quantization. In the beginning of this millennium it dawned on me that the two could (maybe should) be combined by deforming further AdS. The natural way to do so is within the framework of quantum groups, possibly multiparameter since we now have 3 "generations" of elementary particles, which as a fringe benefit might make room for the traditional (compact) internal symmetries. In view of their special properties, quantum groups at roots of unity seem a promising structure, Of course such an approach is at this stage merely a general framework to be developed, and would be only part of the picture if one cannot "plug in" the present QCD dynamics.

In particular with for parameters the algebra of the Abelian group $\frac{\mathbb{Z}}{3\mathbb{Z}}$, at, e.g., sixth root of unity, one might be able to recover $SU(3)$ and "put on solid ground" the Standard Model. Or maybe replace its symmetry with a better one, requiring to "go back to the drawing board" and re-examine half a century of particle physics (from the theoretical, phenomenological and experimental viewpoints). The former alternative is relatively more economical. The latter requires huge efforts, albeit without having to build new machines, which Society is unlikely to give us.

That raises hard mathematical problems. (E.g., the tensor product of two irreducible representations of a quantum group at root of unity is indecomposable.) A solution to part of these has an independent mathematical interest. Combined with the necessary detailed phenomenological study required, that might lead to a re-foundation of half a century of particle physics.

There could also be implications in cosmology, including a possible explanation of dark matter and/or of primordial black holes, which were introduced already in 1974 by Hawking [8] and are now considered as a possible candidate for "dark matter" (see, e.g., [7]). Since that is still an open question (in contradistinction with elementary particles for which there is a solution, even if the problem is occulted) the community might be more receptive to try such ideas.

Acknowledgment

This short panorama benefited from 35 years of active collaboration with Moshe Flato (and coworkers), from close to `\infty` discussions with many scientists (too numerous to be listed individually) and from the hospitality of Japanese colleagues during my so far over 14 years of relatively generous retirement from France, after 42 years in CNRS.

References

[1] E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer, *Massless particles, conformal group, and de Sitter universe*, Phys. Rev. D (3), **23** (6), 1278–1289 (1981).

[2] Michael Atiyah, *On the unreasonable effectiveness of physics in mathematics*. Highlights of mathematical physics (London, 2000), 25–38, Amer. Math. Soc., Providence, RI, 2002.

[3] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization I. Deformations of symplectic structures*, and *II. Physical applications* Ann. Physics **111** 61–110 and 111–151 (1978).

[4] R.E. Behrends, J. Dreitlein, C. Fronsdal, B.W. Lee, *Simple groups and strong interaction symmetries*. Rev. Modern Phys. **34**, 1–40 (1962).

[5] P. Bonneau, M. Flato, M. Gerstenhaber and G. Pinczon, *The hidden group structure of quantum groups: strong duality, rigidity and preferred deformations*, Comm. Math. Phys. **161**, 125–156 (1994).

[6] P. Bonneau, M. Gerstenhaber, A. Giaquinto and D. Sternheimer, *Quantum groups and deformation quantization: explicit approaches and implicit aspects*, J. Math. Phys. **45** (10), 3703–3741 (2004).

[7] B.J. Carr, *Quantum Black Holes as the Link Between Microphysics and Macrophysics*, pp. 85–94 in: 2nd Karl Schwarzschild Meeting on Gravitational Physics (Nicolini, P., Kaminski, M., Mureika, J., and Bleicher, M. Eds.), Springer Proceedings in Physics vol. 208, Springer 2018. (arXiv:1703.08655 [gr-qc].)

[8] B.J. Carr and S.W. Hawking, *Black holes in the early universe*, Mon. Not. R. astr. Soc. **168**, 399–415 (1974).

[9] Séminaire Henri Cartan, 16^{e} année: 1963/64, dirigée par Henri Cartan et Laurent Schwartz. Théorème d'Atiyah–Singer sur l'indice d'un opérateur différential elliptique. Fasc. 1 & 2, Secrétariat mathématique, Paris 1965.

[10] A. Connes, M. Flato and D. Sternheimer, *Closed star-products and cyclic cohomology*, Lett. Math. Phys. **24**, 1–12 (1992).

[11] P.A.M. Dirac, *The relation of Classical to Quantum Mechanics*, pp. 10–31 in Proc. 2nd Can. Math. Congress (Vancouver 1949). U. Toronto Press (1951).

[12] V. Drinfel'd, *Quantum Groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987. Available as: `http://www.mathunion.org/ICM/ICM1986.1/Main/icm1986.1.0798.0820.ocr.pdf`.

[13] M. Flato and C. Fronsdal, *Composite Electrodynamics*, J. Geom. Phys. **5**, 37–61 (1988).

[14] M. Flato, C. Frønsdal and D. Sternheimer, *Singletons, physics in AdS universe and oscillations of composite neutrinos*, Lett. Math. Phys. **48**, 109–119 (1999).

[15] M. Flato and D. Sternheimer, *Remarks on the connection between external and internal symmetries*, Phys. Rev. Letters **15**, 934–936 (1965).

[16] C. Frønsdal, *Singletons and neutrinos*, Lett. Math. Phys. **52**, 51–59 (2000). Conférence Moshé Flato 1999 (Dijon) (`hep-th/9911241`).

[17] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. (2) **79**, 59–103 (1964) and (IV), *ibid.* **99**, 257–276 (1974).

[18] Gerard 't Hooft, *The Grand View of Physics*, Int. J. Mod. Phys. **A23**, 3755–3759 (2008) (`arXiv:0707.4572 [hep-th]`).

[19] M. Kibler, *The Master thesis of Moshé Flato*, Conférence Moshé Flato 1999, Vol. II (Dijon), 177–184, Math. Phys. Stud., **22**, Kluwer Acad. Publ., Dordrecht, 2000.

[20] Daniel Sternheimer. *"The important thing is not to stop questioning", including the symmetries on which is based the standard model.* Geometric methods in physics, 7–37, Trends Math., Birkhäuser Basel, 2014.

[21] E.P. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Comm. Pure Appl. Math. **13**, 1–14 (1960). Reproduced in: *The role of invariance principles in natural philosophy*, pp. ix–xvi in Proc. Internat. School of Phys. "Enrico Fermi", Course XXIX, Varenna. Academic Press 1964.

Daniel Sternheimer
Department of Mathematics
Rikkyo University
Tokyo, Japan

and

Institut de Mathématiques de Bourgogne
Dijon, France

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 139–146

States in Deformation Quantisation: Hopes and Difficulties

Jaromir Tosiek

Abstract. A notion of the state in classical and in quantum physics is discussed. Several classes of continuous linear functionals over different algebras of formal series are introduced. The condition of nonnegativity of functionals over the $*$ algebra is analysed.

Mathematics Subject Classification (2010). 81S30.

Keywords. Deformation quantisation, formal series.

1. Introduction

One of the most fundamental features of physics is that it proposes transformation of the real world into numbers. From this point of view one can say that physical reality consists of two main ingredients: the quantities which are measured called *observables* and the characteristics of a system under consideration known as a *state*. These two components are then combined to give *results*.

There exist several possible realisations of this scheme. In classical statistical physics observables are identified with smooth real functions f on a phase space $\mathcal{M}$, states are represented by densities of probability ϱ and results are mean values calculated as the functional action $\langle \varrho, f \rangle$.

At the quantum level in the Hilbert space model observables are self adjoint operators $\hat{f}$ acting in a space $\mathcal{H}$, states are density operators $\hat{\varrho}$ and results are traces $\mathrm{Tr}(\hat{\varrho} \cdot \hat{f})$. The reader interested in a systematic discussion of these postulates is encouraged to see [1].

However, our expectations in physics are bigger. We not only need a suitable mapping of reality into numbers but we would also like to be able to predict new phenomena. This process of prediction is based on logic and involves mathematical structures in which the sets of observables and of states can be equipped.

We start our contribution with a sketch of connections between a class of functions representing classical observables and functionals being densities of probability. Then we introduce formal series with respect to a deformation parameter λ, substitute a nonabelian $*$ product for the 'usual' multiplication of series and finally build linear functionals representing states. We do that in order to deal with quantum problems in frames of deformation quantisation formalism [2–4].

This formalism of deformation contains some difficulties. First of all, it usually involves infinite number of terms. Thus even elementary calculations for flat systems become rather complicated. Moreover, infinite sums appearing in some expressions may not be convergent.

But on the other hand deformation quantisation works well in every reference system. It thus seems to be a remedy for difficulties present in description of quantum phenomena in gravitational fields. In addition, from the conceptual point of view, it enlightens relationship between classical (undeformed) and quantum (deformed) physics.

2. Classical statistical mechanics

As we have already mentioned, in classical physics we assume that observables are smooth real functions defined on a phase space $\mathcal{M}$ of a system being a symplectic manifold. Thus all observables are elements of a wider structure: the ring of complex-valued smooth functions $(C^\infty(\mathcal{M}), +, \cdot)$ which form an algebra over the field of complex numbers $\mathbb{C}$. The constant function equal to $\mathbf{1}$ at every point of the manifold $\mathcal{M}$ is the identity element of this algebra.

A definition of convergence in the set $C^\infty(\mathcal{M})$ has been adapted from theory of generalised functions (see [5]). We say that the sequence $\{f_n\}_{n=1}^{\infty}$ is *convergent* to a function f_0, if on every compact subset of the manifold $\mathcal{M}$, $\dim \mathcal{M} = 2r$, every sequence of partial derivatives $\left\{ \frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1} q^1 \ldots \partial^{m_{2r}} q^{2r}} f_n \right\}_{n=1}^{\infty}$ is uniformly convergent to the derivative $\frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1} q^1 \ldots \partial^{m_{2r}} q^{2r}} f_0$.

States are represented by the functionals called densities of probabilities ϱ. They are elements of the space of linear continuous functionals $\mathcal{E}'(\mathcal{M})$ over the set of functions $C^\infty(\mathcal{M})$. Every density of probability ϱ is a real functional

$$\forall \ C^\infty(\mathcal{M}) \ni f = \overline{f} \Rightarrow \langle \varrho, f \rangle \in \mathbb{R}. \tag{1}$$

Moreover, the density ϱ has to be nonnegative

$$\forall \ C^\infty(\mathcal{M}) \ni f \ \ \langle \varrho, f \cdot \overline{f} \rangle \geq 0 \tag{2}$$

and normalised

$$\langle \varrho, \mathbf{1} \rangle = 1. \tag{3}$$

A sequence of densities $\{\varrho_n\}_{n=1}^{\infty}$ tends to a functional ϱ_0 if

$$\forall \ C^\infty(\mathcal{M}) \ni f \ \ \lim_{n\to\infty} \langle \varrho_n, f \rangle = \langle \varrho_0, f \rangle. \tag{4}$$

The postulate saying that every density of probability belongs to the space $\mathcal{E}'(\mathcal{M})$ implies that ϱ is of compact support. Many widely used distributions of probability **do not belong** to $\mathcal{E}'(\mathcal{M})$, e.g., the Gaussian distribution. We accept this limitation because the richness of mathematical properties of functionals from $\mathcal{E}'(\mathcal{M})$ provides a perfect opportunity to apply them in modeling of reality.

3. Physical background of formal series calculus

The fundamental difference between classical and quantum physics arises from the fact that observables and states in quantum mechanics depend on a special parameter – *the Planck constant* $\hbar$. Its crucial role is illustrated by the Heisenberg uncertainty principle for the position x and the canonically conjugated momentum p, which for series of independent measurements in classical physics is of trivial form

$$\Delta x \Delta p \geq 0$$

while in quantum mechanics one obtains

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

By Δ we denote the mean square deviation.

For technical reasons in quantum calculations it is convenient to represent observables by their expansions in power series with respect to $\hbar$

$$f \sim \sum_{l=-z}^{\infty} \hbar^l f_l.$$

Notice that at this stage we accept only a finite set of negative powers of $\hbar$.

This series representation usually simplifies considerations but it is the source of two serious problems. The first one is that there is no one to one mapping between smooth functions and their respective power series. The second difficulty is the loss of convergence. Therefore to deal with power series we need to develop a special method known as the *formal series calculus*.

Since foundations of the formal series calculus are purely mathematical, instead of the Planck constant $\hbar$ we will use a parameter λ. We assume that this parameter is real and positive.

At the beginning we extend the field of complex numbers $\mathbb{C}$, namely we introduce a field of formal series of complex numbers

$$\mathbb{C}[\lambda^{-1}, \lambda]] \ni c[[\lambda]] = \sum_{l=-z}^{\infty} \lambda^l c_l, \quad \forall\, l \quad c_l \in \mathbb{C}, \quad z \in \mathcal{N}. \tag{5}$$

A sequence $\{(\sum_{l=-z}^{\infty} \lambda^l c_l)_n\}_{n=1}^{\infty}$ of elements from the field $\mathbb{C}[\lambda^{-1}, \lambda]]$ is convergent to an element $\sum_{l=-z}^{\infty} \lambda^l c_{l0}$, if for every index l the sequence $\{(c_l)_n\}_{n=1}^{\infty}$ of complex numbers approaches c_{l0}.

The set of formal series of smooth functions $C^\infty[\lambda^{-1},\lambda]](\mathcal{M})$ being a stage for constituting the formal series calculus, consists of elements which are of the form

$$\varphi[[\lambda]] = \sum_{l=-z}^{\infty} \lambda^l \varphi_l, \quad \forall\, l \quad \varphi_l \in C^\infty(\mathcal{M}), \quad z \in \mathcal{N}. \tag{6}$$

In $C^\infty[\lambda^{-1},\lambda]](\mathcal{M})$ we define multiplication by scalars from the field $\mathbb{C}[\lambda^{-1},\lambda]]$, complex conjugation and multiplication of series. All of these operations are natural extensions of their $C^\infty(\mathcal{M})$ counterparts. Hence we quote only a formula expressing the product of series.

Multiplication of formal series being a straightforward generalisation of multiplication of functions can be written as

$$\sum_{l=-z}^{\infty} \lambda^l \varphi_l \bullet \sum_{k=-s}^{\infty} \lambda^k \psi_k = \frac{1}{\lambda^{z+s}} \sum_{l=0}^{\infty} \lambda^l \sum_{k=0}^{l} \varphi_{k-z} \psi_{l-k-s}. \tag{7}$$

The set of formal series with the $\bullet$ product constitutes a commutative ring $(C^\infty[\lambda^{-1},\lambda]](\mathcal{M}),\bullet)$.

Moreover, we say that the sequence $\{(\sum_{l=-z}^{\infty}\lambda^l\varphi_l)_n\}_{n=1}^{\infty}$ tends to a series $\sum_{l=-z}^{\infty}\lambda^l\varphi_{l0}$, if for every l the sequence $\{(\varphi_l)_n\}_{n=1}^{\infty}$ is convergent to the function φ_{l0} in the sense of convergence in the space of functions $C^\infty(\mathcal{M})$.

A partial derivative of a series $\sum_{l=-z}^{\infty}\lambda^l\varphi_l$ is calculated as

$$\frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1}q^1\cdots\partial^{m_{2r}}q^{2r}} \sum_{l=-z}^{\infty} \lambda^l \varphi_l := \sum_{l=-z}^{\infty} \lambda^l \frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1}q^1\cdots\partial^{m_{2r}}q^{2r}} \varphi_l$$

and its integral equals

$$\int_{\mathcal{M}} \left(\sum_{l=-z}^{\infty} \lambda^l \varphi_l \right) \omega^r := \sum_{l=-z}^{\infty} \lambda^l \int_{\mathcal{M}} \varphi_l\, \omega^r$$

providing all functions φ_l are summable.

4. States over the commutative ring $(C^\infty[\lambda^{-1},\lambda]](\mathcal{M}),\bullet)$

Let us start from a generalisation of action of any element $T \in \mathcal{E}'(\mathcal{M})$ on a formal series $\sum_{k=-z}^{\infty}\lambda^k\varphi_k$ from $C^\infty[\lambda^{-1},\lambda]](\mathcal{M})$. This generalisation is of the form

$$\left\langle T, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle := \sum_{k=-z}^{\infty} \lambda^k \left\langle T, \varphi_k \right\rangle \in \mathbb{C}[\lambda^{-1},\lambda]]. \tag{8}$$

To be able to talk about the states the three properties have to be satisfied. Reality of functional T means that implication holds

$$\sum_{k=-z}^{\infty} \lambda^k \overline{\varphi_k} = \sum_{k=-z}^{\infty} \lambda^k \varphi_k \Longrightarrow \left\langle T, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle \in \mathbb{R}[\lambda^{-1},\lambda]].$$

Normalisation is natural. It requires only extension of multiplication of functionals by numbers to multiplication by series from $\mathbb{C}[\lambda^{-1}, \lambda]]$.

The notion of nonnegativity is in conflict with the idea of formal series because on one hand we deal with specific real numbers, on the other hand we avoid the question about summability. We propose the following (compromising) definition of nonnegativity.

A generalised function $T \in \mathcal{E}'(\mathcal{M})$ is *nonnegative* if for every admissible value of the parameter λ and every finite series $\sum_{l=-z}^{s} \lambda^l \varphi_l$

$$\left\langle T, \sum_{k_1=-z}^{s} \lambda^{k_1} \overline{\varphi}_{k_1} \bullet \sum_{k_2=-z}^{s} \lambda^{k_2} \varphi_{k_2} \right\rangle \geq 0. \tag{9}$$

This formulation is stronger than the one proposed by Waldmann [6].

It seems to be natural that linear functionals over the ring $(C^\infty[\lambda^{-1},\lambda]](\mathcal{M}),\bullet)$ also may depend on λ. Let us first consider formal series of generalised functions of the form $\sum_{l=-s}^{\infty} \lambda^l T_l$. Their functional action is of the form

$$\left\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle := \frac{1}{\lambda^{s+z}} \sum_{u=0}^{\infty} \lambda^u \sum_{l=0}^{u} \Big\langle T_{l-s}, \varphi_{u-l-z} \Big\rangle. \tag{10}$$

It is required that all supports are contained in a common compact set. Notions of reality, nonnegativity and normalisation condition can be easily adapted to them.

Since we need the formal series calculus to deal with quantum problems, let us consider another set of formal series of functionals.

For systems represented on the phase space $\mathbb{R}^{2r}$ we know that quantum states are represented by the Wigner functions which may contain arbitrary negative powers of λ. Thus it seems to be natural that formal series of generalised functions

$$\sum_{k=1}^{\infty} \lambda^{-k} T_{-k} + \sum_{k=0}^{\infty} \lambda^k T_k$$

should be considered. Unfortunately, such extension is not possible because the functional action

$$\left\langle \sum_{k=1}^{\infty} \lambda^{-k} T_{-k} + \sum_{k=0}^{\infty} \lambda^k T_k, \sum_{l=-z}^{\infty} \lambda^l \varphi_l \right\rangle$$

is not well defined. This observation is probably the weakest point of proposed calculus.

5. States over the algebra $(C^\infty[\lambda^{-1}, \lambda]](\mathcal{M}), *)$

One of the consequences of the Heisenberg uncertainty relation is the fact that the product of quantum observables is noncommutative. Therefore to deal with quantum problems we need another method of multiplication of formal series. This is the so-called $*$ product. Its general form is

$$\varphi * \psi := \sum_{k=0}^{\infty} \lambda^k B_k(\varphi, \psi), \quad \forall\, k \;\; B_k(\varphi, \psi) \in C^{\infty}(\mathcal{M}). \tag{11}$$

We omit here the list of axioms imposed on $\mathbb{C}[\lambda^{-1}, \lambda]]$ – bilinear operators $B_k(\cdot,\cdot)$. This information can be found, e.g., in [4, 7, 8]. An extension of the $*$ product on formal series of functions is straightforward. The space of formal series equipped with the $*$ multiplication constitutes an algebra denoted as $(C^{\infty}[\lambda^{-1}, \lambda]](\mathcal{M}), *)$.

The trace in algebra $(C^{\infty}[\lambda^{-1}, \lambda]](\mathcal{M}), *)$ is of the form

$$\mathrm{Tr}\left(\sum_{k=-z}^{\infty} \lambda^k \varphi_k\right) := \frac{1}{\lambda^r}\int_{\mathcal{M}} \left(\sum_{k=-z}^{\infty} \lambda^k \varphi_k\right) \bullet t[[\lambda]]\, \omega^r,$$

where the series $t[[\lambda]] = \sum_{k=0}^{\infty} \lambda^k t_k$ is called *trace density.*

Since our goal is to introduce quantum states, i.e., some linear continuous functionals over the algebra $(C^{\infty}[\lambda^{-1}, \lambda]](\mathcal{M}), *)$, following Schwartz [5] we consider first functionals which are of the integral form.

$$\mathbb{C}[\lambda^{-1}, \lambda]] \ni \langle \psi, \varphi \rangle_* := \frac{1}{\lambda^r}\int_{\mathcal{M}} (\psi * \varphi) \bullet t[[\lambda]]\, \omega^r.$$

Notice that in general $\langle \psi, \varphi \rangle_* \neq \langle \psi, \varphi \rangle$.

However one can see that this new functional calculus is equivalent to the standard theory of generalised functions with an identification

$$\psi \sim T_{\psi}[[\lambda]] = \frac{1}{\lambda^r} t[[\lambda]] \bullet \sum_{l=0}^{\infty} \lambda^l T_{\psi\, l} \in \mathcal{E}'[\lambda^{-1}, \lambda]](\mathcal{M}), \text{ i.e.,}$$
$$\forall\, \varphi \in C^{\infty}(\mathcal{M}) \;\; \langle \psi, \varphi \rangle_* = \langle T_{\psi}[[\lambda]], \varphi \rangle.$$

Let us see what might be the meaning of states in terms of the $*$ formal series calculus.

Reality of a series $\sum_{l=-s}^{\infty} \lambda^l T_l$ means that if

$$\sum_{k=-z}^{\infty} \lambda^k \overline{\varphi_k} = \sum_{k=-z}^{\infty} \lambda^k \varphi_k$$

then there is

$$\left\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle_* = \overline{\left\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle_*}, \tag{12}$$

To discuss nonnegativity we need the notion of nonnegativity of a formal series of real numbers.

A formal series $\sum_{l=-z}^{\infty} \lambda^l c_l, \;\; \forall\, l \; c_l \in \mathbb{R}$ of real numbers is *nonnegative* if

$$\forall\, \lambda > 0\; \exists\, k \in \mathcal{N}\; \forall\, m > k \;\; \sum_{l=-z}^{m} \lambda^l c_l \geq 0.$$

It is disappointing that we again have to refer to values of sums but at this moment we have no idea how to introduce the notion of nonnegativity for formal series without a reference to numbers.

Applying this suggestion we say that the series $\sum_{l=-s}^{\infty} \lambda^l T_l$ is nonnegative if for every formal series of functions $\sum_{m=-z}^{\infty} \lambda^m \varphi_m$ the inequality

$$\left\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \sum_{m_1=-z}^{\infty} \lambda^{m_1} \overline{\varphi}_{m_1} * \sum_{m_2=-z}^{\infty} \lambda^{m_2} \varphi_{m_2} \right\rangle_* \geq 0$$

holds.

Finally the normalisation condition states that

$$\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \mathbf{1} \rangle_* = 1.$$

What is amazing when we test this list of properties for the most popular example of the $*$ product, i.e., the Moyal product at $\mathbb{R}^2$ [9, 10]

$$\varphi *_M \psi := \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1! n_2!} \left(-\frac{i\lambda}{2}\right)^{n_1} \left(\frac{i\lambda}{2}\right)^{n_2} \frac{\partial^{n_1+n_2} \varphi}{\partial p^{n_1} \partial q^{n_2}} \frac{\partial^{n_1+n_2} \psi}{\partial q^{n_1} \partial p^{n_2}}$$

we arrive at shocking conclusion that generalised functions with compact supports cannot be positive! This observation probably remains true for any local $*$ product. Therefore we deduce that states over formal series cannot be built in a way analogous to classical statistical physics.

6. Conclusions

As we can see, it is extremely difficult to introduce a coherent formal series calculus admitting quantum states. Two crucial facts – impossibility of building formal series of functionals with arbitrary large negative powers of λ and necessity of dealing with functionals with noncompact supports question whether formal series calculus can be successfully incorporated in quantum physics.

Thus the best solution would be to apply convergent expressions. Unfortunately, realisation of such a postulate requires a strict quantisation method which has not been formulated yet.

On the other hand the formal series are frequently useful. Thus at this moment we suggest a compromise – let us use them but simultaneously let us watch if calculations make sense.

References

[1] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States*, Cambridge University Press, Cambridge 2006.

[2] C.K. Zachos, D.B. Fairlie, T.L. Curtright (Eds.), *Quantum Mechanics in Phase Space*, World Scientific Series in 20th Century Physics, vol. 34, World Scientific, Singapore 2005.

[3] Y.S. Kim, M.E. Noz, *Phase Space Picture of Quantum Mechanics*, World Scientific, Singapore 1991.

[4] S. Waldmann, *Poisson-Geometrie und Deformationsquantisierung*, Springer-Verlag, Berlin 2007 (in German).

[5] L. Schwartz, *Méthodes mathématiques pour les sciences physiques*, Hermann, Paris 1965 (in French).

[6] S. Waldmann, *Rev. Math. Phys.* **17**, 15 (2005).

[7] B. Fedosov, *J. Diff. Geom.* **40**, 213 (1994).

[8] B. Fedosov, *Deformation Quantization and Index Theory*, Akademie Verlag, Berlin 1996.

[9] H.J. Groenewold, *Physica* **12**, 405 (1946).

[10] J.F. Plebański, M. Przanowski and J. Tosiek, *Acta Phys. Pol. B* **27**, 1961 (1996).

Jaromir Tosiek
Institute of Physics
Łódź University of Technology
Wólczanska 219
PL-90-924 Łódź, Poland
e-mail: tosiek@p.lodz.pl

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 147–154

Exact Lagrangian Submanifolds and the Moduli Space of Special Bohr–Sommerfeld Lagrangian Cycles

Nikolay A. Tyurin

Abstract. In previous papers we introduced the notion of special Bohr–Sommerfeld Lagrangian cycles on a compact simply connected symplectic manifold with integer symplectic form, and presented the main interesting case: compact simply connected algebraic variety with an ample line bundle such that the space of Bohr–Sommerfeld Lagrangian cycles with respect to a compatible Kähler form of the Hodge type and holomorphic sections of the bundle is finite. The main problem appeared in this way is singular components of the corresponding Lagrangian shadows (or Weinstein skeletons) which are hard to distinguish or resolve. In this note we avoid this difficulty presenting the points of the moduli space of special Bohr–Sommerfeld Lagrangian cycles by exact compact Lagrangian submanifolds on the complements $X \backslash D_\alpha$ modulo Hamiltonian isotopies, where D_α is the zero divisor of holomorphic section α. This correspondence is fair if the Eliashberg conjecture is true, stating that every smooth orientable exact Lagrangian submanifold is regular. In a sense our approach corresponds to the usage of gauge classes of Hermitian connections instead of pure holomorphic structures in the theory of the moduli space of (semi) stable vector bundles.

Mathematics Subject Classification (2010). 53D12, 53D50.

Keywords. Symplectic manifold, algebraic variety, ample divisor, Lagrangian submanifold, Bohr–Sommerfeld condition, Weinstein structure.

1. General theory

Consider (M, ω) – a compact simply connected symplectic manifold of dimension $2n$, endowed with a symplectic form of integer type, $[\omega] \in H^2(M, \mathbb{Z})$. Then there exists a prequantization datum – the pair (L, a), where $L \to M$ is a Hermitian

The author is partially supported by Laboratory of Mirror Symmetry NRU HSE, RF Government grant, ag. No. 14.641.31.0001.

line bundle and $a \in \mathcal{A}_h(L)$ is a Hermitian connection such that the curvature form $F_a = 2\pi i\omega$ (thus the first Chern class $c_1(L) = [\omega]$).

An n-dimensional submanifold $S \subset M$ is called *Lagrangian* iff the restriction $\omega|_S$ identically vanishes; S is called *Bohr–Sommerfeld* Lagrangian (or BS for short) iff the restriction $(L, a)|_S$ admits a covariantly constant section $\sigma_S \in \Gamma(L|_S)$, defined up to $\mathbb{C}^*$. For any chosen smooth section $\alpha \in \Gamma(M, L)$ we say that $S \subset M$ is special with respect to α Bohr–Sommerfeld Lagrangian cycles (or α-SBS for short) iff it is Bohr–Sommerfeld Lagrangian and the restriction $\alpha|_S = e^{ic} f \sigma_S$, where c is a real constant and f is a strictly positive real function on S. In the present paper we consider compact orientable Lagrangian submanifolds only.

It was already shown that the definition above can be reformulated in terms of calibrated Lagrangian geometry. For any smooth section $\alpha \in \Gamma(M, L)$ we define the complex-valued 1-form

$$\rho_\alpha = \frac{\langle \nabla_a \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \Omega^1_{\mathbb{C}}(M \backslash D_\alpha)$$

where $D_\alpha = \{\alpha = 0\} \subset M$ is the zeroset of α. This form satisfies the following properties: its real part is exact being $d(\ln |\alpha|)$, and the imaginary part is a canonical 1-form on the complement $M \backslash D_\alpha$ since $d(\operatorname{Im} \rho_\alpha) = 2\pi\omega$.

In these terms an n-dimensional submanifold $S \subset M$ is α-SBS Lagrangian if and only if the restriction $\operatorname{Im} \rho_\alpha|_S$ identically vanishes (the proof and details can be found in [1]).

Using this "calibrated reformulation" of the definition one proved that any Weinstein neighborhood $\mathcal{O}(S_0)$ of an α-SBS Lagrangian submanifold S_0 cannot contain any other α-SBS Lagrangian submanifold of the same type. It follows that a fixed α admits a discrete set of α-SBS Lagrangian submanifolds of the same topological type.

Recall that the situation stated above is the input of ALAG-program, proposed by A. Tyurin and A. Gorodentsev in [2]: starting with such (M, ω) they constructed certain moduli space of Bohr–Sommerfeld Lagrangian cycles of fixed topological type, denoted as $\mathcal{B}_S$. Such a moduli space is a Fréchet smooth infinite-dimensional real manifold, locally modeled by unobstructed isodrastic deformations of BS Lagrangian submanifolds. To define $\mathcal{B}_S = \mathcal{B}_S(\operatorname{top} S, [S])$ one has to fix the topological type of S and the homology class $[S] \in H_n(M, \mathbb{Z})$ of the corresponding BS submanifolds. Moreover, the BS-level can be shifted up, so one has a series of the moduli space $\mathcal{B}^k_S$ (details see in [2]).

Therefore in the situation presented above we can consider in the direct product $\mathcal{B}_S \times \mathbb{P}\Gamma(M, L)$ certain subset $\mathcal{U}_{\mathrm{SBS}}$ defined by the condition: pair $(S, p) \in \mathcal{U}_{\mathrm{SBS}}$ iff S is α-SBS Lagrangian submanifold where α corresponds to point p in the projectivized space (and of course it is possible to shift the BS-level, getting the corresponding subset in the direct product $\mathcal{B}^k_S \times \mathbb{P}\Gamma(M, L^k)$, but in the present text we leave aside the variation of BS-level).

This subset $\mathcal{U}_{\mathrm{SBS}}$ was studied in [1]; the main result is that the canonical projection $p : \mathcal{U}_{\mathrm{SBS}} \to \mathbb{P}\Gamma(M, L)$ has discrete fibers, has non degenerated differential in smooth points and projects $\mathcal{U}_{\mathrm{SBS}}$ to an open subset of the last projective space. As a corollary one establishes that $\mathcal{U}_{\mathrm{SBS}}$ admits a Kähler structure at smooth points. It seems to be interesting since we have started from pure symplectic situation and came to an object from the Kähler geometry.

This construction can be exploited in the Lagrangian approach to Geometric Quantization. The subset $\mathcal{U}_{\mathrm{SBS}}$ covers the moduli space $\mathcal{B}_S$ of BS Lagrangian cycles, thus it can be regarded as a "complexification" of the last one which looks like an alternative to the complexification $\mathcal{B}_S^{hw}$ given by the introduction of half weights in [2].

2. The case of algebraic varieties

Let X be a compact smooth simply connected algebraic variety which admits a very ample line bundle L; then it can be regarded as a special case of the situation presented above.

Indeed, fixing an appropriate Hermitian structure h on L one induces the corresponding Kähler form ω: any holomorphic section $\alpha \in H^0(X, L)$ in the presence of h defines the function $\psi_\alpha = -\ln|\alpha|_h$ on the complement $X \backslash D_\alpha$ which is a Kähler potential, therefore ω is given by $dId\psi_\alpha$, and the very ampleness condition ensures that whole X is covered by the complements to divisors from the complete linear system $|L| = \mathbb{P}H^0(X, L)$, so ω is globally defined in X, see [3].

Thus one can consider (X, L) as a symplectic manifold with integer symplectic form, and L, the prequantization line bundle, is automatically endowed with a prequantization connection a, compatible with the holomorphic structure on the bundle, when h is fixed. For a holomorphic section α one has $\nabla_a \alpha = \partial_a \alpha$ and consequently the form ρ_α has type (1, 0) with respect to the complex structure. Then one can deduce that the SBS condition with respect to a holomorphic section is equivalent to the following condition: a Lagrangian submanifold $S \subset X$ is α-SBS if and only if it is invariant under the flow generated by the gradient vector field $\operatorname{grad}\psi_\alpha$ (see [4]).

It is well known in algebraic geometry fact: the complement $X \backslash D_\alpha$, described above, is an example of the Stein variety, and since we would like to study Lagrangian geometry of these complements we must follow the key points of the program "From Stein to Weinstein and back", see [5]. The situation we are studying here must be regarded in the framework of the Weinstein manifolds and Weinstien structures, see [5] and [6]. Indeed, the gradient vector field $\operatorname{grad}\psi_\alpha$, which is obviously Liouville, and the function ψ_α give us a Weinstein structure (of course, it just reflects the fact that $X \backslash D_\alpha$ is Stein).

Since we claim that a Lagrangian $S \subset X \backslash D_\alpha$ is α-SBS if and only if it is stable with respect to the gradient flow of ψ_α it follows that such an S must be contained by the base set $B_\alpha \subset X \backslash D_\alpha$ defined as the union of (1) finite critical points of ψ_α and (2) finite trajectories of the gradient flow. Now we can translate our α-SBS condition to the language of Weinstein manifolds and structures: a Lagrangian submanifold $S \subset X$ is α-SBS iff it is a component of the Lagrangian skeleton defined by the Weinstein structure given by $(\operatorname{grad}\psi_\alpha, \psi_\alpha)$ on the complement $X \backslash D_\alpha$. Of course this Weinstein structure is special since it corresponds to a holomorphic section.

Remark. In the previous texts [4] we use the term "Lagrangian shadow of ample divisor" for the Lagrangian components of the Lagrangian skeleton (or Weinstein skeleton), since we would like to emphasize the fact that the corresponding Lagrangian components arise for any very ample divisor; in the theory of Weinstein manifolds which covers much wider situation than our complements $X \backslash D_\alpha$ one says that such a Lagrangian submanifold is *regular*. Below we use this parallel for the modified definition of moduli space of special Bohr–Sommerfled Lagrangian cycles.

The old definition (see [4]) we have tried to exploit for the construction of certain moduli space of SBS Lagrangian cycles over algebraic varieties used to be the following. Take the canonical projection $p : \mathcal{U}_{\mathrm{SBS}} \to \mathbb{P}\Gamma(M, L)$ to the second direct summand from Section 1. Then in the present situation when $M = X$ – an algebraic variety we have a finite-dimensional projective subspace $\mathbb{P}H^0(X, L) \subset \mathbb{P}\Gamma(X, L)$ which corresponds to holomorphic sections. It is not hard to see that the preimage $\mathcal{M}_{\mathrm{SBS}} = p^{-1}(\mathbb{P}H^0(X, L))$ must be finite (and we have proved it for smooth Bohr–Sommerfeld submanifolds in [4]), and we would like to understand it as the "moduli space" of Special Bohr–Sommerfeld Lagrangian cycles.

But the great problem appears with this definition of the moduli space since the components of the Lagrangian skeleton B_α are very far from being smooth Lagrangian submanifolds (or even cycles), and the best case of arboreal singularities (see [9]) doesn't help us in our program. It follows that strictly speaking our coarse "moduli space" must be empty in major cases, and the framework of algebraic geometry does admit no variational freedom to resolve this trouble. In the simplest case, when $H_n(X \backslash D_\alpha, \mathbb{Z}) = \mathbb{Z}$ for generic smooth element D_α of the complete linear system $|L|$ the moduli space $\mathcal{M}_{\mathrm{SBS}}$ can be however correctly defined, as it was done in [4], but in more geometrically interesting cases we face great problem in this way: we must either present a strong theory of desingularization of the components of Lagrangian skeleta doing it however in concordance with the technical details of ALAG or find a different definition of special Bohr–Sommerfeld cycles with respect to holomorphic sections such that these new special submanifolds should be automatically smooth.

Theory of Weinstein manifolds (see [6]) hints how one can avoid these difficulties.

3. "Desingularizing" the definition

In the situation of the previous section the Kähler potential ψ_α defines the structure of the Weinstein manifold on $X \backslash D_\alpha$, given by 1-form $\lambda_\alpha = I d\psi_\alpha$ and ψ_α itself (see [5]); then we can study *exact* compact orientable Lagrangian submanifolds in $X \backslash D_\alpha$. Recall that

Definition 1. A Lagrangian submanifolds $S \subset X \backslash D_\alpha$ is exact if the restriction $\lambda_\alpha|_S$ is an exact form.

Remark that any such an exact S must be automatically Bohr–Sommerfeld in whole X with respect to the corresponding prequantization data. Moreover, we can introduce certain condition on Bohr–Sommerfeld Lagrangian submanifolds in X which is equivalent to the exactness condition on the complement $X \backslash D_\alpha$.

Namely, let $X \supset D_\alpha$ be as above, and the corresponding symplectic form ω evidently represents the cohomology class Poincaré dual to $[D_\alpha] \in H_{2n-2}(X, \mathbb{Z})$. Then we say that a Lagrangian submanifold $S \subset X$ is D-*exact* with respect to D_α (or simply D-exact if the submanifold is clear from the context) iff $D_\alpha \cap S = 0$ and for any oriented loop $\gamma \subset S$ and any compatible oriented disc $K_\gamma \subset X$, bounded by γ, the topological sum of the intersection points $D_\alpha \cap K_\gamma$ equals to the symplectic area of K_γ (note that if K_γ intersects D_α non transversally then we can deform it to have transversal intersection).

Note that the last definition is rather universal: we don't need any Hermitian structure on L or complex structure on X, the property of D-exactness depends on the symplectic form and $2n-2$-dimensional submanifold which represents the homology class, Poincaré dual to $[\omega]$.

Now it is not hard to see that

Proposition 1. *A Lagrangian submanifold $S \subset X \backslash D_\alpha$ is exact with respect to λ_α if and only if S is D-exact with respect to D_α.*

Proof. The proof is straightforward: the calculation of the integral $\int_\gamma \lambda_\alpha$ using the Stocks formula in the presence of poles which correspond to the intersection points in $D_\alpha \cap K_\gamma$ leads to the desired result (note, that S is exact iff the integral vanishes for any loop $\gamma \subset S$). $\square$

In [6] one presents the list of open problems stated in the theory of Weinstein manifolds; and one of these problems hints how the definition of special Bohr–Sommerfeld Lagrangian submanifolds can be modified. Namely the Problem 5.1 from [6] asks: are there non-regular exact Lagrangian submanifolds in $X \backslash D_\alpha$? The *Eliashberg conjecture* suggests the negative answer to this problem. If this conjecture is true (at least not in general situation but in our special case $X \backslash D_\alpha$ when X is an algebraic variety and D_α is a very ample divisor) then it hints how we can "desingularize" the definition of the "coarse" moduli space given above. Moreover, even if the conjecture is not true we still have a correct definition of certain moduli space which we still understand as *modified moduli space of Special Bohr–Sommerfeld cycles.*

Definition 2. The space of pairs $(\{S\}, D_\alpha)$ where $D_\alpha \in |L|$ is an element of the complete linear system $|L| = \mathbb{P}H^0(X, L))$, and $\{S\}$ is a class of compact smooth exact with respect to λ_α Lagrangian submanifolds modulo Hamiltonian isotopies in $X \backslash D_\alpha$, where one fixes the topological type $\operatorname{top} S$ and homology class $[S] \in H_n(X, \mathbb{Z})$, we call the modified moduli space of SBS Lagrangian cycles and denote as $\widetilde{\mathcal{M}}_{\mathrm{SBS}} = \widetilde{\mathcal{M}}_{\mathrm{SBS}}(c_1(L), \operatorname{top} S, [S])$.

For this modified moduli space we have the following

Proposition 2. *The modified moduli space $\widetilde{\mathcal{M}}_{SBS}$ is a smooth open Kähler manifold.*

Proof. The proof is straightforward: since the space $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ admits the forgetful map $p : \widetilde{\mathcal{M}}_{\mathrm{SBS}} \to |L|$, we can describe the smooth structure of $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ and the desired Kähler structure as the lifting from the projective space $|L|$. Indeed, first we prove that the fibers of p are discrete and that the differential of this map is non degenerated (the arguments are essentially the same as in [1]: we study the local picture over a Weinstein neighborhood of a fixed D-exact $S \subset M$). Moreover, the standard neighborhoods covering of $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ are given by Proposition 1 above: taking any D-exact S_0 for certain $D_\alpha = D_0$ we can move the divisor until the deformation D_t for some $t = T$ "reaches" S_0, so before we get non trivial intersection $D_T \cap S_0 \neq \emptyset$ the exact Lagrangian submanifold S_0 states to be exact for all $D_t, t < T$. Thus the differential dp in this setup used to be identical map. $\square$

The Eliashberg conjecture covers the following statement: every class from the group $H_n(X \backslash D_\alpha, \mathbb{Z})$ contains at most one realization by compact smooth exact Lagrangian submanifold modulo Hamiltonian isotopies in $X \backslash D_\alpha$. It follows that the map $p : \widetilde{\mathcal{M}}_{\mathrm{SBS}} \to |L|$ is not a ramification. The projective space $|L|$ is stratified by the rank of the group $H_n(X \backslash D_\alpha, \mathbb{Z})$: over a generic point, for smooth D_α, the rank is maximal, and the structure of the fibers over these points in $|L|$ is the same; then when D_α degenerates to a singular divisor D_s the group turns to be smaller, and no fusion for the classes appears. We illustrate this phenomenon by the simple example below.

Let us illustrate the story by the example which has appeared several times in the previous texts, see [4]. Take $X = \mathbb{CP}^1$ and consider $L = \mathcal{O}(3)$. Study the situation for certain concrete holomorphic section f.e. for the section defined by the polynomial $P_3 = z_0^3 - z_1^3$. It vanishes at three roots of unity p_1, p_2, p_3 which become poles for the function $\psi = -\ln|P_3|$; the last one has exactly 5 finite critical points – 2 local minima $m_1 = [1 : 0], m_2 = [0 : 1]$, and three saddle points s_1, s_2, s_3 at the roots of -1. The base set consists of three lines γ_i each of which joins m_1 and m_2 passing through s_i. Totally we get non smooth simple loops only in the base set: each closed loop is formed by two lines γ_i, γ_j, and at the points m_1, m_2 the loop has corners. Therefore if we are looking for the "old version" of the moduli space $\mathcal{M}_{\mathrm{SBS}}$ we must specialize what singular loops are allowed in our situation. However in this case the specialization can be done: we may say that a singular loop is allowed if it can be transformed by a small deformation to a smooth Bohr–Sommerfeld loop. Then one gets exactly three simple elements for the moduli space.

But our new "desingularized" definition of the moduli space works much better: we claim that there are exactly three smooth exact closed loops on the complement $\mathbb{CP}^1 \backslash \{p_1, p_2, p_3\}$ up to Hamiltonian isotopy. Indeed, for each zero p_i we can take a smooth loop γ_i surrounding p_i only and then "blow" it to bound a disc of symplectic area $\frac{1}{3}$, – it is the desired one. Therefore the moduli space of special Bohr–Sommerfeld Lagrangian cycles $\widetilde{\mathcal{M}}_{\mathrm{SBS}}(S^1, 0, \mathcal{O}(3))$ is organized as follows: over generic point of $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(3)) \backslash Q_4$ where Q_4 is the discriminant of cubic equation, so a hypersurface of degree 4, one has three preimages. But the ramification doesn't appear over the discriminant locus: if two points p_1 and p_2 collide then the corresponding loops around these points collide as well but the resulting loop γ_{12} is Hamilton isotopic to γ_3! Therefore we don't have any ramification. Totally the modified moduli space $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ is isomorphic to the following open variety: take in the direct product $\mathbb{CP}^1 \times \mathbb{P}(H^0(\mathbb{CP}^1, \mathcal{O}(3)))$ with homogeneous coordinates $[x_0 : x_1], [z_0 : \cdots : z_3]$ the hypersurface $Y = \{z_0 x_0^3 + z_1 x_0^2 x_1 + z_2 x_0 x_1^2 + z_3 x_1^3 = 0\}$ and cut off the ramification divisor $\Delta \subset Y$ with respect to the canonical projection $\pi : Y \to \mathbb{P}(H^0(\mathbb{CP}^1, \mathcal{O}(3)))$, then the moduli space $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ is isomorphic to $Y \backslash \Delta$. Note that we get as the moduli space again the familiar picture: algebraic variety minus very ample divisor!

Now we have to explain why we understand the moduli space $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ as a refinement of the previous "coarse" moduli space $\mathcal{M}_{\mathrm{SBS}}$? The relation is based on the Eliashberg conjecture: if it is true then a class of exact compact smooth Lagrangian submanifolds corresponds to a Lagrangian submanifold regular with respect to a Weinstein structure on the same affine variety $X \backslash D_\alpha$. Suppose, that the last Weinstein structure is defined by a smooth section $\alpha \in \Gamma(M, L)$, and there is a family of Weinstein structures, which joins our given structure and the structure defined by α. Then we should have a family of exact Lagrangian submanifolds which starts at our given exact one and reaches the corresponding component (singular, of course) of the Weinstein skeleta. This component presents an element of the coarse moduli space $\mathcal{M}_{\mathrm{SBS}}$, and totally it should give a one-to-one correspondence between elements of $\mathcal{M}_{\mathrm{SBS}}$ and $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$. Note that if the Lagrangian shadow of D_α contains a smooth component, then it is automatically exact and every its small isodrastic deformation lies in the corresponding class of exact Lagrangian submanifolds.

In a sense the presented passage from the components of skeleton to exact Lagrangian submanifolds looks like the standard reduction from $\bar{\partial}$-operators to Hermitian connections in the theory of stable holomorphic vector bundles, see [7]. Indeed, since the quotient space of $\bar{\partial}$ operators modulo locally non compact gauge group is topologically extremely complicated one realizes the holomorphic structures by the gauge classes of Hermitian connections.

The realization of special Bohr–Sommerfeld Lagrangian cycles presented here via D-exact Lagrangian submanifolds modulo Hamiltonian isotopies makes it possible to realize the following "mirror symmetry dream": in [8] one claimed that Lagrangian submanifolds should correspond to vector bundles. This conjectured duality can be realized using the modified moduli space of SBS Lagrangian submanifolds as follows: consider in our given algebraic variety another Lagrangian submanifold $S_0 \subset X$. Then for any point of the moduli space $(\{S\}, D) \in \widetilde{\mathcal{M}}_{\mathrm{SBS}}$ we can take the vector space $HF(S_0; S, \mathbb{C})$ of the Floer cohomology of the pair S_0, S, where S is a smooth D-exact Lagrangian submanifold, representing the class $\{S\}$. Since the Floer cohomology is stable with respect to Hamiltonian isotopies, the vector space doesn't depend on the particular choice of S; moreover, since the moduli space $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ is locally generated by specified Hamiltonian isotopies this implies that globally over $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$ the vector spaces are combined into a complex vector bundle, which we denote as $\mathcal{F}_{S_0}$. The smoothness of the representative S is important here.

Thus we get a functor from the space of Lagrangian submanifolds in X to the set of complex vector bundles on $\widetilde{\mathcal{M}}_{\mathrm{SBS}}$; it is a particular realization of the ideas from [8].

Acknowledgment

I cordially thank Professor Ya. Eliashberg, who greatly inspired me in these humble studies.

References

[1] Nik. Tyurin, *"Special Bohr–Sommerfeld Lagrangian submanifolds"*, Izvestiya: Mathematics, 2016, 80:6, pp. 274–293;

[2] A. Gorodentsev, A. Tyurin, *"Abelian Lagrangian Algebraic Geometry"*, Izvestiya: Mathematics, 2001, 65:3, pp. 437–467;

[3] P. Griffits, J. Harris, *"Principles of algebraic geometry"*, NY, Wiley, 1978;

[4] Nik. Tyurin, arXiv:1508.06804, arXiv:1601.05975, arXiv:1609.00633;

[5] K. Cieliebak, Ya. Eliashberg, *"From Stein to Weinstein and Back-Symplectic geometry of Affine Complex Geometry"*, Colloqium Publications Vol. 59, AMS (2012);

[6] Ya. Eliashberg, *"Weinstein manifolds revisited"*, arXiv: 1707.03442;

[7] S. Donaldson, P. Kronheimer, *"The geometry of four manifolds"*, Clarendon Press, Oxford, 1990;

[8] A. Tyurin, *"Geometric quantization and Mirror symmetry"*, arXiv:math/9902027v.1;

[9] D. Nadler, *"Arboreal singularities"*, arXiv:1309.4122.

Nikolay A. Tyurin
Bogolyubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research (Dubna)

and

Laboratory of Mirror Symmetry and Automorphic Forms
NRU High School of Economics (Moscow), Russia
e-mail: ntyurin@theor.jinr.ru

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 155–164

Star Exponentials in Star Product Algebra

Akira Yoshioka

Dedicated to the memory of Syed Twareque Ali

Abstract. A star product is an associative product for certain function space on a manifold, which is given by deforming a usual multiplication of functions. The star product we consider is given on $\mathbb{C}^n$ in non-formal sense. In the star product algebra we consider exponential elements, which are called star exponentials. Using star exponentials we construct star functions, which are regarded as sections of star algebra bundle over a space of complex matrices. In this note we give a brief review on star products.

Mathematics Subject Classification (2010). Primary 53D55; Secondary 81D30.

Keywords. Star exponential, star product.

1. Star products

The origin of star products can be traced back to Weyl [8], Wigner [9], Moyal [3], related to quantum mechanics. In 1970's, Bayen–Flato–Fronsdal–Licherowicz–Sternheimer [1] gave a concept of deformation quantization or star product, where formal star products are discussed. Formal means that the deformation is constructed in formal power series with respect to the deformation parameter. Many results are published with various applications by means of formal deformation quantization, which is a very general concept and its existence on any Poisson manifold is proved by M. Kontsevich ([2]).

A star product we consider in this note is a star product for certain functions on $\mathbb{R}^n$ or $\mathbb{C}^n$. The star product on $\mathbb{R}^n$ or $\mathbb{C}^n$ can be considered also in non-formal sense, for example we can consider non formal star products for polynomials. We introduce a family of star products which contains noncommutative star products, and also commutative star products. This note is on this product and its extension.

1.1. Definition of star products

First we introduce a biderivation acting on functions as follows.

This work was supported by JSPS KAKENHI Grant Number JP15K04856.

Biderivation. Let Λ be an arbitrary $n \times n$ complex matrix. We then consider a biderivation

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \ldots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \ldots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$$

where $(w_1, \ldots, w_n)$ are the coordinates of $\mathbb{C}^n$. Here the over left (resp. right) arrow means that the derivative $\overleftarrow{\partial}$ (resp. $\overrightarrow{\partial}$) acts to the left (resp. right) function, namely,

$$f\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}g = f\left(\sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}\right)g = \sum_{k,l=1}^{n} \Lambda_{kl}\,\partial_{w_k} f\,\partial_{w_l}\,g.$$

Since Λ is a constant matrix, we can easily calculate the power of the biderivation, for example

$$f(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w})^2 g = \sum_{k_1,k_2,l_1,l_2=1}^{n} \Lambda_{k_1 l_1}\,\Lambda_{k_2 l_2}\,\partial_{w_{k_1}}\,\partial_{w_{k_2}}\,f\,\partial_{w_{l_1}}\,\partial_{w_{l_1}}\,g.$$

Star product. Now for functions f, g we define a star product $f *_{\Lambda} g$ by means of the power series of the above biderivation such that

Definition 1.

$$\begin{aligned} f *_{\Lambda} g &= f \exp\tfrac{i\hbar}{2}\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) g = f\sum_{k=0}^{\infty}\tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^k g \\ &= fg + \tfrac{i\hbar}{2}f\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)g + \cdots + \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k f\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^k g + \cdots \end{aligned}$$

where $\hbar$ is a positive parameter.

Then we see easily

Theorem 2. *For an arbitrary Λ, the star product $*_{\Lambda}$ is well defined on polynomials, and is associative.*

Remark 3.

(i) The star product $*_{\Lambda}$ is a generalization of the well-known products in physics. For example suppose $n = 2m$ and if we put $\Lambda = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (blockwise), then we have the Moyal product, and similarly we have the normal product for $\Lambda = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)$, and the anti-normal product for $\Lambda = \left(\begin{smallmatrix} 0 & -2 \\ 0 & 0 \end{smallmatrix}\right)$, respectively.

(ii) If Λ is a symmetric matrix, the star product $*_{\Lambda}$ is commutative. Furthermore, if Λ is a zero matrix, then the star product is nothing but a usual commutative product.

1.2. Equivalence, Star product algebra bundle and flat connection

Equivalence. Let Λ be an arbitrary $n \times n$ complex matrix. Then $(\mathbb{C}[w], *_{\Lambda})$ is an associative algebra where $\mathbb{C}[w]$ is the set of complex polynomials of the coordinate system $w = (w_1, w_2, \ldots, w_n)$. The algebraic structure of $(\mathbb{C}[w], *_{\Lambda})$ depends only on the skewsymmetric part of Λ. Namely, let Λ_1, Λ_2 be $n \times n$ complex matrices with common skew-symmetric part. Then we have the decomposition

$$\Lambda_1 = \Lambda_- + K_1, \quad \Lambda_2 = \Lambda_- + K_2,$$

where Λ_- is a skew-symmetric matrix and K_1, K_2 are symmetric matrices. Then we have

Theorem 4. *The algebras* $(\mathbb{C}[u,v], *_{\Lambda_1})$ *and* $(\mathbb{C}[u,v], *_{\Lambda_2})$ *are isomorphic with an isomorphism* $I_{K_1}^{K_2} : (\mathbb{C}[u,v], *_{\Lambda_1}) \to (\mathbb{C}[u,v], *_{\Lambda_2})$ *given by the power series of the differential operator* $\partial_w(K_2 - K_1)\partial_w$ *such that*

$$I_{K_1}^{K_2}\ (f) = \exp\left(\tfrac{i\hbar}{4}\partial_w(K_2 - K_1)\partial_w\right)\ (f) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\left(\tfrac{i\hbar}{4}\right)^n \left(\partial_w(K_2 - K_1)\partial_w\right)^n f$$

where $\partial_w(K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}$.

For star products $*_{\Lambda_k}$, $k = 1, 2, 3$ with common skew-symmetric part of Λ_k, a direct calculation gives

Theorem 5. *The isomorphisms satisfy the following chain rule:*

(i) $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = Id$, (ii) $\left(I_{K_1}^{K_2}\right)^{-1} = I_{K_2}^{K_1}$

Star product algebra bundle and flat connection. Let us fix a skew-symmetric matrix Λ_- and consider a family of matrices $\{\Lambda = \Lambda_- + K\}$ with common skew-symmetric part Λ_- where K denotes its symmetric part. Then, by the above theorems we have a family of star products $\{*_{\Lambda}\}$ parameterized by $\{K\}$ whose elements are mutually isomorphic, and since $*_{\Lambda}$ depends only on the symmetric part K we write as $*_{\Lambda} = *_{K}$.

Here we regard this family of star products in the following way: we have an associative algebra $(\mathcal{P}, *)$ determined by Λ_- such that an each algebra $(\mathbb{C}[w], *_{K})$ of the family is regarded as a local expression of $(\mathcal{P}, *)$ at K. Each element $p \in \mathcal{P}$ has a polynomial expression at every K, which is denoted by $: p :_{K}$. Due to the previous theorem of the chain rules of $I_{K_1}^{K_2}$, we have a geometric picture: we have an algebra bundle over the space of symmetric matrices $\pi : \cup_K(\mathbb{C}[w], *_{K}) \to \mathcal{S} = \{K\}$ such that the fiber at K is the algebra $\pi^{-1}(K) = (\mathbb{C}[w], *_{K})$. The bundle has a flat connection ∇ and the element $p \in (\mathcal{P}, *)$ is regarded as a parallel section of the bundle and $: p :_{K}$ is the value at K.

This is a simple translation of the equivalence among the star product algebras. However, this picture plays an important role when we consider star exponentials and star functions below.

2. Star exponential

Now we consider general star product $*_{\Lambda}$, and consider exponential elements of polynomials in star product algebras.

Idea of definition. For a polynomial H of the star product algebra $(\mathbb{C}[w], *_{\Lambda})$, we want to define a star exponential

$$e_{*_{\Lambda}}^{t\frac{H}{\mathrm{i}\hbar}} = \sum_n \frac{t^n}{n!} \left(\tfrac{H}{\mathrm{i}\hbar}\right)^n_{*_{\Lambda}}$$

where $\left(\frac{H}{\mathrm{i}\hbar}\right)^n_{*_{\Lambda}}$ is an nth power of $\frac{H}{\mathrm{i}\hbar}$ with respect to the star product $*_{\Lambda}$. However, the expansion $\sum_n \frac{t^n}{n!} \left(\frac{H}{\mathrm{i}\hbar}\right)^n_{*_{\Lambda}}$ is not convergent in general, and then we consider a star exponential by means of a differential equation.

Definition 6. The star exponential $e_{*_{\Lambda}}^{t\frac{H}{\mathrm{i}\hbar}}$ is given as a solution of the differential equation

$$\tfrac{d}{dt} F_t = \tfrac{H}{\mathrm{i}\hbar} *_{\Lambda} F_t, \quad F_0 = 1.$$

2.1. Star exponential of linear and quadratic polynomials

We are interested in the star exponentials of linear, and quadratic polynomials. For these, we can solve the differential equation explicitly.
Linear case. We denote a linear polynomial by $\sum_{j=1}^n a_j w_j = \langle \boldsymbol{a}, \boldsymbol{w} \rangle$, $a_j \in \mathbb{C}$. This case naive expansion $\sum_n \frac{t^n}{n!} \left(\frac{\langle \boldsymbol{a}, \boldsymbol{w} \rangle}{\mathrm{i}\hbar}\right)^n_{*_{\Lambda}}$ is convergent. Actually we see directly that the nth power with respect to $*_{\Lambda}$ is

$$\langle \boldsymbol{a}, \boldsymbol{w} \rangle^n_{*_{\Lambda}} = \sum_{k=0}^{[n/2]} \tfrac{1}{k!} \left(\tfrac{i\hbar}{4} \boldsymbol{a}\Lambda\boldsymbol{a}\right)^k \frac{n!}{(n-2k)!} \langle \boldsymbol{a}, \boldsymbol{w} \rangle^{n-2k}$$

where $\boldsymbol{a}\Lambda\boldsymbol{a} = \sum_{ij} \Lambda_{ij} a_i a_j$ and the expansion is convergent. Then we have

Proposition 7. *For* $\sum_j a_j w_j = \langle \boldsymbol{a}, \boldsymbol{w} \rangle$

$$e_{*_{\Lambda}}^{t\langle \boldsymbol{a}, \boldsymbol{w} \rangle/(\mathrm{i}\hbar)} = e^{t^2 \boldsymbol{a}\Lambda\boldsymbol{a}/(4\mathrm{i}\hbar)} e^{t\langle \boldsymbol{a}, \boldsymbol{w} \rangle/(\mathrm{i}\hbar)} = e^{t^2 \boldsymbol{a}K\boldsymbol{a}/(4\mathrm{i}\hbar)} e^{t\langle \boldsymbol{a}, \boldsymbol{w} \rangle/(\mathrm{i}\hbar)}$$

where K is the symmetric part of Λ.

Thus the star exponentials are analytic and satisfy the exponential law with respect to the parameter t. By direct calculation we see

Proposition 8. *The star product of the star exponentials is convergent and it holds*

$$e_{*_{\Lambda}}^{\langle \boldsymbol{a}, \boldsymbol{w} \rangle/(\mathrm{i}\hbar)} *_{\Lambda} e_{*_{\Lambda}}^{\langle \boldsymbol{b}, \boldsymbol{w} \rangle/(\mathrm{i}\hbar)} = e^{\boldsymbol{a}(\Lambda_-)\boldsymbol{b}/(2\mathrm{i}\hbar)} e_{*_{\Lambda}}^{\langle \boldsymbol{a}+\boldsymbol{b}, \boldsymbol{w} \rangle/(\mathrm{i}\hbar)}.$$

Thus star exponentials of linear polynomials form a group.

For the linear case, the intertwiners are convergent. Namely, if we write the decomposition as $\Lambda = \Lambda_- + K_1$ we have

Proposition 9. *For any symmetric matrices K_1, K_2, the intertwiner*

$$I_{K_1}^{K_2} = \sum_{n=0}^{\infty} \tfrac{1}{n!} \left(\tfrac{\mathrm{i}\hbar}{4}\right)^n (\partial_w (K_2 - K_1) \partial_w)^n$$

is convergent for a star exponential of linear polynomial and satisfies

$$I_{K_1}^{K_2}\,(e_{*_\Lambda}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}) = e_{*_{\Lambda'}}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}, \quad (\Lambda' = \Lambda_- + K_2).$$

Remark 10. By the above propositions, similarly as polynomial case, for a fixed Λ_- the family of groups $\{e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; a \in \mathbb{C}^n\}_{K\in\mathcal{S}}$ determines a group $\mathcal{G}$. Also we have a group bundle $\pi : \cup_K\{e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; a \in \mathbb{C}^n\} \to \mathcal{S}$ such that the each fiber is the group $\pi^{-1}(K) = \{e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; a \in \mathbb{C}^n\}$. And an element of $\mathcal{G}$ is regarded as a parallel section denoted by $e_*^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}$ of this bundle and a value at K is given by $: e_*^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)} :_K = e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)} = e^{\boldsymbol{a}K\boldsymbol{a}/(4\mathrm{i}\hbar)+\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}$

Quadratic case. For simplicity of formula, we consider the case where Λ is a $2m \times 2m$ complex matrices with the skew symmetric part $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proposition 11. *For a quadratic polynomial $Q = \langle \boldsymbol{w}A, \boldsymbol{w}\rangle$ where A is a $2m \times 2m$ complex symmetric matrix, we have*

$$e_{*_\Lambda}^{t(Q/\mathrm{i}\hbar)} = \frac{2^m}{\sqrt{\det(I-\kappa+e^{-2t\alpha}(I+\kappa))}}e^{\frac{1}{\mathrm{i}\hbar}\langle \boldsymbol{w}\frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J,\boldsymbol{w}\rangle}$$

where $\kappa = KJ$, $\alpha = AJ$ and K is the symmetric part of Λ.

Remark 12. The star exponentials of quadratic polynomials have branching, essential singularities, and also satisfy exponential law with respect to the parameter t whenever they are defined. From these singularities we are trying to derive relations for commutative or noncommutative algebras.

Proposition 13. *We have an explicit formula of the product of star exponentials of quadratic polynomials which contains a square root.*

$$e_{*_\Lambda}^{\langle \boldsymbol{w}A_1,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} *_\Lambda e_{*_\Lambda}^{\langle \boldsymbol{w}A_2,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} = \frac{1}{\sqrt{\det(1-\alpha(A_1,A_2))}}e^{\frac{1}{\mathrm{i}\hbar}\langle \boldsymbol{w}\frac{1}{1-\alpha(A_1,A_2))}A_3(A_1,A_2),\boldsymbol{w}\rangle}$$

where $\alpha(A_1, A_2)$, $A_3(A_1, A_2)$ are certain matrix-valued functions of A_1, A_2 which are explicitly written by means of Cayley transforms of A_1, A_2.

Hence the product is defined when $\det(1-\alpha(A_1, A_2)) \neq 0$ *and associativity holds when $\{A\}$ are sufficiently small. Thus star exponentials of quadratic polynomials form a group-like object, or local group.*

For a quadratic case, since the intertwiner is a parallel transport of a section, we can obtain the intertwiner by solving a certain differential equation. If we write the decomposition as $\Lambda = \Lambda_- + K_1$ we have

Proposition 14. *For any symmetric matrix K_2, the intertwiner $I_{K_1}^{K_2}$ for a star exponential of quadratic polynomial is given as*

$$I_{K_1}^{K_2}\,(e_{*_\Lambda}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}) = \frac{1}{\sqrt{\det(1-\beta(A)(K_2-K_1))}}e_{*_{\Lambda'}}^{\frac{1}{\mathrm{i}\hbar}\langle \boldsymbol{w}\frac{1}{1-\beta(A)(K_2-K_1)}\beta(A),\boldsymbol{w}\rangle}$$

where $\beta(A)$ is a certain matrix-valued function of A and $\Lambda' = \Lambda_- + K_2$.

Remark 15. By the above propositions, similarly as linear case, for a fixed Λ_- the family of group-like objects $\{e_{*_K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}; A \text{ symmetric}\}_{K\in\mathcal{S}}$ determines a group-like object $\mathcal{Q}$.

Also we have a group-like object bundle $\pi : \cup_K\{e_{*_K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}; A \text{symmetric}\} \to \mathcal{S}$ such that the each fiber is $\pi^{-1}(K) = \{e_{*_K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}; A \text{ symmetric}\}$. And an element of $\mathcal{Q}$ is regarded as a parallel section denoted by $e_*^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}$ of this bundle, and a value at K is given by : $e_*^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(i\hbar)} :_K = e_{*_K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}$

2.2. Star functions

By the same way as in the ordinary exponential functions, we can obtain several noncommutative or commutative functions using star exponentials, which we call *star functions.* As is stated in the previous sections, these star functions are given as parallel sections $\mathcal{G}$ or $\mathcal{Q}$ of the group bundle or the group-like object bundle over $\mathcal{S}$, respectively. In this subsection we show some concrete examples of star functions. For more details see Omori–Maeda–Miyazaki–Yoshioka [4, 5].

2.2.1. Linear case. Here we show examples of the simplest case using star product of one variable. We consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2}\overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w).$$

Applying the previous general formulas to the product $*_\tau$ gives

Proposition 16. *For a linear polynomial aw, $a \in \mathbb{C}$, the star exponential and the intertwiner satisfy*

$$\exp_{*_\tau} aw = \exp(aw + (\tau/4)a^2), \quad I_\tau^{\tau'}(\exp_{*_\tau} aw) = \exp_{*_{\tau'}} aw,$$

respectively.

Hence we have the space of parallel sections $\mathcal{G} = \{e_*^{aw}\}$ of the bundles of group over the parameter space $\mathbb{C} = \{\tau\}$.

Star Hermite function. Recall a naive expansion of star exponential for the linear case is convergent, namely

$$: \exp_*(\sqrt{2}tw) :_\tau = \sum_{n=0}^{\infty} : (\sqrt{2}w)_*^n :_\tau \frac{t^n}{n!}.$$

Note, that the explicit formula of star exponential evaluated at $\tau = -1$ gives the generating function of the Hermite polynomials $H_n(w)$, namely

$$: \exp_*(\sqrt{2}tw) :_{\tau=-1} = \exp\left(\sqrt{2}tw - \tfrac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w) \tfrac{t^n}{n!}.$$

Then comparing the both expansions and we obtain

$$H_n(w) =: (\sqrt{2}w)_*^n :_{\tau=-1} .$$

We define star Hermite function (one-parameter deformation of $H_n(w)$) by using parallel sections

$$H_n(w,\tau) =: (\sqrt{2}\,w)_*^n :_\tau, \quad (n=0,1,2,\dots).$$

Then the evaluation of the parallel section $e_*^{\sqrt{2}tw}$ at τ gives a generating function of star Hermite functions, namely

$$: \exp_*(\sqrt{2}\,tw) :_\tau = \sum_{n=0}^{\infty} H_n(w,\tau)\,\tfrac{t^n}{n!}.$$

Trivial identity $\frac{d}{dt}\exp_*(\sqrt{2}\,tw) = \sqrt{2}w{*}\exp_*(\sqrt{2}\,tw)$ evaluated at τ yields the identity

$$\tfrac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}wH_n(w,\tau) = H_{n+1}(w,\tau), \quad (n=0,1,2,\dots)$$

for every $\tau \in \mathbb{C}$, and the exponential law

$$\exp_*(\sqrt{2}sw) * \exp_*(\sqrt{2}tw) = \exp_*(\sqrt{2}(s+t)w)$$

yields the identity

$$\sum_{k+l=n} \tfrac{n!}{k!l!} H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau).$$

Star theta function. We can express the Jacobi's theta functions by using parallel sections of star exponentials $\in \mathcal{G}$. The formula

$$: \exp_* n \text{ i } w :_\tau = \exp(n \text{ i } w - (\tau/4)n^2)$$

shows that for $\mathrm{Re}\,\tau > 0$, the star exponential $: \exp_* n\mathrm{i}\ w :_\tau$ is rapidly decreasing with respect to integer n. Then we can consider summations for τ such that $\mathrm{Re}\,\tau > 0$

$$: \sum_{n=-\infty}^{\infty} \exp_* 2n\mathrm{i}\ w :_\tau = \sum_{n=-\infty}^{\infty} \exp\left(2n\mathrm{i}\ w - \tau n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni\ w}, \quad (q = e^{-\tau})$$

which is convergent and gives Jacobi's theta function $\theta_3(w,\tau)$. Then the infinite sums of parallel sections of $\mathcal{G}$ such as

$$\theta_{1_*}(w) = \tfrac{1}{\mathrm{i}} \sum_{n=-\infty}^{\infty} (-1)^n \exp_*(2n+1)\mathrm{i}\ w, \quad \theta_{2*}(w) = \sum_{n=-\infty}^{\infty} \exp_*(2n+1)\mathrm{i}\ w,$$

$$\theta_{3*}(w) = \sum_{n=-\infty}^{\infty} \exp_* 2ni\ w, \quad \theta_{4*}(w) = \sum_{n=-\infty}^{\infty} (-1)^n \exp_* 2ni\ w$$

are called *star theta functions*. Actually the evaluation of $: \theta_{k*}(w) :_\tau$ at τ with $\mathrm{Re}\,\tau > 0$ gives the Jacobi's theta function $\theta_k(w,\tau)$, $k = 1,2,3,4$ respectively. The exponential law of star exponential yields trivial identities

$$\exp_* 2i\ w * \theta_{k*}(w) = \theta_{k*}(w) \qquad (k=2,3),$$
$$\exp_* 2i\ w * \theta_{k*}(w) = -\theta_{k*}(w) \quad (k=1,4).$$

Then using the evaluation formula : $\exp_* 2i\ w\ :_\tau = e^{-\tau} e^{2i\ w}$ and the product formula directly we see the above trivial identities are equivalent to the quasi periodicity

$$e^{2i\ w-\tau}\theta_k(w+i\ \tau) = \theta_k(w) \qquad (k=2,3),$$
$$e^{2i\ w-\tau}\theta_k(w+i\ \tau) = -\theta_k(w) \quad (k=1,4).$$

$*$-delta functions. Since the star exponential : $\exp_*(itw)\ :_\tau = \exp(itw - \frac{\tau}{4}t^2)$ is rapidly decreasing with respect to t when $\mathrm{Re}\,\tau > 0$. Then the integral of star exponential evaluated at τ

$$:\int_{-\infty}^{\infty} \exp_*(it(w-a)_*)\,dt\ :_\tau = \int_{-\infty}^{\infty} \exp(it(w-a) - \tfrac{\tau}{4}t^2)dt$$

converges for any $a \in \mathbb{C}$. We put a star δ-function

$$\delta_*(w-a) = \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt,$$

which has a meaning at τ with $\mathrm{Re}\,\tau > 0$. It is easy to see for any parallel section of polynomials $p_*(w) \in \mathcal{P}$,

$$p_*(w) * \delta_*(w-a) = p(a)\delta_*(w-a),\ w * \delta_*(w) = 0.$$

Using the Fourier transform we have

$$\theta_{1*}(w) = \tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+\tfrac{\pi}{2}+n\pi), \quad \theta_{2*}(w) = \tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+n\pi)$$
$$\theta_{3*}(w) = \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+n\pi), \qquad \theta_{4*}(w) = \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+\tfrac{\pi}{2}+n\pi).$$

Now, we consider the τ satisfying the condition $\mathrm{Re}\,\tau > 0$. Then we calcultate the integral and obtain $\delta_*(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}(w-a)^2\right)$ and we have

$$\begin{aligned}\theta_3(w,\tau) &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+n\pi) = \tfrac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\tfrac{1}{\tau}\right)\sum_{n=-\infty}^{\infty}\exp\left(-2n\tfrac{1}{\tau}w - \tfrac{1}{\tau}n^2\tau^2\right)\\ &= \tfrac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\tfrac{1}{\tau}\right)\theta_{3*}(\tfrac{2\pi w}{i\tau}, \tfrac{\pi^2}{\tau}).\end{aligned}$$

We also have similar identities for other $*$-theta functions by the similar way.

2.3. Star exponentials of quadratic polynomials

Different from linear case, star exponentials of quadratic polynomials have singularities which are moving, branching, and essential singularities.

Proposition 17. *For a quadratic polynomial $aw^2_{*_\tau} = aw^2 + \frac{a\tau}{2}$, $a \in \mathbb{C}$, the star exponential and the intertwiner satisfy*

$$\exp_{*_\tau} aw^2_{*_\tau} = \frac{1}{\sqrt{1-a\tau}}\exp\left(\frac{1}{1-a\tau}aw^2\right), I_\tau^{\tau'}(\exp_{*_\tau} aw^2_{*_\tau}) = \exp_{*_{\tau'}} aw^2_{*_{\tau'}},$$

respectively, when the star exponential and the intertwiner contain terms of square root then this equality includes a $\pm$ *umbiguity.*

We thus have the space of parallel sections $\mathcal{Q} = \{e_*^{aw_*^2}\}$ of the bundles of group-like objects over the parameter space $\mathbb{C} = \{\tau\}$, respectively. Hence the star exponentials of quadratic polynomials, that is, parallel sections of $\mathcal{Q}$ behave strangely, but are interesting. Here I will show several concrete examples for the simple case, for more examples and details, see the references already cited above.

2.3.1. "Double covering" group. Let us consider a parallel section $e_*^{tw_*^2} \in \mathcal{Q}$. This section has a singular point depending on the parameter τ, actually we see by the evaluation formula at τ that the star exponential

$$:\exp_* tw_*^2 :_\tau = \frac{1}{\sqrt{1-t\tau}} \exp\left(\frac{1}{1-t\tau}\ tw^2\right)$$

has a singularity at $t = 1/\tau$. Thus for small t, the section $e_*^{tw_*^2}$ satisfies the exponential law for every τ, i.e., $\{e_*^{tw_*^2}, t \in \mathbb{C}\}$ forms a local group. On the other hand, for each t, taking an appropriate path in $\tau \in \mathbb{C}$, the parallel transform $I_\tau^{\tau'}$ along the path gives $: e_*^{tw_*^2} :_\tau \mapsto : -e_*^{tw_*^2} :_\tau$. Hence the group-like object $e_*^{tw_*^2} \in \mathcal{Q}$ looks like a double covering group of $\mathbb{C}$.

This also appears when we consider multi-variable case $w = (w_1, \ldots, w_n)$. For example, if we assume that the number of variables is $n = 2$, and the skew-symmetric part is fixed such that $\Lambda_- = J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then for a complex matrix $\Lambda = J + K$, (K symmetric), the associative algebra of polynomial parallel sections $\mathcal{P}$ includes the Lie algebra of $SL(2,\mathbb{C})$, which are given by quadratic polynomials. Exponentiating these quadratic elements one obtains a set of parallel sections $\widetilde{SL(2,\mathbb{C})} \subset \mathcal{Q}$ of the bundle of group-like objects over the space of all symmetric matrices $\{K\}$. The object $\widetilde{SL(2,\mathbb{C})}$ also behaves like a "double covering" group of $SL(2,\mathbb{C})$, which is called a blurred Lie group $\widetilde{SL(2,\mathbb{C})}$. (For more details, see [5]).

2.3.2. Vacuum. Consider a Weyl algebra W of two canonical generators u, v, namely $[v, u] = \mathrm{i}\hbar$. An element $\varpi \in W$ satisfying the relation $\varpi\varpi = \varpi$ and $v\varpi = \varpi u = 0$ is called a vacuum. Vacuum plays an important role in quantum mechanics.

We can construct vacuums in the set of parallel sections $\mathcal{Q}$. For example we consider $n = 2$ and fix the skew-symmetric part of Λ to be J and we set $\Lambda = J + K$, (K symmetric). We write the generators of $\mathcal{P}$ as $w_1 = u, w_2 = v$. Then we see $[v, u]_* = v * u - u * v = \mathrm{i}\hbar$. Then in the group-like parallel sections $\mathcal{Q}$ of star exponentials, we can construct an element $\varpi_{00} \in \mathcal{Q}$ having a property such that $\varpi_{00} * \varpi_{00} = \varpi_{00}$ and $v * \varpi_{00} = \varpi_{00} * u = 0$. We construct ϖ_{00} in the following way. We take a parallel section of star exponential such that $e_*^{2t\frac{u*v}{\mathrm{i}\hbar}} \in \mathcal{Q}$. Then we have $\varpi_{00} = \lim_{t\to-\infty} e_*^{2t\frac{u*v}{i\hbar}}$. For example, for $K = \left(\begin{smallmatrix} 0 & \kappa \\ \kappa & \tau \end{smallmatrix}\right)$, we see

$$:\varpi_{00}:_K = \lim_{t\to-\infty} :e_*^{2t\frac{u*v}{i\hbar}}:_K = \frac{2}{1+\kappa}\exp\left(-\frac{1}{i\hbar(1+\kappa)}(2uv - \tfrac{\tau}{1+\kappa}u^2)\right).$$

Further using this vacuum we can construct generators of Clifford algebra in $\mathcal{Q}$, so we can construct Clifford algebra using parallel sections $\mathcal{Q}$ and $\mathcal{P}$. (See for details, H. Omori, Y. Maeda [6], T. Tomihisa, A. Yoshioka [7].)

Instead of taking a limit, we also obtain a vacuum by a contour integral of a parallel section of $\mathcal{Q}$ around singularities. (For details, see [5].)

2.3.3. Contour integral around singularites. An element of $\mathcal{Q}$, parallel section of star exponential of quadratic polynomials, has branching, essential singularities. Then it is natural to consider the derivation of meaningful relations from these singularities as residues of elements of $\mathcal{Q}$.

As an example, we can construct the Virasoro algebra by using residues. For details, see H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka [4].

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization. I*, Ann. Phys. **111** (1978) 61–110.

[2] M. Kontsevich, *Deformation quantization of Poisson manifolds*, qalg/9709040, Lett. Math. Phys. **66** (2003) 157–206.

[3] J.E. Moyal, *Quantum mechanics as a statistical theory*, Proc. Cambridge Phil. Soc. **45** (1949) 99–124.

[4] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, *Deformation of expression for elements of an algebra*, in Symplectic, Poisson and noncommutative geometry, MSRI publications **62** (2014), 171–209.

[5] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, *Deformation expression for elements of algebra* (I)–(VIII), arXiv math-ph.

[6] H. Omori, Y. Maeda, *Quantum Calculus* (in Japanse), Kinokuniya, 2004.

[7] Tomihisa T. and Yoshioka, *Star Products and Star Exponentials*, J. Geometry and Symmetry in Physics, **19** (2010) 99–111.

[8] H. Weyl, *Quantenmechanik und Gruppentheorie*, Z. Phys. **46** (1927), 1–46.

[9] E.P. Wigner, *On the Quantum Correction For Thermodynamic Equilibrium*, Phys. Rev. **40** (1932), 749–759.

Akira Yoshioka
Department of Mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjyuku-ku
Tokyo 162-8601, Japan
e-mail: yoshioka@rs.kagu.tus.ac.jp

Part IV

Integrable Systems

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 167–180

Beyond Recursion Operators

Yvette Kosmann-Schwarzbach

Abstract. We briefly recall the history of the Nijenhuis torsion of $(1,1)$-tensors on manifolds and of the lesser-known Haantjes torsion. We then show how the Haantjes manifolds of Magri and the symplectic Haantjes structures of Tempesta and Tondo generalize the classical approach to integrable systems in the bi-Hamiltonian and symplectic Nijenhuis formalisms, the sequence of powers of the recursion operator being replaced by a family of commuting Haantjes operators.

Mathematics Subject Classification (2010). Primary 58-03, 01A60, 17B70, 37J99, 70G45; Secondary 53D45.

Keywords. Nijenhuis torsion, Haantjes torsion, higher Nijenhuis torsion, recursion operator, integrable system, Magri–Lenard complex, WDVV equations.

1. Introduction. Haantjes tensors generalize recursion operators

The "Nijenhuis torsion" of a $(1,1)$-tensor field was defined in 1951 by Albert Nijenhuis, a student of the Dutch mathematician J.A. Schouten, while the "Haantjes torsion" of a $(1,1)$-tensor field was defined in 1955 by Johannes Haantjes, another of Schouten's students[1]. If the Nijenhuis torsion vanishes, the Haantjes torsion does also, but the converse is not true in general. Since Nijenhuis tensors, i.e., $(1,1)$-tensors with vanishing Nijenhuis torsion, occur as recursion operators in the theory of integrable systems, one can expect the Haantjes tensors, i.e., $(1,1)$-tensors with vanishing Haantjes torsion, to play a role "beyond recursion operators".

[1] A (1,1)-tensor field on a manifold was called an "affinor of valence two" by Schouten and his contemporaries. We find in the literature and we use indifferently the following expressions for a field of $(1,1)$-tensors: $(1,1)$-tensor [field], mixed tensor [of valence 2], field of endomorphisms of the tangent bundle, field of linear transformations, vector-valued [differential] 1-form, [differential] 1-form with values in the tangent bundle, operator [on vector fields] [on 1-forms].

2. The search for differential concomitants and the Nijenhuis torsion

2.1. Schouten, Haantjes, Nijenhuis

The mathematician Jan A. Schouten (1883–1971) is best known for his contributions to the modern form of the tensor calculus[2], in particular for the differential concomitant that was later called "the Schouten bracket" which he defined in an article that appeared in *Indagationes Mathematicae* in 1940 [22]. His doctoral student, Johannes Haantjes (1909–1956), defended his thesis at the University of Leiden in 1933. Albert Nijenhuis (1926–2015), also a student of Schouten, was awarded a doctorate at the University of Amsterdam in 1951. His article, "X_{n-1}-forming sets of eigenvectors", appeared shortly thereafter in *Indagationes* [18], and four years later he published an article in two parts entitled "Jacobi-type identities for bilinear differential concomitants of certain tensor fields" in the same journal [19]. This second publication of Nijenhuis was preceded, only a few weeks earlier, by an article by Haantjes, "On X_m-forming sets of eigenvectors", which also appeared in *Indagationes* [7].

2.2. The theory of invariants and the question of the integrability of eigenplanes

The discovery of the Nijenhuis torsion followed a search for differential concomitants of tensorial quantities, which had its roots in the theory of invariants, going back to J.J. Sylvester and Arthur Cayley in the mid-19th century. This theory would make use of Sophus Lie's continuous groups and infinitesimal methods, and would later lead to the absolute differential calculus of Gregorio Ricci and Tullio Levi-Civita. It was this search that was extensively carried out by Schouten from the 1920s on. He wrote later [23] that he had "in 1940 succeeded in generalizing Lie's operator by forming a differential concomitant of two arbitrary contravariant quantities". Then he disclosed the method he used to discover his concomitant: it was by requiring that it be a derivation in each argument, which is the essential defining property of what is now called "the Schouten bracket" of contravariant tensors.

Another field of inquiry was the search for conditions that ensure that, given a field of endomorphisms of the tangent bundle of a manifold, assumed to have distinct eigenvalues, the distributions spanned by pairs of eigenvectors are integrable.

2.3. The Nijenhuis torsion

In 1951, Nijenhuis introduced a quantity defined by its components in local coordinates, $H_{\mu\lambda}^{\cdot\cdot\kappa}$, expressed in terms of the components $h_\lambda^{\cdot\kappa}$ of a $(1,1)$-tensor, h, and of their partial derivatives,

$$H_{\mu\lambda}^{\cdot\cdot\kappa} = 2h_{[\mu}^{\cdot\rho}\partial_{|\rho|}h_{\lambda]}^{\cdot\kappa} - 2h_\rho^{\cdot\kappa}\partial_{[\mu}h_{\lambda]}^{\cdot\rho}.$$

[2]See the article by his former and best-known student, Nijenhuis [20].

(The indices between square brackets are to be skew-symmetrized: the opposite term with these indices exchanged is to be added.) He then proved the tensorial character of this quantity [18]. Because of the factor 2, the $H_{\mu\lambda}^{\cdot\cdot\kappa}$ are actually the components of twice what is called the *Nijenhuis torsion* of the $(1,1)$-tensor h, which is a skew-symmetric $(1,2)$-tensor, i.e., a vector-valued differential 2-form.

Remark. The name "torsion" was adopted by Nijenhuis from the theory of complex manifolds, where the "torsion" was defined for an almost complex structure by B. Eckmann and A. Frölicher, also in 1951. However, in the literature, the name "Nijenhuis tensor" is often used for the "Nijenhuis torsion".

2.4. The Nijenhuis torsion without local coordinates

It was also in his 1951 article that Nijenhuis introduced the symmetric bilinear form, depending on a pair of $(1,1)$-tensors, associated by polarization to the quadratic expression of the torsion. Then in 1955 [19], he introduced a bracket notation $[h,k]$ for this symmetric bilinear form, and he found a coordinate-independent formula for this bracket. In particular, the Nijenhuis torsion, $\mathcal{T}_R = [R,R]$, of a $(1,1)$-tensor, R, on a manifold, M, is the $(1,2)$-*tensor* $\mathcal{T}_R$ such that, for all vector fields X and Y on M,

$$\mathcal{T}_R(X,Y) = [RX,RY] - R[RX,Y] - R[X,RY] + R^2[X,Y],$$

Remark. We did not retain the Nijenhuis notation. Our notation is simply related to his by $R = h$ and $\mathcal{T}_R = H$.

2.5. The Frölicher–Nijenhuis bracket

In his 1955 article, Nijenhuis also defined what he called "a concomitant for differential forms with values in the tangent bundle", that is, a graded bracket on the space of vector-valued differential forms of all degrees, extending the bilinear form associated to the torsion, and he proved that this bracket satisfies a graded Jacobi identity. (He also proved that Schouten's brackets of contravariant tensors satisfy a graded Jacobi identity.) This theory would soon be developed in a joint article with A. Frölicher in 1956 [5], and this graded Lie bracket became known as the "Frölicher–Nijenhuis bracket".

In a modern formulation, the Frölicher–Nijenhuis bracket, $[U,V]_{\mathrm{FN}}$, of a vector-valued k-form, U, and a vector-valued ℓ-form, V, is the vector-valued $(k+\ell)$-form, $[U,V]_{\mathrm{FN}}$, satisfying the equation

$$\mathcal{L}_{[U,V]_{\mathrm{FN}}} = [\mathcal{L}_U, \mathcal{L}_V].$$

Here the bracket $[\ ,\ ]$ is the graded commutator of derivations of the algebra of differential forms, and $\mathcal{L}_W = [i_W, \mathrm{d}]$ is the graded commutator of the interior product by a vector-valued form, W, and the de Rham differential.

Also in 1955, there appeared another, very different development of the theory of the Nijenhuis torsion of $(1,1)$-tensors, the Haantjes torsion of $(1,1)$-tensors.

3. The Haantjes torsion

3.1. Haantjes (1909–1956)

The Dutch mathematician Johannes Haantjes, after his doctoral defense in Leiden, was invited by Schouten to join him as his assistant in Delft. From 1934 to 1938, they published several articles in collaboration, on spinors and their role in conformal geometry, and on the general theory of geometric objects, all in German except for one in English, papers that have almost never been cited. After 1938, Haantjes was a lecturer at the Vrije Universiteit in Amsterdam. He was elected to the Royal Dutch Academy of Sciences in 1952, four years before his death at the age of 46. He was among the "distinguished European mathematicians" whom Kentaro Yano in 1982 recalled having met at the prestigious International Conference on Differential Geometry organized in Italy in 1953 [27]. However, for half a century, very few citations of his work appeared in the literature, and his name was nearly forgotten.

3.2. Haantjes's article of 1955

In "X_m-forming sets of eigenvectors" [7], Haantjes considered the case of a field of endomorphims "of class A", i.e., such that the eigenspace of an eigenvalue of multiplicity r be of dimension r. He introduced, in terms of local coordinates, a new quantity whose vanishing did not necessarily imply the vanishing of the Nijenhuis torsion but was necessary and sufficient for the integrability of the distributions spanned by the eigenvectors. From the Nijenhuis torsion H of a $(1,1)$-tensor h, with components $H_{\mu\lambda}^{\cdot\cdot\kappa}$, he obtained the condition he sought as the vanishing of

$$H_{\nu\sigma}^{\cdot\cdot\kappa}h_{.\mu}^{\nu}h_{.\lambda}^{\sigma} - 2H_{\nu[\lambda}^{\cdot\cdot\sigma}h_{.\mu]}^{\nu}h_{.\sigma}^{\kappa} + H_{\mu\lambda}^{\cdot\cdot\nu}h_{.\sigma}^{\kappa}h_{.\nu}^{\sigma}.$$

These are the components of a $(1,2)$-tensor, twice the *Haantjes torsion* of the $(1,1)$-tensor h. The components of the Haantjes torsion of h are of degree 4 in the components of h.

3.3. First citations of Haantjes's article

The 1955 article of Haantjes did not attract the attention of differential geometers or algebraists until the very end of the twentieth century. In fact, it was only cited twice before 1996! In the twenty-first century, the "Haantjes tensor" (i.e., in our terminology, the Haantjes torsion) started appearing, as an object of interest in algebra, in the work of O.I. Bogoyavlenskij [1, 2], and, mostly, in the theory of integrable systems. In 2007, in an article in *Mathematische Annalen*, E.V. Ferapontov and D.G. Marshall presented the Haantjes tensor as a "differential-geometric approach to the integrability" of systems of differential equations, and reformulated the main result of Haantjes's original paper as the theorem, "A system of hydrodynamic type with mutually distinct characteristic speeds is diagonalizable if and only if the corresponding Haantjes tensor [i.e., Haantjes torsion] vanishes identically" [3].

3.4. Haantjes torsion in coordinate-free form

Changing notations, we denote a $(1,1)$-tensor by R, its Nijenhuis torsion by $\mathcal{T}_R$, and the Haantjes torsion of R by $\mathcal{H}_R$. We now formulate an intrinsic characterization of the Haantjes torsion of a $(1,1)$-tensor.

The *Haantjes torsion* of a $(1,1)$-tensor R is the $(1,2)$-*tensor* $\mathcal{H}_R$ such that, for all vector fields X and Y,

$$\mathcal{H}_R(X,Y) = \mathcal{T}_R(RX,RY) - R(\mathcal{T}_R(RX,Y)) - R(\mathcal{T}_R(X,RY)) + R^2(\mathcal{T}_R(X,Y)).$$

Explicitly,

$$\begin{aligned}\mathcal{H}_R(X,Y) = {} & [R^2X,R^2Y] - 2R[R^2X,RY] - 2R[RX,R^2Y] + 4R^2[RX,RY] \\ & + R^2[R^2X,Y] + R^2[X,R^2Y] - 2R^3[RX,Y] - 2R^3[X,RY] + R^4[X,Y].\end{aligned}$$

Next, we shall generalize the definitions of the Nijenhuis torsion and of the Haantjes torsion of a $(1,1)$-tensor field on a manifold to any vector space equipped with a "bracket".

4. Nijenhuis and Haantjes torsions associated to a "bracket"

4.1. Definition

Let $\mu : E \times E \to E$ be a vector-valued skew-symmetric bilinear map on a real vector space E. For each linear map, $R : E \to E$,

(i) the *Nijenhuis torsion* of R is the skew-symmetric $(1,2)$-tensor on E, denoted by $\mathcal{T}_R(\mu)$, such that, for all vectors X and Y in E,

$$\mathcal{T}_R(\mu)(X,Y) = \mu(RX,RY) - R(\mu(RX,Y)) - R(\mu(X,RY)) + R^2(\mu(X,Y)),$$

(ii) the *Haantjes torsion* of R is the skew-symmetric $(1,2)$-tensor on E, denoted by $\mathcal{H}_R(\mu)$, such that, for all vectors X and Y in E,

$$\begin{aligned}\mathcal{H}_R(\mu)(X,Y) = {} & \mathcal{T}_R(\mu)(RX,RY) - R(\mathcal{T}_R(\mu)(RX,Y)) \\ & - R(\mathcal{T}_R(\mu)(X,RY)) + R^2(\mathcal{T}_R(\mu)(X,Y)).\end{aligned}$$

4.2. Lie algebroids

The general definitions of the Nijenhuis torsion and of the Haantjes torsion are applicable when E is the module of sections of a *Lie algebroid*, $A \to M$, and μ is the Lie bracket of sections of A (or to a pre-Lie algebroid in which the bracket of sections does not necessarily satisfy the Jacobi identity), and R is a section of $A \otimes A^*$. There are two important special cases:

(i) $A = TM$ and E is the module of vector fields on a manifold M, μ is the Lie bracket of vector fields and R is a $(1,1)$-tensor, the case originally studied by Haantjes in 1955,

(ii) E is a real Lie algebra with bracket μ, and R is a linear map.

Remark. In the case of TM or, more generally, of a Lie algebroid A over M, the Lie bracket of sections μ is *only* $\mathbb{R}$*-linear, not* $C^\infty(M)$*-linear*. But the torsion $\mathcal{T}_R(\mu)$ of any $(1,1)$-tensor R is $C^\infty(M)$-linear, i.e., it is a $(1,2)$-tensor.

4.3. Haantjes torsion as torsion of the Nijenhuis torsion

From the defining formula of the Haantjes torsion of a linear endomorphism R of E in terms of its Nijenhuis torsion we obtain immediately:

Proposition. *The Haantjes torsion is related to the Nijenhuis torsion by*

$$\mathcal{H}_R(\mu) = \mathcal{T}_R(\mathcal{T}_R(\mu)).$$

This relation suggests the construction by iteration of higher Nijenhuis and Haantjes torsions of a linear endomorphism.

4.4. Higher Nijenhuis torsions

Let R be a linear endomorphism of a vector space E. Then $\mathcal{T}_R$ is the linear endomorphism of $E \otimes \wedge^2 E^*$ such that, for $\nu \in E \otimes \wedge^2 E^*$,

$$\mathcal{T}_R(\nu) = \nu \circ (R \otimes R) - R \circ \nu \circ (R \otimes \mathrm{Id}) - R \circ \nu \circ (\mathrm{Id} \otimes R) + R^2 \circ \nu.$$

For a vector space with bracket μ, set $\mathcal{T}_R^{(1)}(\mu) = \mathcal{T}_R(\mu)$, which is, by definition, the Nijenhuis torsion $\mathcal{T}_R(\mu)$ of R. Define

$$\mathcal{T}_R^{(k+1)}(\mu) = \mathcal{T}_R(\mathcal{T}_R^{(k)}(\mu)), \ \text{ for } \ k \geq 1.$$

The $(1,2)$-tensors $\mathcal{T}_R^{(k)}(\mu)$ are of degree $2k$ in R. We call the skew-symmetric $(1,2)$-tensors, $\mathcal{T}_R^{(k)}(\mu)$, for $k \geq 2$, the *higher Nijenhuis torsions* of R. For any skew-symmetric $(1,2)$-tensor μ, and for all $k, \ell \geq 1$,

$$\mathcal{T}_R^{(k+\ell)}(\mu) = \mathcal{T}_R^{(k)}(\mathcal{T}_R^{(\ell)}(\mu)).$$

4.5. Higher Haantjes torsions

In the preceding notation, the Haantjes torsion of R is

$$\mathcal{H}_R(\mu) = \mathcal{T}_R(\mathcal{T}_R(\mu)) = \mathcal{T}_R^{(2)}(\mu).$$

Set $\mathcal{H}_R^{(1)}(\mu) = \mathcal{H}_R(\mu)$ and define

$$\mathcal{H}_R^{(k+1)}(\mu) = \mathcal{T}_R(\mathcal{H}_R^{(k)}(\mu)), \ \text{ for } \ k \geq 1.$$

The $(1,2)$-tensors $\mathcal{H}_R^{(k)}(\mu)$ are of degree $2(k+1)$ in R. By definition, $\mathcal{H}_R^{(1)}(\mu)$ is the Haantjes torsion $\mathcal{H}_R(\mu)$ of R. We call the skew-symmetric $(1,2)$-tensors, $\mathcal{H}_R^{(k)}(\mu)$, for $k \geq 2$, the *higher Haantjes torsions* which satisfy the very simple relation

$$\mathcal{H}_R^{(k)}(\mu) = \mathcal{T}_R^{(k+1)}(\mu).$$

For any skew-symmetric $(1,2)$-tensor μ, and for all $k, \ell \geq 1$,

$$\mathcal{H}_R^{(k+\ell+1)}(\mu) = \mathcal{H}_R^{(k)}(\mathcal{H}_R^{(\ell)}(\mu)).$$

4.6. A formula for the higher Haantjes torsions

To a $(1,1)$-tensor, Bogoyavlenskij associated a representation of the ring of real polynomials in 3 variables on the space of $(1,2)$-tensors [1]. Expanding the polynomial $(xy-zx-zy+z^2)^{k+1}=(z-x)^{k+1}(z-y)^{k+1}$ furnishes the general formula for the $(k+2)^2$ terms of the expansion of the kth Haantjes torsion of R,

$$\mathcal{H}_R^{(k)}(\mu)(X,Y)=\sum_{p=0}^{k+1}\sum_{q=0}^{k+1}(-1)^{2(k+1)-p-q}C_{k+1}^{p}C_{k+1}^{q}R^{p+q}\mu(R^{k+1-p}X,R^{k+1-q}Y).$$

It remains to be seen what roles, if any, the higher Haantjes torsions can play in geometry and in the theory of integrable systems.

4.7. Properties of the Nijenhuis and Haantjes torsions

If a $(1,1)$-tensor field, R, on a manifold, M, is diagonizable in a local basis, $\left(\frac{\partial}{\partial x^i}\right)$, $i=1,\dots,n$, with eigenvalues $\lambda_i(x^1,\dots,x^n)$, its Nijenhuis torsion satisfies

$$\mathcal{T}_R(\mu)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)=(\lambda_i-\lambda_j)\left(\frac{\partial\lambda_i}{\partial x^j}\frac{\partial}{\partial x^i}+\frac{\partial\lambda_j}{\partial x^i}\frac{\partial}{\partial x^j}\right).$$

Making use of the $C^\infty(M)$-bilinearity of $\mathcal{T}_R(\mu)$, it is easy to prove that, if R is diagonizable, the Haantjes torsion of R vanishes.

If there exists a basis of eigenvectors of R at each point (in particular, if all the eigenvalues of R are simple), the vanishing of the Haantjes torsion of R is *necessary and sufficient* for R to be *diagonalizable* in a system of coordinates.

If $R^2=\alpha\,\mathrm{Id}$, where α is a constant, in particular, if R is an *almost complex structure*, i.e., when $R^2=-\mathrm{Id}$, then the Haantjes torsion is equal to the Nijenhuis torsion, up to a scalar factor,

$$\mathcal{H}_R(\mu)=4\alpha\mathcal{T}_R(\mu),$$

and, more generally, $\mathcal{H}_R^{(k)}(\mu)=(4\alpha)^k\mathcal{T}_R(\mu)$, for $k\geq 1$.

5. Haantjes manifolds and Magri–Lenard complexes

5.1. From Nijenhuis to Haantjes manifolds

In a series of papers written since 2012, Franco Magri has defined the concept of a Haantjes manifold, demonstrated how the concept of a Lenard complex on a manifold extends that of a Lenard chain associated with a bi-Hamiltonian system, related this theory to that of Frobenius manifolds, and developed applications to the study of differential systems [11–14].

A *Nijenhuis manifold* is a manifold endowed with a Nijenhuis tensor, i.e., a tensor whose Nijenhuis torsion vanishes. In the theory of integrable systems, Nijenhuis tensors have also been called *Nijenhuis operators*, since they map vector fields to vector fields, as well as 1-forms to 1-forms, or *hereditary operators*, because they act as recursion operators. It is well known that every power of a

Nijenhuis operator, R, is a Nijenhuis operator. Therefore, in a Nijenhuis manifold, the sequence of powers of the Nijenhuis tensor, $\mathrm{Id}, R, R^2, \ldots, R^k, \ldots$, is a *family of commuting Nijenhuis operators.*

In the new framework, the role of this sequence of powers is played by a *family of Haantjes tensors*, i.e., $(1,1)$-tensors whose Haantjes torsion vanishes. Haantjes tensors are also called *recursion operators.* A *Haantjes manifold* is a manifold endowed with a *family of commuting Haantjes tensors*, $R_1, R_2, \ldots, R_k, \ldots$ In the examples, the family of Haantjes tensors is usually finite, in number equal to the dimension of the manifold, and $R_1 = \mathrm{Id}$. We shall adopt the definition in this restricted sense.

5.2. Magri–Lenard complexes

A *Magri–Lenard complex* on a manifold, M, of dimension n, equipped with n *commuting* $(1,1)$*-tensors,* R_k, $k = 1, \ldots, n$, with $R_1 = \mathrm{Id}$, is defined by a pair (X, θ) such that

(i) the vector fields $R_k X$, $k = 1, \ldots, n$, *commute pairwise*,
(ii) the 1-forms, $\theta R_k R_\ell$, $k, \ell = 1, \ldots, n$, are *closed.*

In particular, θ itself is assumed to be closed and each θR_k is a closed 1-form. The notation θR, where θ is a 1-form and R is a $(1,1)$-tensor, stands for ${}^t R\, \theta$, the $(1,1)$-tensor acting on the 1-form by the dual map.

Magri proved that, under a mild additional condition, if these properties are satisfied, the operators R_k, $k = 1, \ldots, n$, are necessarily Haantjes tensors, so that the underlying manifold of a Magri–Lenard complex is a Haantjes manifold [15].

5.3. Magri–Lenard complexes generalize Lenard chains

In order to show how the Magri–Lenard complexes generalize the Lenard chains of bi-Hamiltonian systems that were already defined by Magri in 1978 [10], we shall first recall how Nijenhuis operators appear in the theory of bi-Hamiltonian systems.

If a vector field, X, leaves a $(1,1)$-tensor, R, invariant, then,

$$0 = (\mathcal{L}_X R)(Y) = \mathcal{L}_X(RY) - R(\mathcal{L}_X Y) = [X, RY] - R[X, Y],$$

for all vector fields Y. Therefore R, when applied to a symmetry Y of the evolution equation, $u_t = X(u)$, yields a new symmetry, RY. If, in addition, R is a Nijenhuis operator, applying the successive powers of R yields a sequence of commuting symmetries, $R^k X$, $k \in \mathbb{N}$, known as a "Lenard chain"[3] and therefore R is a *recursion operator* for each of the evolution equations in the hierarchy, $u_t = (R^k X)(u)$.

The geometric structure underlying the theory of integrable systems is the theory of Poisson–Nijenhuis manifolds, in particular the theory of symplectic Nijenhuis manifolds. If P_1 and P_2 are *compatible Hamiltonian operators*, i.e., Poisson bivectors such that their sum is a Poisson bivector, and if P_1 is invertible, i.e.,

[3] For the story of how the hierarchy of higher Korteweg–de Vries equations became known as a "Lenard chain", named after Andrew Lenard (b. 1927), in papers by Martin Kruskal et al., see Lenard's letter reproduced in [21].

defines a symplectic structure, then $R = P_2 \circ P_1^{-1}$ is a Nijenhuis operator. Thus, (P_2, R) is called a "Poisson–Nijenhuis structure" and (P_1, R) is called a "symplectic Nijenhuis structure". The theory of compatible Poisson structures originated in articles of Gel'fand and Dorfman [6], Fokas and Fuchssteiner [4], Magri and Morosi [16], and was further developed in [8] and [9].

We can now show that there is a Magri–Lenard complex associated to a bi-Hamiltonian system. Let P_1 and P_2 be compatible Hamiltonian operators. A vector field X is called *bi-Hamiltonian* with respect to P_1 and P_2 if there exist *exact* differential 1-forms $\alpha_1 = \mathrm{d}H_1$ and $\alpha_2 = \mathrm{d}H_2$ such that

$$X = P_1(\alpha_1) = P_2(\alpha_2).$$

Assume that P_1 is invertible, then the Nijenhuis operator $R = P_2 \circ P_1^{-1}$ generates a sequence of commuting bi-Hamiltonian vector fields, $R^k X$, the so-called Lenard chain. The sequence of powers of R, $(\mathrm{Id}, R, R^2, \ldots, R^k, \ldots)$, is a family of commuting Nijenhuis operators, and therefore a *family of commuting Haantjes operators.* We set $\theta = \alpha_1$. Then θR and all θR^k are closed 1-forms. Therefore, *the axioms of a Magri–Lenard complex are satisfied.* In addition, the 1-form θ and the recursion operator R are invariant under X.

5.4. A Magri–Lenard complex on $\mathbb{R}^3$

We present an example of a Magri–Lenard complex described by Magri in [14]. On $\mathbb{R}^3$ with coordinates (u_1, u_2, u_3), consider the matrices

$$K = \begin{pmatrix} 0 & 2 & 0 \\ -u_1 & 0 & 2 \\ -\frac{1}{2}u_2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad K^2 + u_1\mathrm{Id} = \begin{pmatrix} -u_1 & 0 & 4 \\ -u_2 & -u_1 & 0 \\ 0 & -u_2 & u_1 \end{pmatrix}.$$

Matrices $K_0 = \mathrm{Id}$, $K_1 = K$, $K_2 = K^2 + u_1\mathrm{Id}$ commute.

Define $\theta_0 = \theta = \mathrm{d}u_1$. We write 1-forms as one-line matrices, and we consider the 1-forms,

$$\begin{aligned} \theta_1 &= \theta_{01} = \theta K = 2\mathrm{d}u_2, & \theta_2 &= \theta_{02} = \theta K_2 = -u_1\mathrm{d}u_1 + 4\mathrm{d}u_3, \\ \theta_{11} &= \theta_1 K = 2\mathrm{d}u_1, & \theta_{12} &= \theta_2 K = -2(u_2\mathrm{d}u_1 + u_1\mathrm{d}u_2), \\ \theta_{22} &= \theta_2 K_2 = u_1^2\mathrm{d}u_1 - 4u_2\mathrm{d}u_2. \end{aligned}$$

All the 1-forms, $\theta K_i K_j$, $0 \leq i, j \leq 2$, are exact, and therefore closed. (Applying the successive powers of K to θ does not yield a sequence of closed forms. While θK^2 and θK^3 are exact, $\theta K^4 = -2u_2\mathrm{d}u_1 - 4u_1\mathrm{d}u_2$ is not closed.)

Let $X = \dfrac{\partial}{\partial u_3}$. The vector fields

$$X_0 = X = \frac{\partial}{\partial u_3}, \quad X_1 = KX = 2\frac{\partial}{\partial u_2}, \quad X_2 = K_2 X = 4\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_3},$$

commute.

Therefore $\left(\mathbb{R}^3, (\mathrm{Id}, K, K_2), \theta = \mathrm{d}u_1, X = \dfrac{\partial}{\partial u_3}\right)$ is a Magri–Lenard complex. In addition, $\mathcal{L}_X\theta = 0$ and $\mathcal{L}_X K = 0$.

Computing the Nijenhuis torsion $\mathcal{T}_K$ of K, we find that $\mathcal{T}_K(e_1, e_2) = e_2$, while $\mathcal{T}_K(e_1, e_3) = e_3$, and $\mathcal{T}_K(e_2, e_3) = 0$. Thus, the vector-valued 2-form $\mathcal{T}_K$ satisfies $i_\theta \mathcal{T}_K = 0$, where $\theta = \mathrm{d}u_1$, but $\mathcal{T}_K$ does not vanish. For a vector X in $\mathbb{R}^3$, let $\mathcal{T}_K(X)$ be the endomorphism of $\mathbb{R}^3$ defined by $Y \mapsto \mathcal{T}_K(X, Y)$. Then

$$\mathcal{T}_K(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathcal{T}_K(e_2) = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{T}_K(e_3) = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We now compute the Haantjes torsion of K. Once we know the components of $\mathcal{T}_K$ and have computed $K^2 = \begin{pmatrix} -2u_1 & 0 & 4 \\ -u_2 & -2u_1 & 0 \\ 0 & -u_2 & 0 \end{pmatrix}$, we can compute the components of $\mathcal{H}_K$:

$$\begin{aligned} \mathcal{H}_K(e_1, e_2) &= \mathcal{T}_K(Ke_1, Ke_2) - K\mathcal{T}_K(Ke_1, e_2) - K\mathcal{T}_K(e_1, Ke_2) + K^2\mathcal{T}_K(e_1, e_2) \\ &= \mathcal{T}_K(-u_1e_2 - \tfrac{1}{2}u_2e_3, 2e_1) - K\mathcal{T}_K(-u_1e_2 - \tfrac{1}{2}u_2e_3, e_2) - K\mathcal{T}_K(e_2, 2e_1) \\ &\qquad - 2u_1e_2 - u_2e_3 = 2u_1e_2 + u_2e_3 - 2u_1e_2 - u_2e_3 = 0, \end{aligned}$$

and similarly, $\mathcal{H}_K(e_1, e_3) = \mathcal{H}_K(e_2, e_3) = 0$. Therefore $\mathcal{H}_K = 0$.

Next, we compute the Nijenhuis and Haantjes torsions of K_2. After computing

$$(K_2)^2 = \begin{pmatrix} (u_1)^2 & -4u_2 & 2u_2u_1 \\ 0 & (u_1)^2 & -4u_2 \\ (u_2)^2 & 0 & (u_1)^2 \end{pmatrix},$$

we evaluate the Nijenhuis torsion of K_2 on the basis vectors and we obtain

$$\mathcal{T}_{K_2}(e_1, e_2) = u_2e_3, \ \mathcal{T}_{K_2}(e_1, e_3) = -2u_1e_3, \ \mathcal{T}_{K_2}(e_2, e_3) = 4e_2.$$

Then we compute the Haantjes torsion of K_2 and we find that it vanishes. Therefore, $\big(\mathbb{R}^3, (\mathrm{Id}, K, K_2)\big)$ is a Haantjes manifold.

Why this example? The matrix K in the preceding example is that of the *integrable system of hydrodynamic type*, $U_t = KU_x$, where $U = \left(\begin{smallmatrix} u_1 \\ u_2 \\ u_3 \end{smallmatrix}\right)$ and u_1, u_2, u_3 are functions of two variables (t, x). Explicitly, this differential system is

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial u_2}{\partial x}, \\ \frac{\partial u_2}{\partial t} &= -u_1\frac{\partial u_1}{\partial x} + 2\frac{\partial u_3}{\partial x}, \\ \frac{\partial u_3}{\partial t} &= -\frac{1}{2}u_2\frac{\partial u_1}{\partial x}. \end{aligned}$$

Another case to which the geometric structure of Haantjes manifolds is applicable is that of the *dispersionless Gel'fand–Dickey equations* defined by the $(1,1)$-tensor, $K = \left(\begin{smallmatrix} 0 & 1 & 0 \\ u_1 & 0 & 1 \\ u_2 & u_1 & 0 \end{smallmatrix}\right)$.

6. WDVV equations and Magri–Lenard complexes

Magri showed how the geometric structures on Haantjes manifolds are related to the solutions of the *WDVV equations*[4] which are the equations satisfied by the partial derivatives of the Hessian, i.e., the matrix of second-order partial derivatives, of a function F of n variables, $(x^1, x^2, \ldots, x^n)$. Let the Hessian matrix of F be denoted by h and assume that the matrix $\dfrac{\partial h}{\partial x^1}$ is invertible. The WDVV equations can be written as the set of nonlinear equations,

$$\frac{\partial h}{\partial x^i}\left(\frac{\partial h}{\partial x^1}\right)^{-1}\frac{\partial h}{\partial x^j} = \frac{\partial h}{\partial x^j}\left(\frac{\partial h}{\partial x^1}\right)^{-1}\frac{\partial h}{\partial x^i}, \quad i, j = 1, \ldots, n,$$

that express the pairwise commutativity of the matrices

$$\left(\frac{\partial h}{\partial x^1}\right)^{-1}\frac{\partial h}{\partial x^i}, \quad i = 1, \ldots, n.$$

Given a solution, F, of the WDVV equations, consider the 1-forms $\theta_{ij} = \mathrm{d}a_{ij}$, $i, j = 1, \ldots, n$, where the $a_{ij} = \dfrac{\partial^2 F}{\partial x^i \partial x^j}$ are the entries of the Hessian matrix h of F. Assume that the 1-forms θ_{1j}, $j = 1, \ldots, n$, are linearly independent, and define operators R_k by the condition

$$\theta_{1j} R_k = \theta_{jk}.$$

Then $R_1 = \mathrm{Id}$ and

$$\theta_{11} R_i R_j = \theta_{1i} R_j = \theta_{ij}.$$

Proposition. *Consider the commuting vector fields* $X_k = \dfrac{\partial}{\partial x^k}$. *Then the operators* R_k *satisfy the relation*

$$X_k = R_k \frac{\partial}{\partial x^1}.$$

Proof. On each of the linearly independent 1-forms $\theta_{1j} = \mathrm{d}a_{1j}$, $j = 1, \ldots, n$, the vector fields $X_k = \dfrac{\partial}{\partial x^k}$ and $R_k \dfrac{\partial}{\partial x^1}$ take the same value, $\dfrac{\partial a_{1j}}{\partial x^k} = \dfrac{\partial a_{jk}}{\partial x^1}$. □

The operators R_k commute because F is assumed to be a solution of the WDVV equations. In fact, $R_k \dfrac{\partial h}{\partial x^1} = \dfrac{\partial h}{\partial x^k}$. Therefore the operators R_k, the vector field $\dfrac{\partial}{\partial x^1}$ and the 1-form θ_{11} define a *Magri–Lenard complex*.

Conversely, consider a *Magri–Lenard complex* (M, R_k, X, θ). Locally, on an open set of the manifold M, the commuting vector fields $X_k = R_k X$ define coordinates x^k, and the closed 1-forms $\theta_{ij} = \theta R_i R_j$ admit local potentials a_{ij},

$$\theta_{ij} = \mathrm{d}a_{ij}.$$

[4] This system of partial differential equations is named after E. Witten, R. Dijkgraaf, E. Verlinde and H. Verlinde.

For $i, j, k = 1, \ldots, n$, consider the functions

$$c_{ijk} = \langle \theta_{ij}, X_k \rangle = \langle \theta R_i Rj, R_k X \rangle = \langle \theta, R_i Rj R_k X \rangle.$$

In local coordinates,

$$c_{ijk} = \langle \theta_{ij}, X_k \rangle = \left\langle \mathrm{d}a_{ij}, \frac{\partial}{\partial x^k} \right\rangle = \frac{\partial a_{ij}}{\partial x^k}.$$

Because the operators R_k commute pairwise, functions c_{ijk} are symmetric. Therefore the functions $\dfrac{\partial a_{ij}}{\partial x^k}$ are symmetric, which implies that the a_{ij} are the second-order partial derivatives of a function $F(x^1, \ldots, x^n)$, $a_{ij} = \dfrac{\partial^2 F}{\partial x^i \partial x^j}$. Then the Hessian of F satisfies the WDVV equations.

7. Lenard–Haantjes chains

To conclude this survey of modern work based on the 1955 article of Haantjes, I must mention recent work of Tempesta and Tondo [24–26].

A *symplectic Haantjes manifold* of dimension $2n$ is a symplectic manifold (M, ω) endowed with a family of n linearly independent Haantjes tensors, $K_0 = \mathrm{Id}, K_1, \ldots, K_{n-1}$, such that:

(i) each map $\omega^\flat \circ K_i : TM \to T^*M$, $i = 0, \ldots, n-1$, is skew-symmetric,
(ii) the K_i's, $i = 0, \ldots, n-1$, generate a $C^\infty(M)$-module of Haantjes tensors,
(iii) for all $i, j = 0, \ldots, n-1$, $K_i K_j$ has a vanishing Haantjes torsion, and $K_i K_j = K_j K_i$.

"Lenard–Haantjes chains" are constructed from a given Hamiltonian H on a symplectic Haantjes manifold by defining Hamiltonians H_j such that

$$\mathrm{d}H_{j+1} = \mathrm{d}H \, K_j.$$

Then the Poisson bracket of any two Hamiltonians H_j in the chain vanishes.

Among the examples given by Tempesta and Tondo are the generalized Stäckel systems, where ω is the canonical symplectic form on $T^*(\mathbb{R}^n)$ with coordinates (q^i, p_i), and the K_j's are diagonal operators defined in terms of the cofactors of a Stäckel matrix, an invertible matrix whose ith row depends only on the coordinate q^i.

Many other applications of the Haantjes tensors can be found in the publications and in the preprints of Tempesta and Tondo, as well as in Magri's articles, published or in progress. The comparison of the methods thus proposed to investigate the geometry of integrable systems remains to be done.

Acknowledgment

I must thank Giorgio Tondo for communicating to me unpublished material on his joint work with Piergiulio Tempesta, and I extend my sincere and friendly thanks to Franco Magri upon whose work, both published and in preprint form, I have largely drawn in this paper.

References

[1] O.I. Bogoyavlenskij, General algebraic identities for the Nijenhuis and Haantjes tensors, *Izv. Ross. Akad. Nauk Ser. Mat.* 68 (2004), no. 6, 71–84; translation in *Izv. Math.* 68 (2004), no. 6, 1129–1141.

[2] O.I. Bogoyavlenskij, Algebraic identity for the Schouten tensor and bi-Hamiltonian systems, *Differential Geom. Appl.* 25 (2007), no. 5, 452–460.

[3] E.V. Ferapontov, D.G. Marshall, Differential-geometric approach to the integrability of hydrodynamic chains: the Haantjes tensor, *Math. Ann.* 339 (2007), 61–99.

[4] A.S. Fokas, B. Fuchssteiner, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D* 4 (1981/82), no. 1, 47–66.

[5] A. Frölicher, A. Nijenhuis, Theory of vector-valued differential forms. I. Derivations of the graded ring of differential forms, *Nederl. Akad. Wetensch. Proc.* Ser. A. 59 = *Indag. Math.* 18 (1956), 338–359.

[6] I.M. Gelfand, I.Ya. Dorfman, The Schouten bracket and Hamiltonian operators, *Funktsional. Anal. i Prilozhen.* 14 (1980), no. 3, 71–74; translation in *Funct. Anal. Appl.* 14 (1980), no. 3, 223–226.

[7] J. Haantjes, On X_m-forming sets of eigenvectors, *Nederl. Akad. Wetensch. Proc. Ser. A.* 58 = *Indag. Math.* 17 (1955), 158–162.

[8] Y. Kosmann-Schwarzbach, F. Magri, Poisson–Nijenhuis structures, *Ann. Inst. Henri Poincaré* 53 (1990), 35–81.

[9] Y. Kosmann-Schwarzbach, V. Rubtsov, Compatible structures on Lie algebroids and Monge–Ampère operators, *Acta Appl. Math.* 109 (2010), no. 1, 101–135.

[10] F. Magri, A simple model of an integrable Hamiltonian system, *J. Math. Phys.* 19 (1978), 1156–1162.

[11] F. Magri, Recursion operators and Frobenius manifolds, *SIGMA Symmetry Integrability Geom. Methods Appl.* 8 (2012), paper no. 076, 7 pages.

[12] F. Magri, Haantjes manifolds, *Journal of Physics: Conference Series* (Physics and Mathematics of Nonlinear Phenomena, 2013) 482 (2014), paper no. 012028, 10 pages.

[13] F. Magri, WDVV equations, Proceedings (Current Problems in Theoretical Physics: Integrability and Nonlinearity in Field Theory, Vietri al Mare 2015), arXiv: 1510.07950.

[14] F. Magri, Haantjes manifolds and Veselov systems, Proceedings (Physics and Mathematics of Nonlinear Phenomena, Gallipoli 2015), *Teor. Mat. Fiz.* 189 (2016), no. 1, 101–114; translation in *Theor. Math. Phys.* 189 (2016), no. 1, 1486–1499.

[15] F. Magri, preprint (2017).

[16] F. Magri, C. Morosi, A geometrical characterization of integrable Hamiltonian systems through the theory of the Poisson–Nijenhuis manifolds, Quaderno S/19, Università di Milano, 1984.

[17] F. Magri, C. Morosi, G. Tondo, Nijenhuis G-manifolds and Lenard bicomplexes: a new approach to KP systems, *Commun. Math. Phys.* 115 (1988), no. 3, 457–475.

[18] A. Nijenhuis, X_{n-1}-forming sets of eigenvectors, *Nederl. Akad. Wetensch. Proc.* Ser. A. 54 = *Indagationes Math.* 13 (1951), 200–212.

[19] A. Nijenhuis, Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I, II, *Nederl. Akad. Wetensch. Proc. Ser. A* 58 = *Indag. Math.* 17 (1955), 390–397, 398–403.

[20] A. Nijenhuis, J.A. Schouten: a master at tensors (28 August 1883–20 January 1971), *Nieuwe Arch. Wisk.* (3) 20 (1972), 1–19.

[21] J. Praught, R.G. Smirnov, Andrew Lenard: a mystery unraveled, *SIGMA Symmetry Integrability Geom. Methods Appl.* 1 (2005), paper no. 005, 7 pages.

[22] J.A. Schouten, Über Differentialkomitanten zweier kontravarianter Grössen, *Nederl. Akad. Wetensch. Proc.* 43 (1940), 449–452 = *Indag. Math.* 2 (1940), 182–185.

[23] J.A. Schouten, On the differential operators of first order in tensor calculus, in: Convegno Internazionale di Geometria Differenziale (1953), Edizioni Cremonese, Roma, 1954, 1–7.

[24] P. Tempesta, G. Tondo, Haantjes structures for the Jacobi–Calogero model and the Benenti system, *SIGMA Symmetry Integrability Geom. Methods Appl.* 12 (2016), paper no. 023, 18 pages.

[25] P. Tempesta, G. Tondo, Haantjes manifolds of classical integrable systems, arXiv: 1405.5118v3 (2016).

[26] P. Tempesta, G. Tondo, Haantjes algebras and diagonalization, preprint (2017).

[27] Kentaro Yano, Selected papers, Morio Obata, ed., North-Holland, Amsterdam, New York, N.Y., Oxford, 1982.

Yvette Kosmann-Schwarzbach
Centre de Mathématiques Laurent Schwartz (retired)
École Polytechnique
F-91128 Palaiseau, France
e-mail: yks@math.cnrs.fr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 181–187

Kepler Problem and Jordan Algebras

Guowu Meng

Abstract. It is reported here that the Jordan algebra approach to the Kepler problem captures the essence of the Kepler problem.

Mathematics Subject Classification (2010). Primary 70GXX; Secondary 32M15.

Keywords. Kepler problem, Jordan algebra, oscillator, Lorentz transformation, future light cone.

1. Introduction

The Kepler problem – a mathematical model for the simplest solar system or the simplest atom – is a completely integrable model. The initial solution of this model, at either the classical level (Newton, 17th century) or the quantum level (Schrödinger, 1920s), was an epoch-making event in science.

The simplicity of the Kepler problem might have misled many people to believe that nothing new about it is yet to be discovered. However, a host of past mathematical discoveries reveals that the mathematical simplicity of this model is quite deceptive. Here is a partial list of past discoveries:

1. **Laplace–Runge–Lenz vector and O(4)-symmetry**. This discovery/rediscovery is associated with historical figures such as Jakob Hermann, Johann Bernoulli, Pierre-Simon Laplace, Josiah Willard Gibbs, Carle Runge, Wilhelm Lenz, Wolfgang Pauli, ...
2. **The S-duality**. This discovery/rediscovery is associated with scientists such as William Rowan Hamilton, Vladimir Fock, J. Moser, ...
3. **The curvature deformed versions**. This discovery/rediscovery is associated with scientists such as Erwin Schrödinger, Leopold Infeld, Peter Higgs, ...
4. **The SO(2,4)-dynamic symmetry**. This discovery/rediscovery is associated with names such as I.A. Malkin and V.I. Manko [1], A.O. Barut and H. Kleinert [2], ...

The author was supported by the Hong Hong Research Grants Council under RGC Project No. 16304014 and the Hong Kong University of Science and Technology under DAG S09/10.SC02.

5. **The magnetized versions**. This discovery/rediscovery is associated with names such as H. McIntosh and A. Cisneros [3], D. Zwanziger [4], T. Iwai [5], G.W. Meng [6], ...

In this talk, I will report that *the Euclidean Jordan algebras capture the essence of the Kepler problem.* Some elaboration on this point is needed here. In general, we say that A captures the essence of B if at least the following three conditions are met: 1) A gives a new insight into B, 2) there is a general theory based on A so that B is just a special example, 3) things become more unified: B and C (something seemingly different from B) are just two different examples of the general theory. In our case here, the Euclidean Jordan algebras is A and the Kepler problem is B. Indeed,

1) The Jordan algebra approach to Kepler problem yields a new insight into the Kepler problem and its magnetized version [7]: the elliptic oriented orbits of magnetized Kepler problems are related to each other via Lorentz transformations and dilations.
2) There is a general theory [7] based on Jordan algebra in which the Kepler problem is just an example associated with the Jordan algebra of complex Hermitian matrices of order 2.
3) An n-dimensional isotropic oscillator is the bounded sector of a Kepler-type problem associated with the Jordan algebra of real symmetric matrices of order n, and the Fradkin tensor [8] of an isotropic oscillator is just the "Laplace–Runge–Lenz vector".

1.1. Motivations

Since the research on the Kepler problem is not topical, some motivations must be provided. At the moment, the speaker can cite at least the following three reasons for this research:

(1) Most mathematical physicists would agree that the Kepler problem is an all-in-one mathematical model with beauty, simplicity, and truth.
(2) The Kepler problem is much deeper than most of us might have thought. Indeed, the signatures of special relativity such as future light cone or Lorentz transformation naturally appear [9] in this non-relativistic dynamical problem.
(3) The Kepler problem might provide some clues and hints for fundamental physics. For example, in our study of this problem, we found that [9] i) a second temporal dimension appears naturally, ii) the magnetic charge is relative.

F. Dyson [10] once pointed out that a research in mathematical physics is always unfashionable, but it might turn out to be extremely fruitful and interesting many years later. History is full of examples like that – from Hamilton's work on Newtonian mechanics to Weyl's work on electromagnetism. This is another motivation for our research on the Kepler problem.

2. Review of the Kepler problem and the Lenz algebra

Recall that, for the Kepler problem, the phase space $T^*\mathbb{R}^3_*$ is a Poisson manifold, and the Hamiltonian, angular momentum, and Lenz vector are

$$\mathrm{H} = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r}, \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{A} = \mathbf{L} \times \mathbf{p} + \frac{\mathbf{r}}{r}$$

respectively. Here $\mathbf{r} \in \mathbb{R}^3_*$ is the position vector and $\mathbf{p}$ is the linear momentum.

In terms of the standard canonical coordinates $x^1, x^2, x^3, p_1, p_2, p_3$ on $T^*\mathbb{R}^3_*$, the Poisson structure can be described by the following basic Poisson bracket relations:

$$\{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0.$$

The fact that $\mathbf{L}$ and $\mathbf{A}$ are constants of motion can be restated as

$$\{\mathbf{L}, \mathrm{H}\} = 0, \quad \{\mathbf{A}, \mathrm{H}\} = 0.$$

Theorem 1. *Let L_i (resp. A_i) be the ith component of $\mathbf{L}$ (resp. $\mathbf{A}$). Then*

$$\boxed{\begin{aligned} \{L_i, \mathrm{H}\} &= 0\ , \\ \{A_i, \mathrm{H}\} &= 0\ , \\ \{L_i, L_j\} &= \epsilon_{ijk} L_k\ , \\ \{L_i, A_j\} &= \epsilon_{ijk} A_k\ , \\ \{A_i, A_j\} &= -2\mathrm{H}\epsilon_{ijk} L_k\ . \end{aligned}} \tag{1}$$

Here $\epsilon_{ijk} = 1$ (resp. -1) if ijk is an even (resp. odd) permutation of 123, and equals to 0 otherwise. A summation over the repeated index k is assumed. So we have $\{L_1, L_2\} = L_3$, $\{L_2, A_3\} = A_1$, and so on.

The Poisson algebra with generators H, $L_1, L_2, L_3, A_1, A_2, A_3$ and relations in Eq. (2) is called the **Lenz algebra**.

3. Review of formally real Jordan algebras

Formally real Jordan algebras [11] were introduced by P. Jordan [12] in the 1930s as the quantum version of the algebra of classical observables. For the finite-dimensional ones, here is the definition.

Definition 2. A finite-dimensional **formally real Jordan algebra** is a finite-dimensional real algebra V with unit e such that, for any two elements a, b in V, we have

1) $ab = ba$ (symmetry),
2) $a(ba^2) = (ab)a^2$ (weakly associative),
3) $a^2 + b^2 = 0 \implies a = b = 0$ (formally real).

The simplest example is $\mathbb{R}$. We shall use $L_a\colon V \to V$ to denote the multiplication by a. It is a fact that condition 3) is equivalent to condition

3′) The "Killing form" $\langle a, b\rangle = \frac{1}{\dim V}\operatorname{tr} L_{ab}$ is positive definite.

So formally real Jordan algebras are also called **Euclidean Jordan algebras**.

3.1. The classification theorem

For the formally real Jordan algebras, the simplest example is the field of real numbers, i.e., $\mathbb{R}$. The more sophisticated example is the algebra $\mathrm{H}_n(\mathbb{R})$ which consists of real symmetric matrices of order n, under the symmetrized matrix multiplication. Next, we have the algebra $\mathrm{H}_n(\mathbb{C})$ ($\mathrm{H}_n(\mathbb{H})$ *resp.*) which consists of complex (quaternionic *resp.*) Hermitian symmetric matrices of order n, under the symmetrized matrix multiplication. However, the algebra $\mathrm{H}_n(\mathbb{O})$ which consists of octonionic Hermitian symmetric matrices of order n is not a formally real Jordan algebra unless $n = 2$ or 3. Finally, there is an infinite series of formally real Jordan algebras which is associate with the Clifford algebra of the Euclidean vector spaces:

$$\Gamma(n) := \mathbb{R} \oplus \mathbb{R}^n$$

whose multiplication rule is given by formula

$$(\alpha, \vec{u})(\beta, \vec{v}) = (\alpha\beta + \vec{u}\cdot\vec{v}, \alpha\vec{v} + \beta\vec{u}).$$

Along with J. von Neumann and E. Wigner, P. Jordan [13] proved the following classification theorem for finite-dimensional Euclidean Jordan algebras.

Theorem 3. *Euclidean Jordan algebras are semi-simple, and the simple ones consist of four infinite series and one exceptional:*

$\mathbb{R}$.
$\Gamma(n) := \mathbb{R} \oplus \mathbb{R}^n$, $n \geq 2$.
$\mathrm{H}_n(\mathbb{R})$, $n \geq 3$.
$\mathrm{H}_n(\mathbb{C})$, $n \geq 3$.
$\mathrm{H}_n(\mathbb{H})$, $n \geq 3$.
$\mathrm{H}_3(\mathbb{O})$.

Some points are worth to mention:

- $\Gamma(0) \cong \mathbb{R}$, $\Gamma(1) \cong \mathbb{R}\oplus\mathbb{R}$, $\Gamma(2) \cong \mathrm{H}_2(\mathbb{R})$, $\Gamma(3) \cong \mathrm{H}_2(\mathbb{C})$, $\Gamma(5) \cong \mathrm{H}_2(\mathbb{H})$, $\Gamma(9) \cong \mathrm{H}_2(\mathbb{O})$.
- Each but the exceptional one is associated with an associative algebra. This is a result of A.A. Albert [14].
- $\mathbb{R}$, $\Gamma(3)$, and $\mathrm{H}_3(\mathbb{O})$ are somewhat special.

3.2. The structure algebra

For a, b in the Jordan algebra V, we let

$$S_{ab} := [L_a, L_b] + L_{ab}, \quad \{abc\} := S_{ab}(c)$$

and $\mathfrak{str}$ be the span of $\{S_{ab} \mid a, b \in V\}$ over $\mathbb{R}$. Since

$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{c\{bad\}},$$

$\mathfrak{str}$ becomes a real Lie algebra – the **structure algebra** of V. For example, (1) $\mathfrak{str} \cong \mathbb{R}$ for $V = \mathbb{R}$, (2) $\mathfrak{str} \cong \mathfrak{so}(1,3) \oplus \mathbb{R}$ for $V = \Gamma(3)$.

This Lie algebra is not simple, actually not even semi-simple, because it has a non-trivial central element: $S_{ee} = L_e$. The good news is that this algebra can be extended to a simple real Lie algebra provided that V is a simple Euclidean Jordan algebra. From hereon V is assumed to be a simple Euclidean Jordan algebra.

3.3. The conformal algebra

Write $z \in V$ as X_z and $\langle w, \rangle \in V^*$ as Y_w.

Definition 4 (J. Tits, M. Koecher, I.L. Kantor, 1960s). The **conformal algebra** $\mathfrak{co}$ is a Lie algebra whose underlying real vector space is $V \oplus \mathfrak{str} \oplus V^*$, and the commutation relations are

$$\boxed{\begin{array}{c}[X_u, X_v] = 0, \quad [Y_u, Y_v] = 0, \quad [X_u, Y_v] = -2S_{uv}, \\ [S_{uv}, X_z] = X_{\{uvz\}}, \quad [S_{uv}, Y_z] = -Y_{\{vuz\}}, \\ [S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}\end{array}} \tag{2}$$

for u, v, z, w in V.

When $V = \Gamma(3)$, $\mathfrak{str} = \mathfrak{so}(3,1) \oplus \mathbb{R}$, $\mathfrak{co} = \mathfrak{so}(4,2)$. When $V = \mathbb{R}$, $\mathfrak{str} = \mathbb{R}$, $\mathfrak{co} = \mathfrak{sl}(2,\mathbb{R})$. In general, $\mathfrak{co}$ is the Lie algebra of the bi-holomorphic automorphism group of the complex domain $V \times \mathrm{i}V_+ \subset V \otimes_{\mathbb{R}} \mathbb{C}$.

4. The universal Kepler problems

Let $\mathcal{TKK}$ be the complexified universal enveloping algebra for the conformal algebra, but with Y_e being formally inverted (the formal two-sided inverse of Y_e shall be denoted by Y_e^{-1}).

Definition 5 (Ref. [15]). The **universal angular momentum** is

$$\begin{aligned} L : V \times V &\to \mathcal{TKK} \\ (u, v) &\mapsto L_{u,v} := [L_u, L_v]. \end{aligned} \tag{3}$$

The **universal Hamiltonian** is

$$H := \frac{1}{2} Y_e^{-1} X_e - (\mathrm{i}Y_e)^{-1}. \tag{4}$$

The **universal Lenz vector** is

$$\begin{aligned} A : V &\to \mathcal{TKK} \\ u &\mapsto A_u := (\mathrm{i}Y_e)^{-1}[L_u, (\mathrm{i}Y_e)^2 H]. \end{aligned} \tag{5}$$

4.1. Universal Lenz algebra

Via the commutation relation for the conformal algebra, one can verify

Theorem 6. *For u, v, z and w in V,*

$$\boxed{\begin{aligned}[L_{u,v}, H] &= 0\ ,\\ [A_u, H] &= 0\ ,\\ [L_{u,v}, L_{z,w}] &= L_{L_{u,v}z,w} + L_{z,L_{u,v}w}\ ,\\ [L_{u,v}, A_z] &= A_{L_{u,v}z}\ ,\\ [A_u, A_v] &= -2HL_{u,v}\ .\end{aligned}} \tag{6}$$

5. Examples

In view of Theorems 1 and 6 we conclude that

$$\begin{gathered}\text{a concrete realization of the conformal algebra}\\ \Downarrow\\ \text{a concrete model of the Kepler type}\end{gathered}$$

To be more precise, we have

$$\begin{aligned}\text{a suitable operator realization} &\implies \text{a quantum model.}\\ \text{a suitable Poisson realization} &\implies \text{a classical model.}\end{aligned}$$

5.1. Kepler problem

The Jordan algebra is $V := \mathrm{H}_2(\mathbb{C})$. An element in V can be written as

$$X = \begin{bmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{bmatrix}.$$

If X has rank one and is semi-positive definite, then $\det X = 0$, i.e.,

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0,$$

moreover $x_0 > 0$. So the set of rank one, semi-positive elements in $\mathrm{H}_2(\mathbb{C})$ is the future light cone $\Lambda_+ \cong \mathbb{R}^3_*$ – the punctured 3D Euclidean space.

One can check that $(C^\infty(T^*\Lambda_+), \{,\})$ provides a suitable Poisson realization of the conformal algebra of V for which the universal Hamiltonian, the universal angular momentum and the universal Lenz vector respectively becomes the Hamiltonian, the angular momentum, and the Laplace–Runge–Lenz vector of the Kepler problem.

5.2. Isotropic oscillator in dimension n

The Jordan algebra is $V := \mathrm{H}_n(\mathbb{R})$, i.e., the Jordan algebra of real symmetric matrices of order n. We let $\mathcal{C}_1$ be the set of rank one, semi-positive elements in

$\mathrm{H}_n(\mathbb{R})$. One can show that $(C^\infty(T^*\mathcal{C}_1), \{,\})$ provides a suitable Poisson realization of the conformal algebra of V, from which one obtains a Kepler-type problem whose bounded sector is isomorphic to the isotropic oscillator in dimension n. Moreover, the Lenz vector gets identified with the Fradkin tensor under the isomorphism.

References

[1] I.A. Malkin and V.I. Man'ko, Symmetry of the Hydrogen Atom, *JETP Letters* **2** (1966), 146–148.

[2] A.O. Barut and H. Kleinert, Transition Probabilities of the H-Atom from Noncompact Dynamical Groups, *Phys. Rev.* **156** (1967), 1541–1545.

[3] H. McIntosh and A. Cisneros, *Degeneracy in the presence of a magnetic monopole*, J. Math. Phys. **11** (1970), 896–916.

[4] D. Zwanziger, *Exactly soluble nonrelativistic model of particles with both electric and magnetic charges*, Phys. Rev. **176** (1968), 1480–1488.

[5] T. Iwai, *The geometry of the* SU(2) *Kepler problem*, J. Geom. Phys. **7** (1990), 507–535.

[6] G.W. Meng, MICZ-Kepler problems in all dimensions, *J. Math. Phys.* **48**, 032105 (2007).

[7] G.W. Meng, *Generalized Kepler Problems I: without Magnetic Charges*, J. Math. Phys. **54** (2013) 012109.

[8] D.M. Fradkin, *Three-dimensional isotropic harmonic oscillator and* SU(3), Am. J. Phys. **33**, no. 3 (1965), 207–211.

[9] G.W. Meng, *Lorentz group and oriented McIntosh–Cisneros–Zwanziger–Kepler orbits*, J. Math. Phys. **53** (2012) 052901.

[10] F.J. Dyson, *Unfashionable pursuits*. In *Symmetries in Particle Physics* (1984), 265–285. Springer.

[11] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*. Oxford: Clarendon Press, 1994.

[12] P. Jordan, *Über die Multiplikation quantenmechanischer Größen*, Z. Phys. **80** (1933), 285–291.

[13] P. Jordan, J. von Neumann and E.P. Wigner, *On an Algebraic Generalization of the Quantum Mechanical Formalism*, Ann. of Math. (1934), 29–64.

[14] A.A. Albert, *On a certain algebra of quantum mechanics*, Ann. of Math. (1934), 65–73.

[15] G.W. Meng, *The Universal Kepler Problems*, J. Geom. Symm. Phys. **36** (2014), 47–57

Guowu Meng
Department of Mathematics
Hong Kong Univ. of Sci. and Tech.
Clear Water Bay, Kowloon, Hong Kong
e-mail: mameng@ust.hk

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 189–195

On Rank Two Algebro-Geometric Solutions of an Integrable Chain

Andrey E. Mironov and Gulnara S. Mauleshova

Abstract. In this paper we consider a differential-difference system which is equivalent to the commutativity condition of two differential-difference operators. We study the rank two algebro-geometric solutions of this system.

Mathematics Subject Classification (2010). Primary 39A14, Secondary 39A23, 17B80.

Keywords. Differential-difference system, Lax representation, algebro-geometric solutions.

1. Introduction and main results

I.M. Krichever and S.P. Novikov [1, 2] proved the existence of the rank $l > 0$ algebro-geometric solution of the Kadomtsev–Petviashvili (KP) equation and the Toda chain. For such solutions the common eigenfunctions of auxiliary commuting operators (differential in the case of KP or difference in the case of the Toda chain) form a rank l vector bundle over the affine part of the spectral curve. They also proved that in the case of the rank two solutions of KP corresponding to elliptic spectral curves there is a remarkable separation of variables. Such solutions are expressed through the solutions of the $1 + 1$ Krichever–Novikov (KN) equation (see (6) below) and the solutions of an ODE [1] (see also [3, formula (22)]).

In this paper we study the rank two algebro-geometric solutions of the following equation

$$[\partial_y - T - f_n(x, y), \partial_x - b_n(x, y)T^{-1} - d_n(x, y)T^{-2}] = 0, \tag{1}$$

where f_n, b_n, d_n are the 4-periodic functions, $f_{n+4} = f_n, b_{n+4} = b_n$, $d_{n+4} = d_n$.

The work was supported by RSF (grant 14-11-00441).

Equation (1) is equivalent to the following 4-periodic chain

$$f_{n,x}(x,y) - b_n(x,y) + b_{n+1}(x,y) = 0, \tag{2}$$

$$f_{n-2}(x,y) - f_n(x,y) + \frac{d_{n,y}(x,y)}{d_n(x,y)} = 0, \tag{3}$$

$$f_{n-1}(x,y) - f_n(x,y) + \frac{b_{n,y}(x,y)}{b_n(x,y)} + \frac{d_n(x,y) - d_{n+1}(x,y)}{b_n(x,y)} = 0. \tag{4}$$

Recall that if two difference operators

$$L_k = \sum_{j=-K_-}^{K_+} u_j(n)T^j, \quad L_m = \sum_{j=-M_-}^{M_+} v_j(n)T^j, \quad n \in \mathbb{Z},$$

where T is the shift operator, $T\psi_n = \psi_{n+1}$, commute, then there is a polynomial $R(z,w)$ such that $R(L_k, L_m) = 0$. The *spectral curve* Γ is defined by the equation $R = 0$. The spectral curve parametrizes the common eigenvalues, i.e., if ψ_n is a common eigenfunction of L_k, L_m

$$L_k\psi_n = z\psi_n, \qquad L_m\psi_n = w\psi_n,$$

then $P = (z,w) \in \Gamma$. The *rank* l of the pair L_k, L_m is

$$l = \dim\{\psi_n : L_k\psi_n = z\psi_n, \quad L_m\psi_n = w\psi_n\}$$

for the general $P = (z,w) \in \Gamma$. The maximal commutative ring of difference operators is isomorphic to the ring of meromorphic functions on a spectral curve (a closed Riemann surface) with poles $q_1, \dots, q_s$. Such operators are called *s-point operators*. In the case of the rank one operators, the eigenfunctions (Baker–Akhiezer functions) can be found explicitly in terms of theta-functions of the Jacobi variety of spectral curves, and coefficients of such operators can be found using eigenfunctions. The case of higher rank is very complicated (higher rank Baker–Akhiezer functions are not found). The one-point rank two operators in the case of elliptic spectral curves were found in [1]. The one-point rank two operators in the case of the hyperelliptic spectral curve

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + c_{2g-1}z^{2g-1} + \dots + c_0 \tag{5}$$

were studied in [4]. In particular, examples of such operators were found for an arbitrary $g > 1$:

1) the operator

$$L_4^\sharp = (T + (r_3n^3 + r_2n^2 + r_1n + r_0)T^{-1})^2 + g(g+1)r_3n, \qquad r_3 \neq 0$$

commutes with a difference operator $L_{4g+2}^\sharp$,

2) the operator

$$L_4^\flat = (T + (r_1\cos(n) + r_0)T^{-1})^2 - 4r_1 \sin\left(\tfrac{g}{2}\right)\sin\left(\tfrac{g+1}{2}\right)\cos\left(n + \tfrac{1}{2}\right), \quad r_1 \neq 0$$

commutes with a difference operator $L_{4g+2}^\flat$.

Following [1, 2], we call the solution f_n, b_n, d_n of (2)–(4) the *algebro-geometric solution of rank two*, if there are one-point rank two commuting difference operators

$$L_4 = \sum_{i=-2}^{2} u_j(n,x,y)T^j, \qquad L_{4g+2} = \sum_{i=-(2g+1)}^{2g+1} v_j(n,x,y)T^j$$

commuting with $\partial_x - b_n(x,y)T^{-1} - d_n(x,y)T^{-2}$ and $\partial_y - T - f_n(x,y)$.

In the next theorem we show that in the case of an elliptic spectral curve given by the equation

$$w^2 = F_1(z) = z^3 + c_2z^2 + c_1z + c_0 \tag{6}$$

there is a separation of variables for rank two genus one solutions (similar to KP) of (2)–(4).

Theorem 1. *Let $\gamma_n = \gamma_n(x)$ and $\wp(y)$ satisfy the equations*

$$\gamma_n' = \frac{F_1(\gamma_n)(\gamma_{n-1} - \gamma_{n+1})}{(\gamma_{n-1} - \gamma_n)(\gamma_n - \gamma_{n+1})}, \tag{7}$$

$$(\wp'(y))^2 = F_1(\wp(y)), \tag{8}$$

and $\gamma_{n+4} = \gamma_n$, then

$$b_n(x,y) = -\frac{\wp'(y)\gamma_n'}{(\wp(y) - \gamma_n)^2},$$

$$d_n(x,y) = \frac{F_1(\gamma_{n-1})F_1(\gamma_n)(\wp(y) - \gamma_{n-2})(\wp(y) - \gamma_{n+1})}{(\gamma_{n-2} - \gamma_{n-1})(\gamma_{n-1} - \gamma_n)^2(\gamma_n - \gamma_{n+1})(\wp(y) - \gamma_{n-1})(\wp(y) - \gamma_n)},$$

$$f_n(x,y) = -\frac{\wp'(y)(\gamma_n - \gamma_{n+1})}{(\wp(y) - \gamma_n)(\wp(y) - \gamma_{n+1})} + g_n(y),$$

$$g_n(y) = \frac{(-1)^n}{\wp'(y)}\big((ns_1 + s_0)\wp^2(y) + (nk_1 + k_0)\wp(y) + (np_1 + p_0)\big)$$

are rank two algebro-geometric solutions of (2)–(4) *corresponding to the spectral curve* (6). *Here s_j, k_j, p_j are constants, $j = 1, 2$.*

Equation (7) has the following Lax representation

$$[L_4, \partial_x - V_{n-1}(x)V_n(x)T^{-2}] = 0,$$

where $L_4 = (T + V_n(x)T^{-1})^2 + W_n(x)$,

$$V_n(x) = \frac{F_1(\gamma_n(x))}{(\gamma_n(x) - \gamma_{n-1}(x))(\gamma_n(x) - \gamma_{n+1}(x))}, \tag{9}$$

$$W_n(x) = -c_2 - \gamma_n(x) - \gamma_{n+1}(x). \tag{10}$$

The operator L_4 commutes with an operator L_6 and L_4, L_6 form a one-point rank two pair of operators with the spectral curve (6). Equation (7) can be considered a difference analogue of KN equation

$$U_t = \frac{48F_1(-\frac{1}{2}(c_2 + U)) - U_{xx}^2 + 2U_xU_{xxx}}{8U_x}. \tag{11}$$

Equation (11), as well as (7), admits the Lax representation related to the rank two (differential) operators corresponding to the elliptic spectral curve. Moreover, (7), as well as (11), appears as an auxiliary equation for the separation variables in the $2+1$ system. For these reasons, we call (7) the Difference Krichever–Novikov equation (DKN).

Difference chains of type (7) were studied in many papers (see, e.g., [5, 6]), but we did not find (7) described in literature.

In Section 2 we study the rank two algebro-geometric solutions of the system

$$\partial_x V_n = V_n(W_{n-1} - W_n + V_{n-1} - V_{n+1}), \tag{12}$$

$$\partial_x W_n = (W_n - W_{n-1})V_n + (W_{n+1} - W_n)V_{n+1}. \tag{13}$$

This system admits a Lax pair (see (17) below). This system is reduced to DKN under the reduction (9), (10) at $g = 1$.

In Section 3 we prove Theorem 1.

2. DKN equation

Let us consider one-point operators of rank two L_4, L_{4g+2} corresponding to the hyperelliptic spectral curve Γ given by (5). Common eigenfunctions of L_4 and L_{4g+2} satisfy the equation

$$\psi_{n+1}(P) = \chi_1(n,P)\psi_{n-1}(P) + \chi_2(n,P)\psi_n(P),$$

where $\chi_1(n,P)$ and $\chi_2(n,P)$ are rational functions on Γ having $2g$ simple poles, depending on n (see [1]). The function $\chi_2(n,P)$ has, in addition, a simple pole at $q = \infty$. To find L_4 and L_{4g+2} it is sufficient to find χ_1 and χ_2. Let σ be the holomorphic involution on Γ, $\sigma(z,w) = \sigma(z,-w)$.
In [4] it was proved that if

$$\chi_1(n,P) = \chi_1(n,\sigma(P)), \qquad \chi_2(n,P) = -\chi_2(n,\sigma(P)),$$

then L_4 has the form

$$L_4 = (T + V_n T^{-1})^2 + W_n, \tag{14}$$

where

$$\chi_1 = -V_n \frac{Q_{n+1}}{Q_n}, \quad \chi_2 = \frac{w}{Q_n}, \quad Q_n(z) = z^g + \alpha_{g-1}(n)z^{g-1} + \cdots + \alpha_0(n),$$

and the polynomial Q satisfies the equation

$$F_g(z) = Q_{n-1}Q_{n+1}V_n + Q_nQ_{n+2}V_{n+1} + Q_nQ_{n+1}(z - V_n - V_{n+1} - W_n). \tag{15}$$

From (15) it follows that Q satisfies also linear equation

$$\begin{aligned} Q_{n-1}V_n + Q_n(z - V_n - V_{n+1} - W_n) - Q_{n+2}(z - V_{n+1} \\ - V_{n+2} - W_{n+1}) - Q_{n+3}V_{n+2} = 0. \end{aligned} \tag{16}$$

At $g = 1$ we have $Q_n = z - \gamma_n$, and equation (15) has the solution

$$V_n = \frac{F_1(\gamma_n)}{(\gamma_n - \gamma_{n-1})(\gamma_n - \gamma_{n+1})}, \qquad W_n = -c_2 - \gamma_n - \gamma_{n+1}.$$

At $g > 1$ it is a very difficult problem to solve equation (15). Moreover to find examples of solutions of (15) is also difficult problem.

At the end of this section we study difference evolution equations related to the operator L_4 (14).

The system (12), (13) has the following Lax representation

$$[(T + V_n(x)T^{-1})^2 + W_n(x), \partial_x - V_{n-1}(x)V_n(x)T^{-2}] = 0. \tag{17}$$

The system (12), (13) is included in the hierarchy of evolution equations of the form

$$[(T + V_n(t_k)T^{-1})^2 + W_n(t_k), \partial_{t_k} - P_1(n, t_k)T^{-2} - \cdots - P_k(n, t_k)T^{-2k}] = 0. \tag{18}$$

These evolution equations define symmetries of (12), (13). At $k = 2$ we have

$$\begin{aligned}\partial_{t_k} V_n = V_n\big(&V_{n-2}V_{n-1} + V_{n-1}V_n - V_nV_{n+1} - V_{n+1}V_{n+2} + V_{n-1}^2 - V_{n+1}^2 \\ &+ W_{n-1}^2 - W_n^2 + 2(V_{n-1} + V_n)W_{n-1} - 2(V_n + V_{n+1})W_n\big),\end{aligned} \tag{19}$$

$$\begin{aligned}\partial_{t_k} W_n = {}&V_{n-1}V_n(W_{n-2} - 2W_{n-1} + W_n) - V_{n+1}V_{n+2}(W_n - 2W_{n+1} + W_{n+2}) \\ &- V_n(W_{n-1} - W_n)(2V_n + W_{n-1} + W_n) \\ &- V_{n+1}(W_n - W_{n+1})(2V_{n+1} + W_n + W_{n+1}).\end{aligned} \tag{20}$$

In the case of the algebro-geometric operator L_4 at $g = 1$, i.e., V_n and W_n have the form (9), (10) the system (12), (13) is reduced to the DKN equation and equation (18) is reduced to the equation from the DKN hierarchy. For example, the system (19), (20) is reduced to

$$\begin{aligned}\partial_{t_k}\gamma_n = V_n\big(&V_{n+1}(W_{n-1} - 2W_n + W_{n+1}) \\ &- V_{n-1}(W_{n-2} - 2W_{n-1} + W_n) + (W_{n-1} - W_n)(2V_n + W_{n-1} + W_n)\big).\end{aligned}$$

At $g > 1$ there is no explicit reduction of (12), (13) since there is no explicit form of L_4. Nevertheless one can find the evolution equation on the polynomial Q_n associated with the algebro-geometric operator L_4. By direct calculation one can check the following lemma.

Lemma 1. *Equation*

$$\partial_x Q_n = V_n(Q_{n+1} - Q_{n-1}) \tag{21}$$

together with (12), (13) *defines a symmetry of* (15) *and* (16).

At $g = 1$ equation (21) is equivalent to DKN.

3. Proof of Theorem 1

In this section we explain how to obtain a rank two algebro-geometric solution of (2)–(4) at $g=1$. A similar method works for KP (see [3, 7]). The main idea is to apply Darboux type transformation to L_4. If Q_n satisfies (15) we have the following factorization (see [4])

$$L_4 - z = \left(T + \chi_2(n+1) - \frac{V_{n-1}V_n}{\chi_1(n-1)}T^{-1}\right)\left(T - \chi_2(n) - \chi_1(n)T^{-1}\right).$$

Let us assume that $\gamma_n = \gamma_n(x)$ and $z = z_0(y)$. After the Darboux transformation we get

$$\tilde{L}_4 = \left(T - \chi_2(n) - \chi_1(n)T^{-1}\right)\left(T + \chi_2(n+1) - \frac{V_{n-1}V_n}{\chi_1(n-1)}T^{-1}\right) + z_0(y).$$

Here $V_n = V_n(x)$ has the form (9),

$$\chi_1(n) = -V_n(x)\frac{z_0(y) - \gamma_{n+1}(x)}{z_0(y) - \gamma_n(x)}, \quad \chi_2(n) = \frac{w(y)}{z_0(y) - \gamma_n(x)}, \quad w^2(y) = F_1(z_0(y)).$$

The operator $\tilde{L}_4$ has the form

$$\tilde{L}_4 = T^2 + A_1(n,x,y)T + A_0(n,x,y) + A_{-1}(n,x,y)T^{-1} + A_{-2}(n,x,y)T^{-2},$$

$$A_1(n,x,y) = \frac{(\gamma_{n+2} - \gamma_n)z_0'(y)}{(z_0(y) - \gamma_n)(z_0(y) - \gamma_{n+2})},$$

$$A_0(n,x,y) = \frac{V_n(z_0(y) - \gamma_{n+1})^2 + V_{n+1}(z_0(y) - \gamma_n)^2 - F_1(z_0(y))}{(z_0(y) - \gamma_n)(z_0(y) - \gamma_{n+1})} + z_0(y),$$

$$A_{-1}(n,x,y) = \frac{(\gamma_{n-1} - \gamma_{n+1})V_n z_0'(y)}{(z_0(y) - \gamma_n)^2},$$

$$A_{-2}(n,x,y) = \frac{V_{n-1}V_n(z_0(y) - \gamma_{n-2})(z_0(y) - \gamma_{n+1})}{(z_0(y) - \gamma_{n-1})(z_0(y) - \gamma_n)}.$$

The operator $\tilde{L}_4$ commutes with $\tilde{L}_6$ and $\tilde{L}_4, \tilde{L}_6$ are operators of rank two with the same spectral curve (6). By direct calculation one can check that if

$$b_n = -\frac{z_0'(y)\gamma_n'}{(z_0(y) - \gamma_n)^2},$$

$$d_n = \frac{F_1(\gamma_{n-1})F_1(\gamma_n)(z_0(y) - \gamma_{n-2})(z_0(y) - \gamma_{n+1})}{(\gamma_{n-2} - \gamma_{n-1})(\gamma_{n-1} - \gamma_n)^2(\gamma_n - \gamma_{n+1})(z_0(y) - \gamma_{n-1})(z_0(y) - \gamma_n)},$$

and $\gamma_n(x)$ satisfies to DKN, then

$$[\tilde{L}_4, \partial_x - b_n(x,y)T^{-1} - d_n(x,y)T^{-2}] = 0.$$

By direct calculation one also can check that if $z_0(y) = \wp(y)$ satisfies (8), $\gamma_{n+4} = \gamma_n$, and $f_n(x,y)$ has the form as in Theorem 1, then

$$[\tilde{L}_4, \partial_y - T - f_n(x,y)] = 0.$$

Moreover $\tilde{L}_6$, $\partial_y - T - f_n(x,y)$, $\partial_x - b_n(x,y)T^{-1} - d_n(x,y)T^{-2}$ pairwise commute. Theorem 1 is proved. $\square$

References

[1] I.M. Krichever, S.P. Novikov, *Two-dimensional Toda lattice, commuting difference operators, and holomorphic bundles*, Russian Math. Surveys, **58**:3 (2003), 473–510.

[2] I.M. Krichever, S.P. Novikov, *Holomorphic bundles over algebraic curves and non-linear equations*, Russian Math. Surveys, **35**:6 (1980), 53–79.

[3] A.E. Mironov, *Commuting self-adjoint differential operators of rank two*, Russian Math. Surveys, **71**:4 (2016), 751–779.

[4] G.S. Mauleshova, A.E. Mironov *Commuting difference operators of rank two*, Russian Math. Surveys, **70**:3 (2015), 557–559.

[5] A.V. Mikhailov, A.B. Shabat, R.I. Yamilov *The symmetry approach to the classification of non-linear equations. Complete lists of integrable systems*, Russian Math. Surveys, **42**:4 (1987), 1–63.

[6] D. Levi, P. Winternitz, R.I. Yamilov, *Symmetries of the Continuous and Discrete Krichever–Novikov Equation*, SIGMA, **7** (2011), 097, 16 pp.

[7] G.A. Latham, E. Previato, *Darboux transformations for higher-rank Kadomtsev–Petviashvili and Krichever–Novikov equations*, Acta Appl. Math., **39**:1–3 (1995), 405–433.

Andrey E. Mironov and Gulnara S. Mauleshova
Sobolev Institute of Mathematics
4 Acad. Koptyug Avenue
630090, Novosibirsk

and

Novosibirsk State University
1, Pirogova str.
630090, Novosibirsk, Russia
e-mail: mironov@math.nsc.ru
mauleshova_gs@mail.ru

Part V

Differential Geometry and Physics

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 199–205

The Dressing Field Method of Gauge Symmetry Reduction: Presentation and Examples

Jeremy Attard

Abstract. This paper is a presentation of a recent method of gauge symmetry reduction, distinct from the well-known gauge fixing, the bundle reduction theorem or even the Spontaneous Symmetry Breaking Mechanism (SSBM). Given a symmetry group G acting on a fiber bundle and its naturally associated fields (Ehresmann (or Cartan) connection, curvature, ...) there are situations where it is possible to erase (in whole or in part) the G-action by just reconfiguring these fields, i.e., by making a mere change of field variables in order to get new *composite* fields on which G (or a subgroup) does not act anymore. Two examples are presented in this paper: the re-interpretation of the BEGHHK (Higgs) mechanism without calling on a SSBM, and the top-down construction of Tractor and Twistor bundles and connections in the framework of conformal Cartan geometry.

Mathematics Subject Classification (2010). Primary 99Z99; Secondary 00A00.

Keywords. Gauge theory, symmetry reduction, spontaneous symmetry breaking, Higgs mechanism, twistors.

This paper stems from a joint work of the author and his colleague Jordan François and his supervisors Serge Lazzarini and Thierry Masson.

1. Introduction

The fundamental interactions are described in the framework of gauge theories, the geometric content of which is a principal fibre bundle P over a smooth (spacetime) manifold M, with structure Lie group G, together with associated vector bundles $E = P \times_\rho \mathbb{V}$ where ρ is a representation of G on the vector space $\mathbb{V}$. A (classical) matter field is then represented by a section ξ of E, while the interacting bosons fields are connections ω on P, and act on matter fields via the associated covariant derivative $D\xi = d\xi + \rho(\omega)\xi$. The curvature associated to a connection ω will be denoted $\Omega := d\omega + \frac{1}{2}[\omega, \omega]$. These notations will be used throughout the paper.

The central notion of a gauge theory is that of *local symmetry.* The latter is implemented by the local action of G, i.e., by the action of the (infinite-dimensional) *gauge group* $\mathfrak{G} = \{g : M \to G\}$.

Fields ξ and connections ω transform under the action of $g \in \mathfrak{G}$ as $\omega^g = g^{-1}\omega g + g^{-1}dg$ and $\xi^g = \rho(g)^{-1}\xi$. Knowing this, one can then write a physical theory by choosing a functional integral $S[\omega, \xi]$ which has the property to be invariant under this action. The theory is said to be *gauge invariant.* The physical meaning of this feature is that two fields in the same gauge orbit are physically equivalent, i.e., are indistinguishable by any physical experiment. The gauge symmetry is the translation of an intrinsic mathematical redundancy of our formalism.

The description of fundamental interactions on which modern physics is built reduces, then, to the choice of symmetry Lie groups. Electroweak and strong interactions are ruled by the Lie group $U(1) \times SU(2) \times SU(3)$. Regarding the gravitational interaction, the fundamental symmetry group of General Relativity (GR) is the group of diffeomorphisms of the base manifold. Let us remark that one can also write GR under the form of a gauge theory[1], in which is added a local symmetry ruled by the local action of the Lorentz group SO(1,3). This can be done in the framework of Cartan geometry.

Although the symmetry group G is central and unavoidable in the construction of a gauge theory, one often needs to reduce its action, i.e., passing to a theory with less symmetry. There can be several reasons for that. For example, for a quantization purpose: the gauge symmetry group produces infinities in the path integral over all fields. Also, e.g., in the case of the electroweak sector of the Standard Model (SM) ($G = U(1) \times SU(2)$), the constraint imposed by the symmetry group is such that mass terms are not allowed, *a priori*, in the action. Thus, since massive particles are observed, one has to find a way to re-write the same theory but with a smaller symmetry group.

There exist many well-known ways of reducing a symmetry. The simplest is *gauge fixing*: since all fields in a given gauge orbit are equivalent, one just can choose a particular one – which renders the computations easier, for example; the physical results should be, by definition, independent of the choice of gauge. Another one, which applies in the case of the electroweak sector, as one shall see, is the *spontaneous symmetry breaking.* In this case, the symmetry reduction is thought as a physical phenomenon, like a phase transition, induced by the fact that the ground state has less symmetry than the theory of which it is a solution.

The recent method of symmetry reduction presented in this paper is called the *dressing field method.* It is a systematic way of finding, if they exist, new fields which are invariant under the action of the gauge group $\mathfrak{G}$ or one of its subgroup. This method turns out to be a mere change of field variables. This change is performed with the help of a dressing field u which does not, in general, belong to

---

[1]In the sense of using a connection on a fibre bundle.

the gauge group $\mathfrak{G}$. Thus, it is neither a gauge transformation nor a gauge fixing: the new field variables, called *composite fields*, belong in general to representation spaces – for the action of the remaining gauge (sub)group – different than the original variables.

The aim of the present paper is first to present in a nutshell the formalism of the method and then to give two examples of application. The first one is the reinterpretation of the spontaneous symmetry breaking in the electroweak sector of the SM as being a dressing field symmetry reduction. This gives a new physical interpretation to the BEGHHK (Higgs) mechanism. The second one is the reconstruction of Twistors (and Tractors), in the field of conformal geometry, starting from the conformal Cartan geometry and applying the dressing field to erase a part of the conformal group to end up with the transformations found in the usual constructions. This offers a new insight into the geometric nature of these objects. These are examples among many others. The interested reader will find a more complete and detailed presentation of the dressing field method and its applications in [4].

2. The dressing field method in a nutshell

The elements of the gauge group $\mathfrak{G}$ can also be seen as G-valued fields defined on P. Such an element g is then transformed under the action of another element h as $g^h = h^{-1}gh$. Let K be a subgroup of G, possibly G itself. A dressing field is a locally defined G-valued field u on P, which transforms under a gauge transformation $k \in K$ as $u^k = k^{-1}u$. Thus, $u \notin \mathfrak{G}$.

The existence of such a field ensures that the following *composite fields*:

- $\omega^u := u^{-1}\omega u + u^{-1}du$,
- $\Omega^u := u^{-1}\Omega u$,
- $\xi^u := \rho(u)^{-1}u$,

are then K-gauge invariant as it can be checked by a straightforward computation. The fact that these fields are now K-invariant is interpreted saying that actually, the subgroup K does not act anymore on the fields.

Thus, if one re-writes the theory (i.e., the gauge invariant action $S[\omega, \xi]$) in the new variables, one gets a theory for which the K-symmetry has been erased. It is a mere reconfiguration of the fields which redistributes the degrees of freedom of the theory. The latter are computed as follows: let $\#\mathrm{TOT}$, $\#\Phi$, $\#\mathrm{G}$ and $\#(\Theta = 0)$ be respectively the total number of degrees of freedom, the degrees of freedom related to the fields (ω and ξ) of the theory, the dimension of the symmetry group G, and the number of constraint equations. Then:

$$\#\mathrm{TOT} = \#\Phi – \#G – \#(\Theta = 0).$$

For example, if the operation of dressing leaves invariant the constraint equations, in the new variables the theory will have less symmetry and then necessarily "less fields", i.e., less degrees of freedom coming from the fields.

Let us present now the two examples announced in the introduction.

3. The Higgs mechanism of the SM as a dressing field reduction.

In the Standard Model, the electroweak sector is governed by the symmetry group $G = U(1) \times SU(2)$. The $SU(2)$-symmetry prevents the action from having mass terms for the weak bosons. Thus, one has to find a way to erase the $SU(2)$-symmetry. Let us present first the usual version, as developed by many authors in the 60s. Then, one presents another interpretation, developed in [1], based on the dressing field method.

3.1. Usual viewpoint

The idea is to suppose the existence of a complex scalar field $\Phi : M \to \mathbb{C}^2$ embedded in the potential $V(\Phi) = -\mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$, with $\lambda > 0$, which spontaneously gets the value Φ_{min} which minimizes the potential $V(\Phi)$. This value depends on the form of the potential, i.e., of the sign of μ^2. For $\mu^2 > 0$, $\Phi_{\text{min}} = 0$ and the choice $\Phi = 0$ is unique, and viewed as a point in $\mathbb{C}^2$, it is still $SU(2)$-invariant. However, for $\mu^2 < 0$, $\Phi_{\text{min}} \neq 0$, and Φ has to "make a choice" (hence the term *spontaneous*) between a subset of corresponding points in $\mathbb{C}^2$. It turns out that a particular point is no more $SU(2)$-invariant, and this phenomenon breaks the symmetry. The scalar field then reads $\Phi = \Phi_{\text{min}} + H$, and the fluctuation H is interpreted as a particle, the Higgs particle which has been discovered in the LHC in 2012. The constant part Φ_{min} couples with other fields, giving them mass.

Thus, the generation of masses in the usual viewpoint is deeply related to the $SU(2)$-symmetry breaking. One shall see now that it is actually possible to reinterpret the Higgs mechanism without calling on a spontaneous symmetry breaking, but merely by viewing it as a dressing.

3.2. $SU(2)$-erasing without symmetry breaking

Let us take the same initial data as in the usual viewpoint. We are going to show that $SU(2)$ is actually always erasable by a dressing, u, built out of the scalar field Φ.

The first step is to write the polar decomposition of Φ as an element of $\mathbb{C}^2$: there exists $u \in SU(2)$ such that $\Phi = u\eta$, with $\eta = \begin{pmatrix} 0 \\ \|\Phi\| \end{pmatrix}$, with $\|\Phi\|^2 := \Phi^\dagger\Phi \in \mathbb{R}^+$. Due to the gauge transformation of Φ under $SU(2)$ (as a scalar doublet), the new variable η is invariant under $SU(2)$, and u transforms as: $u \to \beta^{-1}u$, with $\beta \in SU(2)$. Thus, u is a dressing field. From this point, we already know that it is possible to erase the $SU(2)$-symmetry by dressing, whatever the value of μ^2 is. $\eta = u^{-1}\Phi$ is the $SU(2)$-invariant composite field which takes the place of the original scalar field.

Now, one can generate masses for the weak fields by making η fluctuating around its value which minimizes $V(\eta)$. For $\mu^2 > 0$, $\eta_{\text{min}} = 0$ and mass terms are identically zero. For $\mu^2 < 0$, $\eta_{\text{min}} \neq 0$, and moreover, there is no more actual "choice": η being a positive real number, $V(\eta)$ is now a mere one-variable real-valued function. Thus, the term "spontaneously" is no more relevant. Writing $\eta = \eta_{\text{min}} + H$ leads to the same conclusions as in the usual viewpoint.

The only difference is in the physical interpretation: here, one has seen that the generation of masses is totally decorrelated from the symmetry breaking. The latter is not seen as a physical phenomenon which historically occurred. Rather, the original $SU(2)$-symmetry appears to be an artifact due to field variables in which the theory is originally written, and can be structurally erased by using new field variables. The dressing field method is a systematic way of finding such new field variables which simplify the theory.

Let us now present another example for which the dressing field method applies: the top-down construction of Twistor (and Tractor) bundles and connections.

4. Twistors as composite fields from conformal Cartan geometry

The whole construction being technical, one only sketches it. The interested reader is highly recommended to take a look at [2] (Tractors) and [3] (Twistors).

4.1. The usual bottom-up construction

Twistors are for conformal geometry what spinors are for Lorenztian geometry. These objects are usually obtained following a "bottom-up construction" over a conformal manifold $(M, [g])$, where $[g] = \{\lambda g, \lambda \in \mathcal{C}^\infty(M, \mathbb{R}^{+*})\}$, for a Lorentzian metric g. As in Penrose's work, for example ([5, 6]), one takes a $\mathbb{C}^2$-valued field ω^B which satisfies the Twistor equation: $\nabla_{AA'}\omega^B - \frac{1}{2}\delta_A^B \nabla_{CA'}\omega^C = 0$, and then constructs a closed system by introducing another $\mathbb{C}^2$-valued field $\pi_{A'} := \frac{i}{2}\nabla_{CA'}\omega^C$:

- $\nabla_{AA'}\omega^B + i\frac{1}{2}\delta_A^B\pi_{A'} = 0$,
- $\nabla_{AA'}\pi_{B'} - i\overline{P}_{AA'BB'}\omega^B = 0$,

with $\overline{P}_{AA'BB'}$ corresponding to the Schouten tensor $P_{ab} = -\frac{1}{2}(R_{ab} - \frac{1}{6}Rg_{ab})$. One then encompasses the whole construction in the new definitions:

- $Z^\alpha := (\omega^B, \pi_{A'}) \in \mathbb{C}^4$,
- $\nabla^T_{AA'}Z^\alpha = 0$ with $\nabla^T_{AA'} := \nabla_{AA'}\mathbb{I}_4 + \begin{pmatrix} 0 & i\delta_A^B \\ -i\overline{P}_{AA'BB'} & 0 \end{pmatrix}$,

where $\nabla^T_{AA'}$ is the Twistor connection and Z^α is a $\mathbb{C}^4$-valued Twistor. By construction, these equations are conformally invariant (i.e., well defined on $(M, [g])$). Yet, the objects like $\nabla^T_{AA'}$ and Z^α are conformally covariant, and one can compute the corresponding transformation laws under a conformal rescaling of the metric. One can then consider a general Twistor Z^α, i.e., an object such that $\nabla^T_{AA'}Z^\alpha$ does not necessarily vanish, and which transforms with the same laws, which are:

$$\hat{Z}^\alpha = \begin{pmatrix} \mathbb{I}_2 & 0 \\ i\Upsilon_{AA'} & \mathbb{I}_2 \end{pmatrix} Z^\alpha \text{ and } \widehat{\nabla^T_{AA'}Z^\alpha} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ i\Upsilon_{AA'} & \mathbb{I}_2 \end{pmatrix} \nabla^T_{AA'}Z^\alpha$$

under $g_{\mu\nu} \to z^2 g_{\mu\nu}$, with $\Upsilon_{AA'}$ corresponding to $\Upsilon_a = \partial_a \ln(z)$.

Twistors are thus sections of a vector bundle with fibre $\mathbb{C}^4$, transforming under the action of a certain group represented by elements of the form

$$\begin{pmatrix} \mathbb{I}_2 & 0 \\ i\Upsilon_{AA'} & \mathbb{I}_2 \end{pmatrix}.$$

One now presents another construction which is based on the dressing field method.

4.2. A top-down construction via dressing field method

A Cartan geometry over a manifold M is a way of implementing local external symmetries (like Lorentz, projective, ...) in the form of usual internal ones. It is used to write gravitation theories (i.e., theories in which the geometry of the base manifold is dynamic) in the form of usual gauge theories. From a mathematical point of view, a Cartan geometry expresses a given geometric structure over M into the form a principal bundle endowed with a so-called Cartan connection, slightly different from the usual notion of the Ehresmann connection.

For example, the data of a Lorentzian metric on a manifold M is equivalent to the data of a torsion-free Cartan connection over a H-principal bundle, H being the Lorentz group. Following the same idea, a conformal manifold $(M, [g])$ can be seen as a normal conformal Cartan connection ϖ over a H-principal bundle P, with H defined as follows. Let $G := SO(2,4)/(\pm\mathbb{I})$ be the conformal group, and $M_0 := S^1 \times S^3/\mathbb{Z}_2$ the conformal compactification of Minkowski spacetime, which is homogeneous with respect to the action of G. The corresponding Lie group H is then the stabilizer of a point of M_0. One takes then the complex $\mathbb{C}^4$-representation of these groups. A Twistor should be a section of the associated vector bundle $P \times_{\overline{H}} \mathbb{C}^4$, where $\overline{H}$ is the complex representation of H. It turns out that as it stands, the structure does not reproduce the Twistor space and connection, for it does not imply the same transformation law.

To recover it, one has to apply the dressing field method, with dressing fields built out of the conformal Cartan connection. In doing so, one can erase some parts of the original structure group H, and end up with composite fields which transform under a modified transformation law corresponding to the residual action of H. Twistors as previously defined then appear from this construction, with a slight modification: the residual symmetry group does not act (on the composite fields) through a representation of the Weyl group $\mathbb{R}^*_+$, but via something more complicated called a 1-α-cocyle, see [3], Section 4.2.2.

The fact is that in our procedure, no arbitrary choice is made: one just takes the "rigid" normal conformal Cartan geometry and applies to it the dressing field method. Everything is "already there". One is just playing with objects which naturally belong to the geometry. On the contrary, in the usual construction, some ansatz are taken to simplify the transformation laws, rendering the construction more arbitrary, even if it remains, of course, totally coherent.

Finally, let us remark that the same thing has been done also for Tractors, which appear to be merely the real version of Twistors ([2]).

5. Conclusion

In this presentation of the dressing field method, it has been shown that it can apply in a quite wide range of different cases. Once one works on a principal bundle equipped with a connection (of Erhesmann or Cartan type), one can try to investigate if it is possible to build a dressing field out of the fields to erase the action of the symmetry group or one of its subgroups. A first hint can be given by counting how are initially distributed the degrees of freedom, and how/if they could be distributed differently. Then, if the answer is positive, one can start to search for a field transforming on the right way to be a dressing field.

More than just giving a way of simplifying the writing of physical theories, it often offers a new insight into some already known constructions. In the case of the SM, it gives a natural and new interpretation of the generation of masses. In the case of Twistors (or even Tractors), it offers a new view of the geometric nature of these objects and of their underlying structure.

Acknowledgment

Many thanks go to Jordan François and Serge Lazzarini for having read this paper and suggested useful corrections.

References

[1] T. Masson and J.-C. Wallet, *A remark on the spontaneous symmetry breaking in the standard model*, arXiv:1001.1176 (2010).

[2] J. Attard and J. François, *Tractors and Twistors from conformal Cartan geometry: a gauge theoretic approach. I. Tractors*, arXiv:1609.07307, (2017).

[3] J. Attard and J. François, *Tractors and Twistors from conformal Cartan geometry: a gauge theoretic approach. II. Twistors*, Classical and Quantum Gravity, (2017).

[4] J. Attard, J. François, T. Masson and S. Lazzarini, *The dressing field method of gauge symmetry reduction: a review with examples*, Journal of Geometric Method in Modern Physics, (2017).

[5] R. Penrose, *Twistor algebra*, J. Math. Phys., **8**, 345 (1967).

[6] R. Penrose, *Twistor quantization and curved space-time*, Int. J. Theor. Phys., **1**, 61–99 (1968).

[7] J. François, S. Lazzarini and T. Masson, *Residual Weyl symmetry out of conformal geometry and its BRS structure*, JHEP **2015** (09) 195.

[8] J. Attard, J. François and S. Lazzarini, *Weyl Gravity and Cartan Geometry*, Phys. Rev. D, **93**, 085032 (2016).

Jeremy Attard
Centre de Physique Théorique
Campus de Luminy, Case 907
163 Avenue de Luminy
F-13288 Marseille Cedex 9, France
e-mail: jeremy.attard@cpt.univ-mrs.fr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 207–214

A Differential Model for B-type Landau–Ginzburg Theories

Elena Mirela Babalic, Dmitry Doryn, Calin Iuliu Lazaroiu and Mehdi Tavakol

Abstract. We describe a mathematically rigorous differential model for B-type open-closed topological Landau–Ginzburg theories defined by a pair (X, W), where X is a non-compact Kählerian manifold with holomorphically trivial canonical line bundle and W is a complex-valued holomorphic function defined on X and whose critical locus is compact but need not consist of isolated points. We also show how this construction specializes to the case when X is Stein and W has finite critical set, in which case one recovers a simpler mathematical model.

Mathematics Subject Classification (2010). 81T45, 18Axx, 55N30.

Keywords. Topological field theory, category theory, sheaf cohomology.

1. Axiomatics of two-dimensional oriented open-closed TFTs

Classical oriented open-closed topological LG (Landau–Ginzburg) theories of type B are classical field theories defined on compact oriented Riemann surfaces with corners and parameterized by pairs (X, W), where X is a non-compact Kählerian manifold and $W : X \to \mathbb{C}$ is a non-constant holomorphic function defined on X and called the superpotential. Previous work in the Mathematics literature assumed algebraicity of X and W, being mostly limited to very simple examples such as $X = \mathbb{C}^d$ and generally assumed that the critical points of W are isolated, in which case topological D-branes can be described by matrix factorizations. We do not impose such restrictions since there is no Physics reason to do so. This leads to a much more general description.[1]

[1] We use the results, notations and conventions of [1, 2]. In our terminology "off-shell" refers to an object defined at cochain level while "on-shell" refers to an object defined at cohomology level. The Physics constructions and arguments behind this work can be found in [3, 4].

A *non-anomalous quantum oriented two-dimensional open-closed topological field theory* (*TFT*) can be defined axiomatically [5] as a symmetric monoidal functor from a certain symmetric monoidal category $\mathrm{Cob}_2^{\mathrm{ext}}$ of labeled 2-dimensional oriented cobordisms with corners to the symmetric monoidal category $\mathrm{vect}^s_{\mathbb{C}}$ of finite-dimensional supervector spaces defined over $\mathbb{C}$. The objects of the category $\mathrm{Cob}_2^{\mathrm{ext}}$ are finite disjoint unions of oriented circles and oriented segments while the morphisms are oriented cobordisms with corners between such, carrying appropriate labels on boundary components (labels which can be identified with the topological D-branes). By definition, the *closed sector* of such a theory is obtained by restricting the monoidal functor to the subcategory of $\mathrm{Cob}_2^{\mathrm{ext}}$ whose objects are disjoint unions of circles and whose morphisms are ordinary cobordisms (without corners). It was shown in [5] that such a functor can be described equivalently by an algebraic structure which we shall call a *TFT datum*. We start by describing certain simpler algebraic structures, which form part of any such datum:

Definition 1. A pre-TFT datum is an ordered triple $(\mathcal{H}, \mathcal{T}, e)$ consisting of:

1. A finite-dimensional unital and supercommutative superalgebra $\mathcal{H}$ defined over $\mathbb{C}$ (called the *bulk algebra*), whose unit we denote by $1_{\mathcal{H}}$
2. A Hom-finite $\mathbb{Z}_2$-graded $\mathbb{C}$-linear category $\mathcal{T}$ (called the *category of topological D-branes*), whose composition of morphisms we denote by $\circ$ and whose units we denote by $1_a \in \mathrm{End}_{\mathcal{T}}(a) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathcal{T}}(a,a)$ for all objects $a \in \mathrm{Ob}\,\mathcal{T}$
3. A family $e = (e_a)_{a\in \mathrm{Ob}\,\mathcal{T}}$ consisting of even $\mathbb{C}$-linear *bulk-boundary maps* $e_a : \mathcal{H} \to \mathrm{Hom}_{\mathcal{T}}(a,a)$ defined for each object a of $\mathcal{T}$

such that the following conditions are satisfied:

- For any object $a \in \mathrm{Ob}\,\mathcal{T}$, the map e_a is a unital morphism of $\mathbb{C}$-superalgebras from $\mathcal{H}$ to the endomorphism algebra $(\mathrm{End}_{\mathcal{T}}(a), \circ)$
- For any two objects $a, b \in \mathrm{Ob}\,\mathcal{T}$ and for any $\mathbb{Z}_2$-homogeneous elements $h \in \mathcal{H}$ and $t \in \mathrm{Hom}_{\mathcal{T}}(a,b)$, we have: $e_b(h) \circ t = (-1)^{\deg h\, \deg t} t \circ e_a(h)$.

Definition 2. A *Calabi–Yau supercategory of parity* μ is a pair $(\mathcal{T}, \mathrm{tr})$, where:

1. $\mathcal{T}$ is a $\mathbb{Z}_2$-graded and $\mathbb{C}$-linear Hom-finite category
2. $\mathrm{tr} = (\mathrm{tr}_a)_{a\in\mathrm{Ob}\,\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\mathrm{tr}_a : \mathrm{End}_{\mathcal{T}}(a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree μ

such that the following conditions are satisfied:

- For any two objects $a, b \in \mathrm{Ob}\,\mathcal{T}$, the $\mathbb{C}$-bilinear pairing $\langle\cdot,\cdot\rangle_{a,b} : \mathrm{Hom}_{\mathcal{T}}(a,b) \times \mathrm{Hom}_{\mathcal{T}}(b,a) \to \mathbb{C}$ defined through:
$$\langle t_1, t_2\rangle_{a,b} \stackrel{\mathrm{def.}}{=} \mathrm{tr}_b(t_1 \circ t_2)\ ,\ \forall t_1 \in \mathrm{Hom}_{\mathcal{T}}(a,b)\ ,\ \forall t_2 \in \mathrm{Hom}_{\mathcal{T}}(b,a)$$
is non-degenerate
- For any $a, b \in \mathrm{Ob}\,\mathcal{T}$ and any $\mathbb{Z}_2$-homogeneous elements $t_1 \in \mathrm{Hom}_{\mathcal{T}}(a,b)$ and $t_2 \in \mathrm{Hom}_{\mathcal{T}}(b,a)$, we have:
$$\langle t_1, t_2\rangle_{a,b} = (-1)^{\deg t_1\, \deg t_2} \langle t_2, t_1\rangle_{b,a}\ .$$

If only the second condition above is satisfied, we say that $(\mathcal{T}, \mathrm{tr})$ is a *pre-Calabi–Yau supercategory* of parity μ.

Definition 3. A *TFT datum* of parity μ is a system $(\mathcal{H}, \mathcal{T}, e, \mathrm{Tr}, \mathrm{tr})$, where:

1. $(\mathcal{H}, \mathcal{T}, e)$ is a pre-TFT datum
2. $\mathrm{Tr} : \mathcal{H} \to \mathbb{C}$ is an even $\mathbb{C}$-linear map (called the *bulk trace*)
3. $\mathrm{tr} = (\mathrm{tr}_a)_{a \in \mathrm{Ob}\,\mathcal{T}}$ is a family of $\mathbb{C}$-linear maps $\mathrm{tr}_a : \mathrm{End}_{\mathcal{T}}(a) \to \mathbb{C}$ of $\mathbb{Z}_2$-degree μ (called the *boundary traces*)

such that the following conditions are satisfied:

- $(\mathcal{H}, \mathrm{Tr})$ is a supercommutative Frobenius superalgebra. This means that the pairing induced by Tr on $\mathcal{H}$ is non-degenerate
- $(\mathcal{T}, \mathrm{tr})$ is a Calabi–Yau supercategory of parity μ
- The following condition (known as the *topological Cardy constraint*) is satisfied for all $a, b \in \mathrm{Ob}\,\mathcal{T}$:

$$\mathrm{Tr}(f_a(t_1) f_b(t_2)) = \mathrm{str}(\Phi_{ab}(t_1, t_2)) \ , \ \forall t_1 \in \mathrm{End}_{\mathcal{T}}(a) \ , \ \forall t_2 \in \mathrm{End}_{\mathcal{T}}(b) \ .$$

  Here, str denotes the supertrace on the finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{T}}(a, b))$ and:
  - The $\mathbb{C}$-linear *boundary-bulk* map $f_a : \mathrm{End}_{\mathcal{T}}(a) \to \mathcal{H}$ of $\mathbb{Z}_2$-degree μ is defined as the adjoint of the bulk-boundary map $e_a : \mathcal{H} \to \mathrm{End}_{\mathcal{T}}(a)$ with respect to the non-degenerate traces Tr and tr_a:

$$\mathrm{Tr}(h f_a(t)) = \mathrm{tr}_a(e_a(h) \circ t) \ , \ \forall h \in \mathcal{H} \ , \ \forall t \in \mathrm{End}_{\mathcal{T}}(a)$$

  - For any $a, b \in \mathrm{Ob}\,\mathcal{T}$ and any $t_1 \in \mathrm{End}_{\mathcal{T}}(a)$ and $t_2 \in \mathrm{End}_{\mathcal{T}}(b)$, the $\mathbb{C}$-linear map $\Phi_{ab}(t_1, t_2) : \mathrm{Hom}_{\mathcal{T}}(a, b) \to \mathrm{Hom}_{\mathcal{T}}(a, b)$ is defined through:

$$\Phi_{ab}(t_1, t_2)(t) \stackrel{\text{def.}}{=} t_2 \circ t \circ t_1 \ , \ \forall t \in \mathrm{Hom}_{\mathcal{T}}(a, b) \ .$$

2. B-type open-closed Landau–Ginzburg theories

Definition 4. A *Landau–Ginzburg pair* of dimension d is a pair (X, W) where:

1. X is a non-compact Kählerian manifold of complex dimension d which is *Calabi–Yau* in the sense that the canonical line bundle $K_X \stackrel{\text{def.}}{=} \wedge^d T^* X$ is holomorphically trivial.
2. $W : X \to \mathbb{C}$ is a *non-constant* complex-valued holomorphic function.

The *signature* $\mu(X, W)$ of a Landau–Ginzburg pair (X, W) is defined as the mod 2 reduction[2] of the complex dimension d of X.

The *critical set of* W is defined as the set of critical points of W:

$$Z_W \stackrel{\text{def.}}{=} \{p \in X | (\partial W)(p) = 0\} \ .$$

[2]We denote by $\hat{k} \in \mathbb{Z}_2$ the mod 2 reduction of any integer $k \in \mathbb{Z}$.

2.1. The off-shell bulk algebra

Let (X, W) be a Landau–Ginzburg pair with $\dim_{\mathbb{C}} X = d$. The *space of polyvector-valued forms* is defined through:

$$\mathrm{PV}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathrm{PV}^{i,j}(X) = \bigoplus_{i=-d}^{0} \bigoplus_{j=0}^{d} \mathcal{A}^j(X, \wedge^{|i|} TX) \ ,$$

where $\mathcal{A}^j(X, \wedge^{|i|}TX) \equiv \Omega^{0,j}(X, \wedge^{|i|}TX)$. The *twisted Dolbeault differential* induced by W is defined through $\delta_W \stackrel{\text{def.}}{=} \boldsymbol{\overline{\partial}} + \boldsymbol{\iota}_W : \mathrm{PV}(X) \to \mathrm{PV}(X)$, where $\boldsymbol{\overline{\partial}}$ is the Dolbeault operator of $\wedge TX$ (which satisfies $\boldsymbol{\overline{\partial}}(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i,j+1}(X)$), while $\boldsymbol{\iota}_W \stackrel{\text{def.}}{=} -\mathbf{i}(\partial W) \lrcorner$ is the contraction with the holomorphic 1-form $-\mathbf{i}\partial W \in \Gamma(X, T^*X)$ (which satisfies $\boldsymbol{\iota}_W(\mathrm{PV}^{i,j}(X)) \subset \mathrm{PV}^{i+1,j}(X)$). Here $\mathbf{i}$ denotes the imaginary unit. Notice that $(\mathrm{PV}(X), \boldsymbol{\overline{\partial}}, \boldsymbol{\iota}_W)$ is a bicomplex since $\boldsymbol{\overline{\partial}}^2 = \boldsymbol{\iota}_W^2 = \boldsymbol{\overline{\partial}}\boldsymbol{\iota}_W + \boldsymbol{\iota}_W\boldsymbol{\overline{\partial}} = 0$.

Definition 5. The *twisted Dolbeault algebra of polyvector-valued forms* of the LG pair (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear dg-algebra $(\mathrm{PV}(X), \delta_W)$, where $\mathrm{PV}(X)$ is endowed with the total $\mathbb{Z}$-grading.

Definition 6. The *cohomological twisted Dolbeault algebra* of (X, W) is the supercommutative $\mathbb{Z}$-graded $\mathrm{O}(X)$-linear algebra defined through:

$$\mathrm{HPV}(X, W) = \mathrm{H}(\mathrm{PV}(X), \delta_W) \ .$$

An analytic model for the off-shell bulk algebra

Definition 7. The *sheaf Koszul complex* of W is the following complex of locally-free sheaves of $\mathcal{O}_X$-modules[3]:

$$(Q_W): \ 0 \to \wedge^d TX \stackrel{\boldsymbol{\iota}_W}{\to} \wedge^{d-1} TX \stackrel{\boldsymbol{\iota}_W}{\to} \cdots \stackrel{\boldsymbol{\iota}_W}{\to} \mathcal{O}_X \to 0 \ ,$$

where $\mathcal{O}_X$ sits in degree zero and we identify the exterior power $\wedge^k TX$ with its locally-free sheaf of holomorphic sections.

Proposition 8. *Let $\mathbb{H}(Q_W)$ denote the hypercohomology of the Koszul complex Q_W. There exists a natural isomorphism of $\mathbb{Z}$-graded $\mathrm{O}(X)$-modules:*

$$\mathrm{HPV}(X, W) \cong_{\mathrm{O}(X)} \mathbb{H}(Q_W) \ ,$$

where $\mathrm{HPV}(X, W)$ is endowed with the total $\mathbb{Z}$-grading. Thus:

$$\mathrm{H}^k(\mathrm{PV}(X), \delta_W) \cong_{\mathrm{O}(X)} \mathbb{H}^k(Q_W), \ \forall k \in \{-d, \ldots, d\} \ .$$

Moreover, we have $\mathbb{H}^k(Q_W) = \bigoplus_{i+j=k} \mathbf{E}_\infty^{i,j}$, where $(\mathbf{E}_r^{i,j}, \mathbf{d}_r)_{r \geq 0}$ is a spectral sequence which starts with:

$$\mathbf{E}_0^{i,j} := \mathrm{PV}^{i,j}(X) = \mathcal{A}^j(X, \wedge^{|i|}TX), \ \mathbf{d}_0 = \boldsymbol{\overline{\partial}} \ , \ (i = -d, \ldots, 0, \ j = 0, \ldots, d) \ .$$

[3]We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by $\mathrm{O}(X) = \Gamma(X, \mathcal{O}_X)$ the ring of holomorphic functions on X. Here Γ denotes taking holomorphic sections, while Γ_∞ denotes taking smooth sections.

2.2. The category of topological D-branes

Definition 9. A *holomorphic vector superbundle* on X is a $\mathbb{Z}_2$-graded holomorphic vector bundle defined on X, i.e., a complex holomorphic vector bundle E endowed with a direct sum decomposition $E = E^{\hat{0}} \oplus E^{\hat{1}}$, where $E^{\hat{0}}$ and $E^{\hat{1}}$ are holomorphic sub-bundles of E.

Definition 10. A *holomorphic factorization* of W is a pair $a = (E, D)$, where $E = E^{\hat{0}} \oplus E^{\hat{1}}$ is a holomorphic vector superbundle on X and $D \in \Gamma(X, \mathrm{End}^{\hat{1}}(E))$ is a holomorphic section of the bundle $\mathrm{End}^{\hat{1}}(E) = \mathrm{Hom}(E^{\hat{0}}, E^{\hat{1}}) \oplus \mathrm{Hom}(E^{\hat{1}}, E^{\hat{0}}) \subset \mathrm{End}(E)$ which satisfies the condition $D^2 = W \, \mathrm{id}_E$.

2.3. The full TFT data

Definition 11. The *twisted Dolbeault category of holomorphic factorizations* of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg-category $\mathrm{DF}(X, W)$ defined as follows:

- The objects of $\mathrm{DF}(X, W)$ are the holomorphic factorizations of W
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$ and $a_2 = (E_2, D_2)$, the Hom spaces:

$$\mathrm{Hom}_{\mathrm{DF}(X,W)}(a_1, a_2) \stackrel{\text{def.}}{=} \mathcal{A}(X, \mathrm{Hom}(E_1, E_2))$$

  are endowed with the total $\mathbb{Z}_2$-grading and with the twisted differentials $\boldsymbol{\delta}_{a_1,a_2} \stackrel{\text{def.}}{=} \boldsymbol{\overline{\partial}}_{a_1,a_2} + \boldsymbol{\mathfrak{d}}_{a_1,a_2}$, where $\boldsymbol{\overline{\partial}}_{a_1,a_2}$ is the Dolbeault differential of $\mathrm{Hom}(E_1, E_2)$, while $\boldsymbol{\mathfrak{d}}_{a_1,a_2}$ is the *defect differential*:

$$\boldsymbol{\mathfrak{d}}_{a_1,a_2}(\rho \otimes f) = (-1)^{\mathrm{rk}\,\rho} \rho \otimes (D_2 \circ f) - (-1)^{\mathrm{rk}\,\rho + \sigma(f)} \rho \otimes (f \circ D_1)$$

- The composition of morphisms $\circ : \mathcal{A}(X, \mathrm{Hom}(E_2, E_3)) \times \mathcal{A}(X, \mathrm{Hom}(E_1, E_2)) \to \mathcal{A}(X, \mathrm{Hom}(E_1, E_3))$ is determined uniquely by the condition:

$$(\rho \otimes f) \circ (\eta \otimes g) = (-1)^{\sigma(f)\ \mathrm{rk}\,\eta} (\rho \wedge \eta) \otimes (f \circ g)$$

  for all pure rank forms $\rho, \eta \in \mathcal{A}(X)$ and all pure $\mathbb{Z}_2$-degree elements $f \in \Gamma_\infty(X, \mathrm{Hom}(E_2, E_3))$ and $g \in \Gamma_\infty(X, \mathrm{Hom}(E_1, E_2))$, where $\sigma(f)$ is the degree of f.

We have (omitting indices): $\boldsymbol{\delta}^2 = \boldsymbol{\overline{\partial}}^2 = \boldsymbol{\mathfrak{d}}^2 = \boldsymbol{\overline{\partial}} \circ \boldsymbol{\mathfrak{d}} + \boldsymbol{\mathfrak{d}} \circ \boldsymbol{\overline{\partial}} = 0\,.$

Definition 12. The *cohomological twisted Dolbeault category of holomorphic factorizations* of (X, W) is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear category defined as the total cohomology category of $\mathrm{DF}(X, W)$:

$$\mathrm{HDF}(X, W) \stackrel{\text{def.}}{=} \mathrm{H}(\mathrm{DF}(X, W))\ .$$

Theorem 13. *Suppose that the critical set Z_W is compact. Then the cohomology algebra* $\mathrm{HPV}(X, W)$ *of* $(\mathrm{PV}(X), \delta_W)$ *is finite-dimensional over* $\mathbb{C}$ *while the total cohomology category* $\mathrm{HDF}(X, W)$ *of* $\mathrm{DF}(X, W)$ *is Hom-finite over* $\mathbb{C}$*. Moreover, one can define*[4] *a bulk trace* $\mathrm{Tr} : \mathrm{HPV}(X, W) \to \mathbb{C}$*, boundary traces* tr_{a_1,a_2} :

[4] Explicit expressions for $\mathrm{Tr}, \mathrm{tr}, e$ can be found in [1].

$\mathrm{Hom}_{\mathrm{HPV}(X,W)}(a_1, a_2) \to \mathbb{C}$ *and bulk-boundary maps* e_a*:*

$$\mathrm{HPV}(X, W) \to \mathrm{End}_{\mathrm{HDF}(X,W)}(a)$$

such that the system:

$$(\mathrm{HPV}(X, W), \mathrm{HDF}(X, W), \mathrm{Tr}, \mathrm{tr}, e)$$

obeys the defining properties of a TFT datum except for non-degeneracy of the bulk and boundary traces and for the topological Cardy constraint.

Conjecture 14. Suppose that the critical set Z_W is compact. Then the quintuplet $(\mathrm{HPV}(X, W), \mathrm{HDF}(X, W), \mathrm{Tr}, \mathrm{tr}, e)$ is a TFT datum and hence defines a quantum open-closed TFT.

3. B-type Landau–Ginzburg theories on Stein manifolds

When X is a Stein manifold, Cartan's theorem B states that the higher sheaf cohomology $\mathrm{H}^i(X, \mathcal{F})$ vanishes when $i > 0$ for any coherent analytic sheaf $\mathcal{F}$.

3.1. An analytic model for the on-shell bulk algebra

Theorem 15. *Suppose that* X *is Stein. Then the spectral sequence defined previously collapses at the second page* $\mathbf{E}_2$ *and* $\mathrm{HPV}(X, W)$ *is concentrated in non-positive degrees. For all* $k = -d, \ldots, 0$*, the* $\mathrm{O}(X)$*-module* $\mathrm{HPV}^k(X, W)$ *is isomorphic with the cohomology at position* k *of the following sequence of finitely-generated projective* $\mathrm{O}(X)$*-modules:*

$$(\mathcal{P}_W): \ 0 \to \mathrm{H}^0(X, \wedge^d TX) \stackrel{\boldsymbol{\iota}_W}{\to} \cdots \stackrel{\boldsymbol{\iota}_W}{\to} \mathrm{H}^0(X, TX) \stackrel{\boldsymbol{\iota}_W}{\to} \mathrm{O}(X) \to 0 \ ,$$

where $\mathrm{O}(X)$ *sits in position zero.*

Remark 16. Assume that X is Stein. Then the critical set Z_W is compact iff it is finite, which implies $\dim_{\mathbb{C}} Z_W = 0$.

Let $\mathcal{J}_W \stackrel{\mathrm{def.}}{=} \mathrm{im}\,(\boldsymbol{\iota}_W : TX \to \mathcal{O}_X)$ be the critical sheaf of W, $Jac_W \stackrel{\mathrm{def.}}{=} \mathcal{O}_X/\mathcal{J}_W$ be the Jacobi sheaf of W and $\mathrm{Jac}(X, W) \stackrel{\mathrm{def.}}{=} \Gamma(X, Jac_W)$ be the Jacobi algebra of (X, W).

Proposition 17. *Suppose that* X *is Stein and* $\dim_{\mathbb{C}} Z_W = 0$*. Then* $\mathrm{HPV}^k(X) = 0$ *for* $k \neq 0$ *and there exists a natural isomorphism of* $\mathrm{O}(X)$*-modules:*

$$\mathrm{HPV}^0(X, W) \simeq_{\mathrm{O}(X)} \mathrm{Jac}(X, W) \ .$$

3.2. An analytic model for the category of topological D-branes

Definition 18. The *holomorphic dg-category of holomorphic factorizations* of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg-category $\mathrm{F}(X,W)$ defined as follows:

- The objects are the holomorphic factorizations of W
- Given two holomorphic factorizations $a_1 = (E_1, D_1)$, $a_2 = (E_2, D_2)$, let:

  $$\mathrm{Hom}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, \mathrm{Hom}(E_1, E_2))$$

  be the space of morphisms between such, endowed with the $\mathbb{Z}_2$-grading:

  $$\mathrm{Hom}^{\kappa}_{\mathrm{F}(X,W)}(a_1, a_2) = \Gamma(X, \mathrm{Hom}^{\kappa}(E_1, E_2)) \, , \ \forall \kappa \in \mathbb{Z}_2$$

  and with the differentials $\mathfrak{d}_{a_1,a_2}$ determined uniquely by the condition:

  $$\mathfrak{d}_{a_1,a_2}(f) = D_2 \circ f - (-1)^{\kappa} f \circ D_1 \, , \ \forall f \in \Gamma(X, \mathrm{Hom}^{\kappa}(E_1, E_2)) \, , \ \forall \kappa \in \mathbb{Z}_2$$

- The composition of morphisms is the obvious one.

Theorem 19. *Suppose that X is Stein. Then* $\mathrm{HDF}(X,W)$ *is equivalent with the total cohomology category* $\mathrm{HF}(X,W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{F}(X,W))$ *of* $\mathrm{F}(X,W)$ *.*

Definition 20. A *projective analytic factorization* of W is a pair (P, D), where P is a finitely-generated projective $\mathrm{O}(X)$-supermodule and $D \in \mathrm{End}^{\hat{1}}_{\mathrm{O}(X)}(P)$ is an odd endomorphism of P such that $D^2 = W \, \mathrm{id}_P$.

Definition 21. The *dg-category* $\mathrm{PF}(X,W)$ *of projective analytic factorizations* of W is the $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear dg-category defined as follows:

- The objects are the projective analytic factorizations of W
- Given two projective analytic factorizations (P_1, D_1) and (P_2, D_2) of W, set $\mathrm{Hom}_{\mathrm{PF}(X,W)}((P_1, D_1), (P_2, D_2)) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$ endowed with the obvious $\mathbb{Z}_2$-grading and with the $\mathrm{O}(X)$-linear odd differential

  $$\mathfrak{d} := \mathfrak{d}_{(P_1,D_1),(P_2,D_2)}$$

  determined uniquely by the condition:

  $$\mathfrak{d}(f) = D_2 \circ f - (-1)^{\deg f} f \circ D_1 \ , \ \forall f \in \mathrm{Hom}_{\mathrm{O}(X)}(P_1, P_2)$$

- The composition of morphisms is the obvious one.

Definition 22. The *cohomological category* $\mathrm{HPF}(X,W)$ *of analytic projective factorizations* of W is the total cohomology category $\mathrm{HPF}(X,W) \stackrel{\mathrm{def.}}{=} \mathrm{H}(\mathrm{PF}(X,W))$, which is a $\mathbb{Z}_2$-graded $\mathrm{O}(X)$-linear category.

Theorem 23. *Suppose that X is Stein. Then* $\mathrm{HDF}(X,W)$ *and* $\mathrm{HPF}(X,W)$ *are equivalent as $\mathrm{O}(X)$-linear $\mathbb{Z}_2$-graded categories.*

When X is Stein and Z_W is compact, the category of topological D-branes of the B-type Landau–Ginzburg theory can be identified with $\mathrm{HPF}(X,W)$.

Acknowledgment

This work was supported by the research grant IBS-R003-S1.

References

[1] E.M. Babalic, D. Doryn, C.I. Lazaroiu, M. Tavakol, *Differential models for B-type open-closed topological Landau–Ginzburg theories*, arXiv:1610.09103v3 [math.DG], Commun. Math. Phys. 361 (3), 1169–1234 (2018).

[2] E.M. Babalic, D. Doryn, C.I. Lazaroiu, M. Tavakol, *On B-type open-closed Landau–Ginzburg theories defined on Calabi–Yau Stein manifolds*, arXiv:1610.09813v3 [math. DG], Commun. Math. Phys. 362 (1), 129—165 (2018).

[3] C.I. Lazaroiu, *On the boundary coupling of topological Landau–Ginzburg models*, JHEP **05** (2005) 037.

[4] M. Herbst, C.I. Lazaroiu, *Localization and traces in open-closed topological Landau–Ginzburg models*, JHEP **05** (2005) 044.

[5] C.I. Lazaroiu, *On the structure of open-closed topological field theories in two dimensions*, Nucl. Phys. B **603** (2001), 497–530.

Elena Mirela Babalic and Calin Iuliu Lazaroiu
Center for Geometry and Physics
Institute for Basic Science
7 Cheongam-ro, Nam-gu, Pohang
Republic of Korea 37673

and

Horia Hulubei National Institute for Physics
and Nuclear Engineering (IFIN-HH)
Str. Reactorului no.30, P.O.BOX MG-6
RO-077125 Bucharest-Magurele, Romania
e-mail: mirela@ibs.re.kr
calin@ibs.re.kr

Dmitry Doryn and Mehdi Tavakol
Center for Geometry and Physics
Institute for Basic Science
7 Cheongam-ro, Nam-gu, Pohang
Republic of Korea 37673
e-mail: dmitry@ibs.re.kr
mehdi@ibs.re.kr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 215–222

On the Dirac Type Operators on Symmetric Tensors

Bogdan Balcerzak

Abstract. In the paper we define the Dirac type operators on anchored vector bundles given by a skew-symmetric 2-tensor. The Weyl module structure is defined in the case of the symmetric bundle. The decomposition of the symmetric Dirac operator into the sum of the symmetric covariant derivative and symmetric coderivative is presented.

Mathematics Subject Classification (2010). Primary 58H05; Secondary 17B66, 53C05, 58J60.

Keywords. Dirac operator, anchored vector bundle, Lie algebroid, symmetric covariant derivative, symmetric product, Weyl algebra.

1. Introduction and basic notions

An **anchored vector bundle** over a manifold M we call a vector bundle A over M equipped with a homomorphism of vector bundles $\rho_A : A \to TM$ over the identity, called an **anchor** (see [9, 10]). In particular, any Lie algebroid in the sense of Pradines [11] is an anchored vector bundle. Lie algebroids were introduced by Pradines as infinitesimal objects associated with Lie groupoids. A **Lie algebroid** over a manifold M is actually an anchored vector bundle A with an anchor ρ_A equipped with a real Lie algebra structure $[\cdot,\cdot]$ in the module $\Gamma(A)$ of sections of A associated with the anchor by the following Leibniz identity:

$$[a, f \cdot b] = f \cdot [a, b] + (\rho_A \circ a)(f) \cdot b$$

for $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$. For general theory of Lie algebroids we refer to [7, 8].

The main purpose of the work is to define some the Dirac type operator $\mathcal{D}^s$ induced by a skew-symmetric 2-tensor for an anchored vector bundle equipped with two connections. The bundles considered here are equipped with the Weyl module structure which comes from a skew-symmetric 2-tensor on A^*. In particular case, it is considered the symmetric covariant derivative d^s which is defined as the symmetrization of the connection up to a constant. The operator d^s can be written in the Koszul form using the symmetric product for a given A-connection in A. We show that, as in the classical case (cf. [2, 3]), $\mathcal{D}^s$ is the sum of the symmetric derivative d^s and the symmetric coderivative d^{s*}. The problem of ellipticity of $\mathcal{D}^s$ is also mentioned.

Let us first introduce the concept of linear connections associated with the given anchored vector bundle.

Now let A be an anchored vector bundle over a manifold M with an anchor ρ_A and let E a vector bundle over M.

We denote by $\mathcal{A}(E)$ the Lie algebroid of the vector bundle E (cf. [7, 8]). The module $\mathcal{CDO}(E)$ of sections of $\mathcal{A}(E)$ is the space of all $\mathbb{R}$-linear operators $l : \Gamma(E) \to \Gamma(E)$ such that there is (unique) $X \in \Gamma(TM)$ such that $l(f \cdot \nu) = f \cdot l(\nu) + X(f) \cdot \nu$ for $f \in C^\infty(M)$, $\nu \in \Gamma(E)$. By a **linear A-connection** in the vector bundle E we mean $C^\infty(M)$-linear operator of modules

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

commuting (on the level of sections) with anchors. This definition extends usual notion of linear connection in a vector bundle in the case where A is the tangent bundle to M.

Denote the vector bundles $\bigotimes^k A^*$ and $\bigotimes A^*$ by $A^{*\otimes k}$ and by $A^{*\otimes}$, respectively. Take two A-connections

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

and

$$\nabla^A : \Gamma(A) \longrightarrow \mathcal{CDO}(A).$$

Observe that ∇ and ∇^A define in a natural way the A-connection $\nabla \otimes \nabla^A$ in the bundle $A^* \otimes E$. Next, by the Leibniz rule we extend this connection to the whole tensor bundle $A^{*\otimes} \otimes E$. The extended connection will be denoted by the same symbol ∇. Equivalently,

$$(\nabla_a \zeta)(a_1, \ldots, a_k) = \nabla_a(\zeta(a_1, \ldots, a_k)) - \sum_{j=1}^{k} \zeta(a_1, \ldots, \nabla^A_a a_j, \ldots, a_k)$$

for $\zeta \in \Gamma(A^{*\otimes k} \otimes E)$, $a, a_1, \ldots, a_k \in \Gamma(A)$.

The connection ∇ can be treated as the operator

$$\nabla : \Gamma(A^{*\otimes k} \otimes E) \to \Gamma(A^{*\otimes k+1} \otimes E)$$

by the convention

$$(\nabla \zeta)\,(a_1, a_2, \ldots, a_{k+1}) = (\nabla_{a_1} \zeta)\,(a_2, \ldots, a_{k+1})\,,$$

$\zeta \in \Gamma(A^{*\otimes k} \otimes E)$, $a, a_1, \ldots, a_k \in \Gamma(A)$.

2. The notion of the Dirac type operator determined by a skew-symmetric tensor

Let

$$\pi : \Gamma(A^*) \times \Gamma(A^*) \to C^\infty(M)$$

be a skew-symmetric bilinear form and let

$$\sharp : A^* \longrightarrow A$$

be the homomorphism of the dual bundles determined by the contraction with respect to π:

$$\sharp(\omega) = i_\omega \pi \quad \text{for} \quad \omega \in \Gamma(A^*).$$

Observe that $\sharp$ is an isomorphism if and only if π is non degenerate.

Define the bundle of Weyl algebras with respect to π to be

$$W(A^*) = A^{*\otimes}/I$$

where

$$I = \langle a^* \otimes b^* - b^* \otimes a^* + 2\pi(a^*, b^*) : \; a^*, b^* \in A^* \rangle$$

(cf. [5]).

Let E be any vector bundle over M and let

$$W(\omega) : E \longrightarrow E, \qquad \omega \in \Gamma(A^*)$$

be the Weyl multiplication in the Weyl algebra – i.e.,

$$W(\omega)\; W(\eta) - W(\eta)\; W(\omega) = -2\pi(\omega, \eta) \quad \text{for} \quad \omega, \eta \in \Gamma(A^*).$$

In this way, we have the Weyl module:

$$W : W(A^*) \otimes E \longrightarrow E, \qquad W(\omega, e) = W(\omega)e.$$

Define the **Dirac operator** $\mathcal{D}^s$ as the composition

$$\mathcal{D}^s = W \circ \nabla : \Gamma(E) \longrightarrow \Gamma(E).$$

Let $(e_1, \ldots, e_n)$ and $(\alpha_1, \ldots, \alpha_n)$ be local dual frames of A and A^*, respectively. Since (cf. [1])

$$\nabla\zeta = \sum_{j=1}^{n} \alpha_j \otimes \nabla_{e_j}\zeta$$

for $\zeta \in \Gamma(A^{*\otimes k} \otimes E)$, the Dirac operator $\mathcal{D}^s$ is given by

$$\mathcal{D}^s \nu = \sum_{j=1}^{n} W(\alpha_j)\; \nabla_{e_j}\nu \tag{1}$$

for $\nu \in \Gamma(E)$.

3. The Weyl module of the bundle of symmetric tensors with values in a vector bundle and the Dirac operator on symmetric tensors

The main subject of this section is the Weyl module of the bundle of symmetric tensors with vector bundle values. Here the Weyl action is a linear combination of multiplication and substitution operators restricted to the symmetric bundle.

Let F be any vector bundle over M. Let $\mathscr{S}^k(A,F)$ denote the module of sections of the tensor product $E = \mathsf{S}^k A^* \otimes F$ of the kth symmetric power $\mathsf{S}^k A^*$ of A^* and the bundle F, $\mathsf{S}A^* = \bigoplus_{k\geq 0} \mathsf{S}^k A^*$,

$$\mathscr{S}(A,F) = \bigoplus_{k\geq 0} \mathscr{S}^k(A,F),$$

and let

$$\mathscr{S}(A) = \bigoplus_{k\geq 0} \mathscr{S}^k(A) = \bigoplus_{k\geq 0} \Gamma(\mathsf{S}^k A^*).$$

Moreover, let

$$\mu(\omega) = \omega \odot (\cdot)$$

be the operator of symmetric multiplication by $\omega \in \mathscr{S}(A)$ and let

$$i_a : \mathscr{S}^k(A,F) \to \mathscr{S}^{k-1}(A,F)$$

denote the substitution operator on $\mathscr{S}(A,F)$ with respect to the first argument.

Define a family $\{W_t\}_{t\in\mathbb{R}}$ of operators

$$W_t : W(A^*) \otimes \mathscr{S}(A,F) \to \mathscr{S}(A,F)$$

by the formulas:

$$W_t(\omega,\zeta) = \mu(\omega)\zeta + t \cdot i_{\sharp(\omega)}\zeta$$

for $\omega \in \Gamma(A^*)$ and $\zeta \in \mathscr{S}^k(A,F)$.

By the following identities on $\mathscr{S}(A,F)$:

$$i_a\ i_b = i_b\ i_a,$$
$$\mu(\omega)\ \mu(\eta) = \mu(\eta)\ \mu(\omega),$$
$$\mu(\omega)\ i_a - i_a\ \mu(\omega) = i_a(\omega)$$

for $\omega,\eta \in \Gamma(A^*)$, $a,b \in \Gamma(A)$, we obtain the following useful

Lemma 1.

$$W_t(\omega,\cdot)\ W_s(\eta,\cdot) - W_s(\eta,\cdot)\ W_t(\omega,\cdot) = (t+s)\ \pi(\omega,\eta)\ \ \mathrm{id}_{\mathscr{S}(A,F)}$$

for any $t,s \in \mathbb{R}$, $\omega,\eta \in \Gamma(A^*)$.

Consider two particular operators

$$W^- = W_{-1}, \quad W^+ = W_{+1}$$

and use the notation

$$W^-(\eta) = W^-(\eta,\cdot), \quad W^+(\eta) = W^+(\eta,\cdot) \ \text{ for } \ \eta \in \Gamma(A^*).$$

Then, for $\omega, \eta \in \Gamma(A^*)$ we have

$$W^-(\omega)\ W^-(\eta) - W^-(\eta)\ W^-(\omega) = -2\ \pi(\omega,\eta)\ \mathrm{id}_{\mathscr{S}(A,F)},$$
$$W^+(\omega)\ W^+(\eta) - W^+(\eta)\ W^+(\omega) = 2\ \pi(\omega,\eta)\ \mathrm{id}_{\mathscr{S}(A,F)},$$
$$W^-(\omega)\ W^+(\eta) = W^+(\eta)\ W^-(\omega).$$

The action

$$W^- : W(A^*) \otimes (\mathsf{S}(A^*) \otimes F) \longrightarrow \mathsf{S}(A^*) \otimes F$$

defines a Weyl module structure on $\mathsf{S}(A^*) \otimes F$.

Now, let

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(F)$$

and

$$\nabla^A : \Gamma(A) \longrightarrow \mathcal{CDO}(A)$$

be two A-connections, which tensor product we extend by the Leibniz rule to the A-connection ∇ in the tensor bundle $\mathsf{S}(A^*) \otimes F$ of the symmetric bundle and F.

In view of (1), the local formula for the Dirac operator $\mathcal{D}^s$ determined by W^- and ∇ is of the form

$$\mathcal{D}^s \xi = \sum_{j=1}^{n} \left(\alpha_j \odot \nabla_{e_j} \xi - i_{\sharp(\alpha_j)} \nabla_{e_j} \xi\right)$$

for $\xi \in \Gamma(\mathsf{S}^k A^* \otimes F)$, and where $(e_1, \ldots, e_n)$ and $(\alpha_1, \ldots, \alpha_n)$ are local dual frames of A and A^*, respectively.

4. The Dirac symmetric operator, the symmetric derivative and coderivative

We will describe the Dirac type operator $\mathcal{D}^s$ in the language of the symmetric covariant derivative and the symmetric coderivative.

In the module $\mathscr{S}(A,F)$ we define the operator d^s of degree $+1$ by

$$d^s : \mathscr{S}^k(A,F) \longrightarrow \mathscr{S}^{k+1}(A,F),$$
$$(d^s \xi)(a_1, \ldots, a_{k+1}) = \sum_{j=1}^{k+1} \left(\nabla_{a_j} \xi\right)(a_1, \ldots, \widehat{a}_j, \ldots, a_{k+1})$$

for $\xi \in \mathscr{S}^k(A,F)$, $a_1, \ldots, a_{k+1} \in \Gamma(A)$. The operator d^s is called the **symmetric covariant derivative** for ∇. One can see that

$$d^s = (k+1) \cdot (\mathrm{Sym} \circ \nabla) | \mathscr{S}^k(A,F),$$

where Sym is the *symmetrizer* defined by

$$(\mathrm{Sym}\, \zeta)(a_1, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \zeta(a_{\sigma(1)}, \ldots, a_{\sigma(k)}).$$

The operator d^s in the case of tangent bundles was introduced by Sampson in [14], where a symmetric version of Chern's theorem is considered. One can check that d^s can be described by the following Koszul type form:

$$\begin{aligned}\left(d^s\xi\right)\left(a_1,\dots,a_{k+1}\right) = &\sum_{j=1}^{k+1}\nabla_{a_j}\left(\xi\left(a_1,\dots,\widehat{a}_j,\dots,a_{k+1}\right)\right)\\ &-\sum_{i<j}\xi\left(\left\langle a_i : a_j\right\rangle,a_1,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_{k+1}\right),\end{aligned}$$

$\xi \in \mathscr{S}^k(A,F)$, $a_1,\dots,a_{k+1} \in \Gamma(A)$, and where

$$\langle a : b\rangle = \nabla^A_a b + \nabla^A_b a$$

for $a, b \in \Gamma(A)$. The symmetric $\mathbb{R}$-bilinear form

$$\langle\cdot:\cdot\rangle : \Gamma(A)\times\Gamma(A) \longrightarrow \Gamma(A), \quad \langle a : b\rangle = \nabla^A_a b + \nabla^A_b a$$

is called the **symmetric product** defined by the connection ∇^A. The symmetric product in the case $A = TM$ was first introduced by Crouch in [4]. Observe that

$$\langle a : f\cdot b\rangle = f\cdot\langle a : b\rangle + (\varrho_A\circ a)(f)\cdot b$$

and

$$\langle f\cdot a : b\rangle = f\cdot\langle a : b\rangle + (\varrho_A\circ b)(f)\cdot a$$

for $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$. Therefore, $\langle\cdot:\cdot\rangle$ is a symmetric bracket in $\Gamma(A)$ satisfying the Leibniz kind rules.

Lewis in [6] gives some interesting geometrical interpretation of the symmetric product associated with the geodesically invariant property of a distribution. A smooth distribution D on a manifold M with an affine connection ∇^{TM} is *geodesically invariant* if for every geodesic $c : I \to M$ with the property $c'(s) \in D_{c(s)}$ for some $s \in I$, we have $c'(s) \in D_{c(s)}$ for every $s \in I$. Lewis proved in [6] that a distribution D on a manifold M equipped with an affine connection ∇^{TM} is geodesically invariant if and only if the symmetric product induced by ∇^{TM} is closed under D.

The symmetric bracket plays an important rule in research under mechanical control systems. Respondek and Ricardo in [12, 13] examine mechanical control systems with the geodesic accessibility property, i.e., mechanical control systems for which the smallest distribution on the configuration manifold, containing the input vector fields and closed under the symmetric product, is of full rank at each point. They characterized the local mechanical state equivalence of two geodesically accessible mechanical systems using some families of structure functions defined via the symmetric bracket.

By the **symmetric coderivative** d^{s*} we mean the restriction of the coderivative operator given in $A^{*\otimes}\otimes F$ to the space of symmetric tensors, i.e.,

$$d^{s*} = \nabla^*|_{\mathscr{S}^k(A,F)} : \mathscr{S}^k(A,F) \longrightarrow \mathscr{S}^{k-1}(A,F),$$

where

$$\nabla^* : \Gamma(A^{*\otimes k}\otimes F) \longrightarrow \Gamma(A^{*\otimes k-1}\otimes F)$$

is given by

$$\nabla^*\zeta = -\operatorname{tr}(\nabla\zeta) = -\sum_{j=1}^{n} i_{\sharp(\alpha_j)}(\nabla_{e_j}\zeta).$$

Since one can prove that

$$d^s = \sum_{j=1}^{n} \mu(\alpha_j) \circ \nabla_{e_j}$$

we see that the symmetric Dirac operator $\mathcal{D}^s$ is equal to

$$\mathcal{D}^s = \sum_{j=1}^{n} W^-(\alpha_j) \circ \nabla_{e_j} = \sum_{j=1}^{n} \left(\mu(\alpha_j) - i_{\sharp(\alpha_j)}\right) \circ \nabla_{e_j} = d^s + d^{s*}.$$

Let A be transitive Lie algebroid, i.e., the anchor ρ_A is surjective, $x \in M$, $\nu \in \mathsf{S}^k A^*_x \otimes F_x$, $\xi \in \mathscr{S}^k(A, F)$ and $\omega \in A^*_x$ such that $\omega = (d^A f)(x)$ for some $f \in C^\infty(M)$ satisfying $f(x) = 0$ and $\xi(x) = \nu$. Since symbols of d^s and d^{s*} are respectively given by

$$\begin{aligned}
\sigma_{d^s}(\omega,\nu) &= d^s(f\xi)(x) = \left(d^A f \odot \xi + f d^s \xi\right)(x) = \omega \odot \nu,\\
\sigma_{d^{s*}}(\omega,\nu) &= d^{s*}(f\xi)(x) = \left(-i_{\#(d^A f)}\xi\right)(x) = -i_{\#(\omega)}\nu,
\end{aligned}$$

we conclude that the symbol of $\mathcal{D}^s$ is the Weyl multiplication by ω:

$$\sigma_{\mathcal{D}^s}(\omega,\nu) = W^-(\omega)\nu.$$

Therefore $\mathcal{D}^s$ is of elliptic type.

References

[1] B. Balcerzak, A. Pierzchalski, *Generalized gradients on Lie algebroids.* Ann. Global. Anal. Geom. **44**, no. 3 (2013), 319–337, doi: 10.1007/s10455-013-9368-y.

[2] B. Balcerzak, A. Pierzchalski, *On Dirac operators on Lie algebroids.* Differential Geom. Appl. **35** (2014), 242–252, doi:10.1016/j.difgeo.2014.06.001.

[3] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators.* Springer, 2004.

[4] P. Crouch, *Geometric structures in systems theory.* IEEE Proc. D Control Theory Appl. **128** (5) (1981), 242–252, doi: 10.1049/ip-d.1981.0051.

[5] K. Habermann, L. Habermann, *Introduction to Symplectic Dirac Operators.* Lect. Notes in Math. **1887**, Springer, 2006.

[6] A.D. Lewis, *Affine connections and distributions with applications to nonholonomic mechanics.* Reports Math. Phys. **42**, No. 1/2 (1998), 135–164.

[7] K.C.H. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry.* London Math. Soc. Lecture Note Ser. **124**, Cambridge Univ. Press, 1987.

[8] K.C.H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids.* London Math. Soc. Lecture Note Ser. **213**, Cambridge Univ. Press, 2005.

[9] P. Popescu, *Categories of Modules with Differentials.* J. Algebra **185** (1996), 50–73.

[10] M. Popescu, P. Popescu: *Geometric objects defined by almost Lie structures.* In: Lie algebroids and related topics, Banach Centre Publ. **54** (2001), 217–233.

[11] J. Pradines, *Théorie de Lie pour les groupoïdes différentiables, calcul différentiel dans la catégorie des groupoïdes infinitésimaux*. C. R. Acad. Sci. Paris **264** (1967) 245–248.

[12] W. Respondek, S. Ricardo, When is a control system mechanical? J. Geom. Mechanics **2** (3) (2010), 265–302, doi:10.3934/jgm.2010.2.265.

[13] W. Respondek, S. Ricardo, *Equivariants of mechanical control systems*. SIAM J. Control Optim. **51** (4) (2013), 3027–3055, doi: 10.1137/11086104X.

[14] J.H. Sampson, *A thorem of Chern*. Trans. Amer. Math. Soc. **177** (1973), 141–153.

Bogdan Balcerzak
Institute of Mathematics
Łódź University of Technology
ul. Wólczańska 215
PL-90-924 Łódź, Poland
e-mail: bogdan.balcerzak@p.lodz.pl

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 223–228

Surfaces Which Behave Like Vortex Lines

Marián Fecko

Abstract. In general setting of theory of integral invariants, due to Poincaré and Cartan, one can find a d-dimensional integrable distribution (given by a possibly higher-rank form) whose integral surfaces behave like vortex lines: they move with (abstract) fluid. Moreover, in a special case they reduce to true vortex lines and, in this case, we get the celebrated Helmholtz theorem.

Mathematics Subject Classification (2010). Primary 53Z05; Secondary 76B47.

Keywords. Poincaré and Cartan integral invariant, vortex line, Helmholtz theorem.

1. Introduction

In hydrodynamics, *vortex lines* are field lines of the *vorticity* vector field $\boldsymbol{\omega}$, which is curl of velocity field $\mathbf{v}$.

A classical theorem due to Helmholtz says that, in the case of ideal and barotropic fluid that is only subject to conservative forces, vortex lines "move with the fluid" (see Ref. [1] and Refs. [2, 3]; one also says that the lines are "frozen into the fluid" or that "vortex lines are material lines").

Hydrodynamics of ideal fluid may be viewed, albeit it is not quite standard, as an application of the theory of *integral invariants* due to Poincaré and Cartan (see Refs. [4, 5], or, in modern presentation, Refs. [6–8]). Then, the original Poincaré version of the theory refers to the *stationary* (time-independent) flow, described by the stationary Euler equation, whereas Cartan's extension embodies the full, possibly time-dependent, situation.

In this picture, one can base a proof of the Helmholtz theorem upon the concept of a *distribution*. Namely, first, vortex lines are identified with integral surfaces of a 1-dimensional integrable distribution, defined in terms of the appropriate 2-form. Second, the structure of the (Euler) equation of motion immediately reveals that the 2-form is *Lie-invariant* w.r.t. the flow of the fluid. So, third, the corresponding distribution is invariant as well and, consequently, its integral surfaces are invariant w.r.t. the flow of the fluid. And this is exactly what the Helmholtz statement says.

Now, it turns out that the same reasoning may be repeated within the *general integral invariant* setting (so beyond even the "n-dimensional Riemannian hydrodynamics", discussed, e.g., in [9]). What differs is that we have an integrable distribution based upon a possibly *higher-degree* Lie-invariant differential form, there. In particular, the distribution may be *higher-dimensional* and, consequently, its integral surfaces become then higher-dimensional, too. Nevertheless, they still obey the Helmholtz-like rule of "moving with the fluid" (i.e., the *abstract* flow in the general theory translates the integral surfaces into one another).

2. Integral invariants – Poincaré and Cartan

Before considering the main subject of the paper, let us briefly recall key concepts and state main results of Poincaré and Cartan on general theory of integral invariants. See Ref. [8] in this volume or, for a more detailed account, Ref. [7].

2.1. Poincaré integral invariants

Following *Poincaré*, one starts from a manifold M endowed with dynamics (time evolution) given by a *vector field* v (via its flow)

$$(M, \Phi_t \leftrightarrow v) \qquad \textit{phase space} \tag{1}$$

Now, consider integrals of a k-form α over various k-chains (k-dimensional surfaces) c on M. Compare the value of the integral of α over the original c and the integral over $\Phi_t(c)$. If, *for any chain* c, the two integrals are equal, we call it (absolute) integral invariant:

$$\int_{\Phi_t(c)} \alpha = \int_c \alpha \qquad \Leftrightarrow \qquad \int_c \alpha \quad \text{is } \textit{integral invariant.} \tag{2}$$

If we only restrict to k-*cycles* (i.e., chains whose boundaries vanish, $\partial c = 0$), we speak of *relative* integral invariants. It turns out that one can recognize the relative invariant by the *differential* equation

$$i_v d\alpha = d\beta, \tag{3}$$

i.e., the following statement is true

$$i_v d\alpha = d\beta \qquad \Leftrightarrow \qquad \oint_c \alpha = \quad \textit{relative} \text{ invariant.} \tag{4}$$

2.2. Cartan integral invariants

Cartan proposed to study dynamics on $M \times \mathbb{R}$ (*extended* phase space; *time* coordinate is added) rather than on M. Analogs of the forms α and β (from the Poincaré theory) are combined into a *single* k-form

$$\sigma = \hat{\alpha} + dt \wedge \hat{\beta}. \tag{5}$$

Here, $\hat{\alpha}$ and $\hat{\beta}$ are the most general *spatial* forms on $M \times \mathbb{R}$. (In coordinate presentation, they do not contain the dt factor. They may be, however, *time-dependent*, i.e., their *components* may depend on time.) In a similar way, the dynamical vector field v sits in the combination

$$\xi = \partial_t + v. \tag{6}$$

Then, according to Cartan, one has to replace the crucial equation of Poincaré, viz. Eq. (3), with

$$i_\xi d\sigma = 0. \tag{7}$$

And the main statement of Poincaré, viz. Eq. (4), takes the form

$$\boxed{i_\xi d\sigma = 0} \qquad \Leftrightarrow \qquad \oint_c \sigma = \quad \textit{relative} \text{ invariant.} \tag{8}$$

It turns out that the proof of (8) does not use any details of the decomposition. The *structure* of equation (7) is all one needs. One can check that

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta} \tag{9}$$

(the term $\mathcal{L}_{\partial_t}\hat{\alpha}$ is new w.r.t. (3)). Here $\hat{d}$ denotes the *spatial* exterior derivative. (In coordinate presentation – as if the variable t in components was *constant*.) So, the equation

$$\boxed{\mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta}} \tag{10}$$

is *the* equation that (possibly) *time-dependent* forms $\hat{\alpha}$ and $\hat{\beta}$ are to satisfy in order that the integral of σ is to be a relative integral invariant.

3. Surfaces and their motion

3.1. Stationary case

Return back to equation (3). Application of d on both sides results in

$$\mathcal{L}_v(d\alpha) = 0, \qquad \text{i.e.,} \qquad \Phi_t^*(d\alpha) = d\alpha \qquad \Phi_t \leftrightarrow v \tag{11}$$

So, the form $d\alpha$ is *invariant* w.r.t. the flow Φ_t.

Let us define a *distribution* $\mathcal{D}$ in terms of $d\alpha$:

$$\mathcal{D} := \{\text{vectors } w \text{ such that } \ i_w d\alpha = 0 \ \text{ holds}\}. \tag{12}$$

[Motivation for this definition comes from *hydrodynamics*. Namely, see Ref. [8] in this volume, integral submanifolds of this distribution for the particular choice $\alpha = \tilde{v} \equiv g(v, \ \cdot\ )$, where v is the velocity field in hydrodynamics, are one-dimensional and coincide with *vortex lines*.]

Due to the Frobenius criterion the distribution is integrable. Indeed, let $w_1, w_2 \in \mathcal{D}$, i.e., $i_{w_1} d\alpha = 0$ and $i_{w_2} d\alpha = 0$. Then, because of the identity

$$i_{[w_1,w_2]} = [\mathcal{L}_{w_1}, i_{w_2}] \equiv \mathcal{L}_{w_1} i_{w_2} - i_{w_2}\mathcal{L}_{w_1} \tag{13}$$

(see, e.g., Ref. [11]) plus Cartan's formula

$$i_u d + d i_u = \mathcal{L}_u \tag{14}$$

one immediately sees that

$$i_{[w_1,w_2]}d\alpha = 0, \tag{15}$$

i.e., $[w_1, w_2] \in \mathcal{D}$, too. So $\mathcal{D}$ is integrable. Since the distribution $\mathcal{D}$ is invariant w.r.t. $\Phi_t \leftrightarrow v$, its integral surfaces are invariant w.r.t. $\Phi_t \leftrightarrow v$, too. But this means that a "Helmholtz-*like*" theorem is true: *whenever* we encounter the *general* context of the Poincaré integral invariants, the integral surfaces of the distribution $\mathcal{D}$ are *frozen into* "the fluid".

3.2. General, non-stationary case

Now, a question arises whether or not a similar statement is true in a much more complex, *time-dependent*, situation. The answer turns out to be still positive, although the proof is more involved.

Let us start with the application of d on (7). It results in

$$\mathcal{L}_\xi(d\sigma) = 0, \qquad \text{i.e.,} \qquad \Phi_\tau^*(d\sigma) = d\sigma \qquad \Phi_\tau \leftrightarrow \xi. \tag{16}$$

So, $d\sigma$ is *invariant* w.r.t. the flow.

Define the distribution $\mathcal{D}$ (on $M \times \mathbb{R}$, now) in terms of annihilation of as many as *two* exact forms:

$$\mathcal{D} \quad \leftrightarrow \quad i_w d\sigma = 0 = i_w dt. \tag{17}$$

So, we are interested in *spatial* vectors ($i_w dt = 0$) which, in addition, annihilate $d\sigma$.

The new distribution $\mathcal{D}$ is *integrable* as well. The Frobenius criterion shows this easily, again: We assume

$$i_{w_1}d\sigma = 0 = i_{w_1}dt \qquad i_{w_2}d\sigma = 0 = i_{w_2}dt \tag{18}$$

and, using (13) and (14), we see that

$$i_{[w_1,w_2]}d\sigma = 0 = i_{[w_1,w_2]}dt. \tag{19}$$

So, our new distribution $\mathcal{D}$ (on $M \times \mathbb{R}$) defined via annihilation of $d\sigma$ and dt is *integrable* and *invariant* w.r.t. the flow. Consequently, its integral submanifolds (surfaces) are *frozen into* "the fluid".

What is not yet clear, however, is the exact relation of this result to the result of the time-independent case from Section 3.1. (Recall that the distribution considered there was spanned by vectors which annihilate $d\alpha$ rather than $d\sigma$.)

It is here where Eq. (7) comes to rescue again, now in a more subtle way. Indeed, applying d on (5) and then using the decomposed version (10) of (7), we can write

$$d\sigma = \hat{d}\hat{\alpha} + dt \wedge (\mathcal{L}_{\partial_t}\hat{\alpha} + \hat{d}\hat{\beta}) \qquad \text{always} \tag{20}$$

$$= \hat{d}\hat{\alpha} + dt \wedge (-i_v \hat{d}\hat{\alpha}) \qquad \textit{on solutions}. \tag{21}$$

Now, let w be arbitrary *spatial* vector. Denote, for a while, $i_w\hat{d}\hat{\alpha} =: \hat{b}$ (it is a *spatial* 1-form). Then, from (21),

$$i_w d\sigma = \hat{b} - dt \wedge i_v \hat{b} \tag{22}$$

from which immediately

$$i_w(d\sigma) = 0 \qquad \Leftrightarrow \qquad \hat{b} \equiv i_w \hat{d}\hat{\alpha} = 0. \tag{23}$$

This says that we can, alternatively, describe the distribution $\mathcal{D}$ as consisting of those *spatial* vectors which, in addition, annihilate $\hat{d}\hat{\alpha}$ (rather than $d\sigma$, as it is expressed in the definition (17)). But this means that we speak of "the same" distribution as in (12). (The language of σ is more advantageous for proving invariance of the distribution w.r.t. the flow as well as for its integrability, whereas the "decomposed" language of $\hat{\alpha}$ and $\hat{\beta}$ is needed for identification of the distribution as the one from the time-independent case.) So, the Helmholtz-like statement from Section 3.1 is also true in the general, time-dependent, case. (Notice that the system of the surfaces, if regarded as living on M, looks, in general, different in different times. This is because its generating object, the form $\hat{d}\hat{\alpha}$, depends on time.)

[*On solutions* in Eq. (21) means on solutions of equation (7) or, equivalently, of (10). In hydrodynamics, (7) turns out to be (see Ref. [8] in this volume) nothing but the *Euler* equation, i.e., the *equation of motion* of ideal fluid. So, the fact that vortex lines are frozen into the fluid is only true in the case of *real dynamics* of the fluid. It is, unlike the Helmholtz statement on strength of vortex tubes, a *dynamical*, rather than kinematical, statement.]

4. Conclusions

Theory of integral invariants due to Poincaré and Cartan enables one, when applied to hydrodynamics, to get a simple and convincing proof of Helmholtz' classical theorem on motion of vortex lines. Moreover, this approach reveals that, actually, there is *a generalization* of the phenomenon still in the original theory (prior to application to hydrodynamics). In this case, vortex *lines* are to be replaced by appropriate distinguished *surfaces.*

Acknowledgment

This research was supported by VEGA project V-16-087-00.

References

[1] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welcher der Wirbelbewegungen entsprechen.* Journal für die reine und angewandte Mathematik, 55 (1858), 25–55.

[2] J.Z. Wu, H.Y. Ma, M.D.Zhou, *Vorticity and Vortex Dynamics.* Springer-Verlag Berlin Heidelberg, 2006.

[3] K.S. Thorne, R.D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics.* Princeton University Press, 2017.

[4] H. Poincaré, *Les Méthodes nouvelles de la Mécanique Céleste,* (*III, Invariants intégraux*). Gauthier-Villars et fils, 1899.

[5] E. Cartan, *Leçons sur les invariants intégraux*. Hermann, 1922.

[6] V. Arnold, *Mathematical Methods of Classical Mechanics*. Springer-Verlag, 1989.

[7] M. Fecko, *Modern geometry in not-so-high echelons of physics: Case studies*. Acta Physica Slovaca **63**, No.5 (2013), 261–359. (arXiv:1406.0078 [physics.flu-dyn])

[8] M. Fecko, *Integral invariants* (*Poincaré–Cartan*) *and hydrodynamics*. VI School on Geometry and Physics, Białowieza 2017, 377–382. (this volume)

[9] V. Arnold, B. Khesin, *Topological Methods in Hydrodynamics*. Springer-Verlag, 1998.

[10] M. Fecko, *A generalization of vortex lines*. Journal of Geometry and Physics **124**, (2018), 64–73.

[11] M. Fecko, *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2006.

Marián Fecko
Department of Theoretical Physics and Didactics of Physics
Comenius University in Bratislava
Mlynská dolina F2
SK-84248 Bratislava, Slovakia
e-mail: fecko@fmph.uniba.sk

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 229–235

On the Spin Geometry of Supergravity and String Theory

C.I. Lazaroiu and C.S. Shahbazi

Abstract. We summarize the main results of our recent investigation of bundles of real Clifford modules and briefly touch on some applications to string theory and supergravity.

Mathematics Subject Classification (2010). 53C27, 15A66.

Keywords. Spin geometry, Clifford bundles.

1. Introduction

A full global understanding of real spinor fields in supergravity and string theories requires a characterization of those real vector bundles S over the space-time (M, g) of which such fields can be sections. Existence of a globally-defined Clifford multiplication on (M, g) is the minimal ingredient needed to construct a Dirac operator and hence to formulate the fermion kinetic action. Hence, one can take Clifford multiplication as the fundamental ingredient needed to describe the physics of spinor fields. One has a choice between *inner* and *outer* Clifford multiplications[1]. The former is a morphism $TM \otimes S \to S$ obeying the Clifford property, while the latter is a morphism $TM \otimes S \to S'$, where S' is a vector bundle which need not be isomorphic with S.

Below, we consider exclusively real vector bundles endowed with inner Clifford multiplication, hence S will be a bundle of modules over the Clifford bundle $\mathrm{Cl}(TM, g)$ of the space-time (M, g), which is assumed to be connected. In this case, the fiber S_p of S at every point p of M carries a representation of the algebra $\mathrm{Cl}(T_pM, g_p)$. This gives a unital morphism $\gamma : \mathrm{Cl}(TM, g) \to \mathrm{End}(S)$ of bundles of associative algebras, which we shall call the *structure morphism* of S. For technical reasons, we also require that γ be *weakly-faithful*, which means that the restriction of γ to the vector bundle $TM \subset \mathrm{Cl}(TM, g)$ is injective. For brevity of language,

[1]Outer Clifford multiplication arises, for example, in the theory of Pin structures, in which situation it sometimes allows one to define a “modified” Dirac operator.

we define a *real pinor bundle*[2] to be such a weakly-faithful bundle S of Clifford modules. This leads us to consider the following mathematical questions:

- *Is every real pinor bundle associated to a spin structure? If not, to what principal bundle is it associated?*
- *What is the topological obstruction to existence of a pinor bundle on a pseudo-Riemannian manifold (M, g) of arbitrary signature (p, q)?*

These questions were answered in [1]. The results of loc. cit. show that, in general, real pinor bundles are associated not to spin structures but to more general spinorial structures, which we call *real Lipschitz structures*[3]. The second question was completely solved in [1] for so-called *elementary real pinor bundles*, defined as those real pinor bundles whose fiberwise Clifford representation is irreducible.

2. Real Lipschitz structures and their relation to real pinor bundles

Let (V, h) be a quadratic vector space which is isomorphic with each fiber of the tangent bundle (TM, g). A representation $\eta : \mathrm{Cl}(V, h) \to \mathrm{End}_{\mathbb{R}}(S_0)$ of the Clifford algebra $\mathrm{Cl}(V, h)$ in a finite-dimensional real vector space S_0 is called *weakly-faithful* if the restriction of η to the subspace V of $\mathrm{Cl}(V, h)$ is injective. In this case, the *real Lipschitz group* $\mathrm{L}(\eta)$ of η is defined as the group consisting of all invertible operators a acting in S_0 whose adjoint action preserves the subspace $\eta(V)$ of $\mathrm{End}_{\mathbb{R}}(S)$:

$$\mathrm{L}(\eta) \stackrel{\text{def}}{=} \{a \in \mathrm{Aut}_{\mathbb{R}}(S_0) | \, \mathrm{Ad}(a)(\eta(V)) = \eta(V)\} \ .$$

The *vector representation* of $\mathrm{L}(\eta)$ is the group morphism $\mathrm{Ad}_0 : \mathrm{L}(\eta) \to \mathrm{O}(V, h)$ defined through:

$$\mathrm{Ad}_0(a) \stackrel{\text{def}}{=} (\eta|_V)^{-1} \circ \mathrm{Ad}(a)|_{\eta(V)} \circ (\eta|_V) \ .$$

A *real Lipschitz structure of type* η on (M, g) is an Ad_0-reduction (Q, τ) of the principal bundle $P(M, g)$ of pseudo-orthogonal frames of (TM, g), i.e., a pair formed of a principal $\mathrm{L}(\eta)$-bundle Q over M and an Ad_0-equivariant fiber bundle map $\tau : Q \to P(M, g)$. A bundle (S, γ) of Clifford modules over (M, g) is weakly-faithful iff each fiberwise Clifford representation $\gamma_p : \mathrm{Cl}(T_pM, g_p) \to \mathrm{End}(S_p)$ (where $p \in M$) is weakly-faithful. Since M is connected, all fiberwise Clifford representations γ_p are *unbasedly* isomorphic[4] with each other and hence with some fiducial weakly-faithful Clifford representation $\eta : \mathrm{Cl}(V, h) \to \mathrm{End}_{\mathbb{R}}(S_0)$, where S_0 is a vector space which models the fibers of S. The representation η (considered up to unbased isomorphism of representations) is called the *type* of (S, γ). One has the following key result:

[2]The word "pinor" refers to the fact that we consider bundles of modules over the fibers of $\mathrm{Cl}(TM, g)$ rather than over the fibers of $\mathrm{Cl}^{\mathrm{ev}}(TM, g)$.
[3]This follows the terminology introduced by T. Friedrich and A. Trautman [2] for the case of *complex* vector bundles with Clifford multiplication.
[4]This means that they are isomorphic in a certain category which is defined in [1] and which has more morphisms than the usual category of representations.

Theorem 1 ([1]). *There exists an equivalence of categories between the groupoid of real Lipschitz structures of type η and the groupoid of real pinor bundles of type η defined over (M,g). In particular, the underlying vector bundle S of every real pinor bundle (S,γ) of type η is associated to the principal bundle Q of a Lipschitz structure (Q,τ) which has type η.*

This implies that (M,g) admits a real pinor bundle of type η iff it admits a real Lipschitz structure of type η, and that the classifications of these two kinds of objects up to the corresponding notion of isomorphism agree.

One can show that any irreducible real Clifford representation is weakly-faithful and that all such representations of $\mathrm{Cl}(V,h)$ belong to the same *unbased* isomorphism class, which is determined by the signature (p,q) of (V,h). A real pinor bundle (S,γ) is called *elementary* if its fibers are *irreducible* as real Clifford representations, which amounts to the requirement that its type η is irreducible. The real Lipschitz groups of irreducible Clifford representations are called *elementary*, as are the real Lipschitz structures whose type is given by such representations. For each quadratic vector space (V,h), there exists an essentially unique elementary real Lipschitz group L, determined up to isomorphism by the signature (p,q) of (V,h). Moreover, the nature of this group depends only on $p-q \mod 8$. One has $\mathrm{L} \simeq \mathbb{R}_{>0} \times \mathcal{L}$, where $\mathcal{L}$ is a natural subgroup called the *reduced Lipschitz group*, which can be constructed using the so-called "Lipschitz norm". Elementary real Lipschitz groups were classified in [1], the result being summarized in Table 1. A *reduced elementary Lipschitz structure* is defined like a Lipschitz structure, but using the group $\mathcal{L}$ (and the restriction of Ad_0 to $\mathcal{L}$) instead of L. The groupoid of elementary real Lipschitz structures is equivalent with that of reduced elementary real Lipschitz structures, so the latter is also equivalent with the groupoid of elementary real pinor bundles. When $pq \neq 0$, $\mathcal{L}$ is neither compact nor connected. It

$p-q$ mod 8	$\mathcal{L}$	$\mathfrak{G}(p,q)$
$0,2$	$\mathrm{Pin}(p,q)$	1
$3,7$	$\mathrm{Spin}^o(p,q) \stackrel{\mathrm{def}}{=} \mathrm{Spin}(p,q)\cdot \mathrm{Pin}_2^{\alpha_{p,q}}$	$\mathrm{O}(2,\mathbb{R})$
$4,6$	$\mathrm{Pin}^q(p,q) \stackrel{\mathrm{def}}{=} \mathrm{Pin}(p,q)\cdot \mathrm{Sp}(1)$	$\mathrm{SO}(3,\mathbb{R})$
1	$\mathrm{Spin}(p,q)$	1
5	$\mathrm{Spin}^q(p,q) \stackrel{\mathrm{def}}{=} \mathrm{Spin}(p,q)\cdot \mathrm{Sp}(1)$	$\mathrm{SO}(3,\mathbb{R})$

TABLE 1. Reduced elementary Lipschitz groups in signature (p,q). The sign factor $\alpha_{p,q}$ equals -1 when $p-q\equiv_8 3$ and $+1$ when $p-q\equiv_8 7$ and we use the notation $\mathrm{Pin}_2^+ \stackrel{\mathrm{def}}{=} \mathrm{Pin}(2,0)$ and $\mathrm{Pin}_2^- \stackrel{\mathrm{def}}{=} \mathrm{Pin}(0,2)$. The last column lists the characteristic group. The symbol "$\cdot$" denotes direct product of groups divided by a central $\mathbb{Z}_2$ subgroup.

is clear from Table 1 that the conditions for existence of an elementary real Lipschitz structure are generally *weaker* (and sometimes considerably so) than those for existence of a spin structure. Every elementary Lipschitz group has a so-called *characteristic representation*, which is naturally associated to it as explained in [1]. The image of this representation is the so-called *characteristic group* $\mathfrak{G}(V,h)$, whose isomorphism type is listed in the last column of Table 1. Accordingly, an elementary Lipschitz structure (Q,τ) induces a principal *characteristic bundle* E (with structure group $\mathfrak{G}(p,q)$), which is associated to Q through the characteristic representation of the corresponding Lipschitz group; this bundle can be non-trivial only when $p-q \not\equiv_8 0,1,2$. For $p-q\equiv_8 0,1,2$, a Lipschitz structure is either a Spin or Pin structure and hence is of the classical type studied for example in [3]. When $p-q\equiv_8 5$, it is a Spin^q structure in general signature (the positive-definite case $q=0$ of such was studied in [4]). The cases $p-q\equiv_8 4,6$ lead to Pin^q structures, which are a slight extension of Spin^q structures to non-orientable pseudo-Riemannian manifolds. The cases $p-q\equiv_8 3,7$ lead to what we call Spin^o structures, which appear to be new.

The characteristic bundle of a Spin^o-structure is a principal $\mathrm{O}(2)$ bundle, which suggests that it may be relevant to situations where spinors are *charged* under a $\mathrm{O}(2)$ gauge group rather than under a $\mathrm{U}(1)$ group. This fact may be relevant to understand the worldvolume theories of non-orientable D-branes. Let:

$$\sigma := \sigma_{p,q} \stackrel{\text{def}}{=} (-1)^{q+\left[\frac{d}{2}\right]} = \begin{cases} (-1)^{\frac{p-q}{2}} & \text{if } d = \text{ even} \\ (-1)^{\frac{p-q-1}{2}} & \text{if } d = \text{ odd} \end{cases} = \begin{cases} +1 & \text{if } p-q\equiv_4 0,1 \\ -1 & \text{if } p-q\equiv_4 2,3 \end{cases}$$

Let $\mathrm{w}_1^{\pm}(M)$ be the modified Stiefel–Whitney classes of (M,g) introduced in [3]; these classes depend on g but we don't indicate this in the notation. The topological obstructions to existence of elementary real Lipschitz structures (and hence of elementary real pinor bundles) on (M,g) are as follows [1]:

- In the "normal simple case" ($p-q\equiv_8 0,2$), (M,g) admits an elementary real pinor bundle iff $(M,-\sigma g)$ admits a Pin structure, which requires that the following condition is satisfied:
$$\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) + \mathrm{w}_1^{\sigma}(M)^2 + \mathrm{w}_1^-(M)\mathrm{w}_1^+(M) = 0\,.$$
- In the "complex case" ($p-q\equiv_8 3,7$), (M,g) admits an elementary real pinor bundle iff it admits a Spin^o-structure, which happens iff there exists a principal $\mathrm{O}(2,\mathbb{R})$-bundle E on M such that the following two conditions are satisfied:
$$\begin{aligned} \mathrm{w}_1(M) &= \mathrm{w}_1(E) \\ \mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) &= \mathrm{w}_2(E) + \mathrm{w}_1(E)(p\mathrm{w}_1^+(M) + q\mathrm{w}_1^-(M)) \\ &\qquad + \left[\delta_{\alpha,-1} + \frac{p(p+1)}{2} + \frac{q(q+1)}{2}\right]\mathrm{w}_1(E)^2\ , \end{aligned}$$
where $\alpha \stackrel{\text{def}}{=} \alpha_{p,q}$.

- In the "quaternionic simple case" ($p-q \equiv_8 4, 6$), (M, g) admits an elementary real pinor bundle iff $(M, -\sigma g)$ admits a Pin^q-structure, which happens iff there exists a principal $\mathrm{SO}(3,\mathbb{R})$-bundle E on M such that the following condition is satisfied:

$$\mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) + \mathrm{w}_1^\sigma(M)^2 + \mathrm{w}_1^-(M)\mathrm{w}_1^+(M) = \mathrm{w}_2(E) \; .$$

- In the "normal non-simple case" ($p - q \equiv_8 1$), (M, g) admits an elementary real pinor bundle iff it admits a Spin structure, which requires that the following two conditions are satisfied:

$$\mathrm{w}_1(M) = 0 \, , \qquad \mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) = 0 \, .$$

- In the "quaternionic non-simple case" ($p - q \equiv_8 5$), (M, g) admits an elementary real pinor bundle iff it admits a Spin^q-structure, which happens iff there exists a principal $\mathrm{SO}(3,\mathbb{R})$-bundle E over M such that the following conditions are satisfied:

$$\mathrm{w}_1(M) = 0 \, , \qquad \mathrm{w}_2^+(M) + \mathrm{w}_2^-(M) = \mathrm{w}_2(E) \, .$$

3. Applications to string theory and supergravity

The results of reference [1] can be applied to study the spinorial structures needed to formulate various supergravity theories. In this section, we sketch a simple application to M-theory, obtaining a no-go result regarding the global interpretation of its spinor fields.

Consider M-theory on an eleven-dimensional Lorentzian manifold of "mostly plus" signature $(p, q) = (10, 1)$. The low energy limit is given by eleven-dimensional supergravity, whose supersymmetry generator is a 32-component real spinor ϵ. The gravitino Killing spinor equation contains terms with an odd number of gamma matrices acting on ϵ, implying that the whole Clifford algebra $\mathrm{Cl}(T_xM, g_x)$ at a point $x \in M$ must act on the value of ϵ at x. If one assumes that ϵ is a global section of a vector bundle S endowed with *inner* Clifford multiplication, it follows that each fiber S_p must carry a real irreducible representation of $\mathrm{Cl}(T_pM, g_p)$ and hence that S is an elementary real pinor bundle. Since $p-q = 9 \equiv_8 1$, we are in the normal simple case. Hence (M, g) admits an elementary real pinor bundle S if and only if it is oriented and spin. Since $\mathrm{w}_2^-(M) = 0$, the corresponding topological obstruction can be written as follows:

$$\mathrm{w}_1^+(M) = \mathrm{w}_1^-(M) \, , \qquad \mathrm{w}_2^+(M) = 0 \, .$$

We conclude that, in signature $(10, 1)$, the supersymmetry parameter can be interpreted as a global section of an elementary real pinor bundle iff the space-time is orientable and spin. Of course, M-theory can in fact be defined on Lorentzian eleven-manifolds admitting a Pin structure [5, 6], but that construction involves a bundle with external Clifford multiplication, which leads to a modified Dirac operator as in [7].

4. Future directions

The results of [1] open up various directions for further research. Here we list some questions which may be worth pursuing:

- Reference [1] classifies bundles of irreducible modules over $\mathrm{Cl}(M, g)$. It would be interesting to classify bundles of faithful real Clifford modules over $\mathrm{Cl}(M, g)$ and of irreducible or faithful real Clifford modules over the even sub-bundle $\mathrm{Cl}^{\mathrm{ev}}(M, g)$, since such bundles may also be relevant to string theory and supergravity.
- It would be interesting to study the index theorem for general bundles of real Clifford modules, without assuming that (M, g) is spin.
- One could consider extending Wang's classification [8] beyond the case of spin manifolds, characterizing manifolds admitting sections of an elementary real pinor bundle which are parallel w.r.t. a connection lifting the Levi-Civita connection on (M, g) and a fixed connection on the characteristic bundle.
- Killing and generalized Killing spinors were studied in the literature [9–11], usually on manifolds carrying a fixed Spin or Spin^c structure. Using our results, this could be extended to the most general pseudo-Riemannian manifolds admitting elementary real pinor bundles.
- One could apply our results to the spin geometry of branes in string and M-theory. As shown in reference [12], the worldvolume of orientable D-branes in the absence of H-flux admits a Spin^c-structure. In the unorientable case, this may become a Lipschitz structure.
- Our results may be useful to globally characterize the local spinor bundles appearing in exceptional generalized geometry [13], obtaining the topological obstructions to their existence.

Acknowledgment

The work of C.I.L. is supported by grant IBS-R003-S1. The work of C.S.S. is supported by the ERC Starting Grant 259133 – Observable String.

References

[1] C.I. Lazaroiu and C. Shahbazi, *Real pinor bundles and real Lipschitz structures*, arXiv:1606.07894 [math.DG].

[2] T. Friedrich and A. Trautman, *Spin spaces, Lipschitz groups and spinor bundles*, Ann. Global Anal. Geom. **18** (2000) 3–4, 221–240.

[3] M. Karoubi, *Algèbres de Clifford et K-théorie*, Ann. Sci. Ec. Norm. Sup. **1** (1968) 2, 161 – 270.

[4] M. Nagase, *Spin^q structures*, J. Math. Soc. Japan **47** (1995) 1.

[5] E. Witten, *On flux quantization in M theory and the effective action*, J. Geom. Phys. **22** (1997) 1.

[6] E. Witten, *The "Parity" Anomaly On An Unorientable Manifold*, arXiv:1605.02391.

[7] M. Cahen, S. Gutt and A. Trautman, *Pin structures and the modified Dirac operator*, Journ. Geom. Phys. **17** (1995), 283–297.

[8] M.Y. Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989) 1, 59–68.

[9] C. Bar, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993) 3, 509–521.

[10] A. Moroianu and U. Semmelmann, *Generalized Killing spinors on Einstein manifolds*, Int. J. Math. **25** (2014) 4, 1450033.

[11] A. Moroianu, *Parallel and Killing Spinors on* Spin^c *Manifolds*, Commun. Math. Phys. **187** (1997) 2, 417–427.

[12] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, Asian J. Math. **3** (1999) 819 [hep-th/9907189].

[13] P. Pires Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, JHEP **0809** (2008) 123, [arXiv:0804.1362 [hep-th]].

C.I. Lazaroiu
IBS – Center for Geometry and Physics
Pohang, Republic of Korea 37673
e-mail: calin@ibs.re.kr

C.S. Shahbazi
Institut de Physique Théorique
CEA-Saclay, France
e-mail: carlos.shabazi-alonso@cea.fr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 237–244

Conic Sub-Hilbert–Finsler Structure on a Banach Manifold

F. Pelletier

Abstract. A Hilbert–Finsler metric $\mathcal{F}$ on a Banach bundle $\pi : E \to M$ is a classical Finsler metric on E whose fundamental tensor is positive definite. Following [5], a conic Hilbert–Finsler metric $\mathcal{F}$ on E is a Hilbert–Finsler metric which is defined on an open conic submanifold of E. In the particular case where we have a (strong) Riemannian metric g on E, then $\sqrt{g}$ is a natural example of Hilbert–Finsler metric on E. According to [1], if, moreover, we have an anchor $\rho : E \to TM$ we get a sub-Riemannian structure on M, that is, g induces a "singular" Riemannian metric on the distribution $\mathcal{D} = \rho(E)$ on M. By analogy, a sub-Hilbert–Finsler structure on M is the data of a conic Hilbert–Finsler metric $\mathcal{F}$ on a Banach bundle $\pi : E \to M$ and an anchor $\rho : E \to TM$. Of course, we get a "singular" conic Hilbert–Finsler metric on $\mathcal{D} = \rho(E)$. In the finite-dimensional sub-Riemannian framework, it is well known that "normal extremals" are projections of Hamiltonian trajectories, and any such extremal is locally minimizing (relatively to the associated distance). Analogous results in the context of sub-Riemannian Banach manifold were obtained in [1] by Arguillère. By an adaptation of his arguments, we generalize these properties to the sub-Hilbert–Finsler framework.

Mathematics Subject Classification (2010). Primary 37K05, 53D30, 58B25. Secondary 30C35.

Keywords. Banach manifold, Banach anchored bundle, Hilbert–Finsler metric, sub-Riemannian, Hamiltonian field, normal extremal, geodesic.

1. Introduction

Finite-dimensional sub-Riemannian geometry is a large research domain which concerns some variational problems in Physics and in control theory. Recall that, classically, a sub-Riemannian structure on a manifold is a Riemannian bundle $(\mathcal{D}, g)$, where $\mathcal{D}$ is a sub-bundle of the tangent bundle TM of a manifold M. Therefore, for each *horizontal* curve, i.e., tangent to $\mathcal{D}$, we can define its g-length.

When any two points can be joined by a horizontal curve, as in Riemannian geometry, we can define a distance d_g which leads to the notion of geodesic for this structure. On the opposite of Riemannian framework, in sub-Riemannian geometry, it is well known that not all such geodesics are the projections of Hamiltonian trajectories (for a natural associated Hamiltonian on T^*M). Such projections are called normal geodesics and minimize d_g locally (see for instance [6]).

These results were firstly extended ([9]) to the case where $\mathcal{D}$ is the range of an almost injective anchor $\rho : E \to TM$ of a Riemannian bundle $\pi : E \to M$, that is the restriction ρ_x to a fiber $E_x = \pi^{-1}(x)$ is an injective linear map for each x in an open dense set of M. A more large generalization called *sub-Finsler structure* was introduced in [4]: the authors consider a bundle $\pi : E \to M$ provided with a Finsler metric $\mathcal{F}$ and a morphism $\rho : E \to TM$ (the anchor). On the range $\mathcal{D}_x = \rho(E_x)$, we obtain a Finsler metric F_x. We can look again for horizontal curves and if any two points can be joined by a horizontal curve, we can associate a semi-distance $d_{\mathcal{F}}$. By application of classical optimal control theory arguments, the authors prove that geodesics which are projections of Hamiltonian trajectories minimize $d_{\mathcal{F}}$ locally.

The concept of sub-Riemannian geometry in an infinite-dimensional context was introduced in [3] for a sub-bundle $\mathcal{D}$ of the tangent bundle TM of a Riemannian "convenient manifold" M with application to Fréchet Lie groups. Essentially motivated by *mathematical analysis of shapes*, the particular Banach context was recently studied by S. Arguillère in his PhD thesis and, more precisely, formalized in [1]. Essentially, he considers a Banach bundle $\pi : E \to M$ provided with an anchor $\rho : E \to TM$ and a strong Riemannian metric g on E. In this situation, we can again associate a "generalized" distance d_g on M and a notion of geodesic. Among all his results, the author proves that in this Banach framework, "the normal geodesics" are still projections of Hamiltonian trajectories and any such a geodesic minimizes d_g locally.

The concept of conic Finsler metric on the tangent bundle of a manifold was firstly introduced by Bryant in [2] and, more recently, by Javaloyes and M. Sanchez [5]. *The purpose of this work* is to propose a generalization of this concept to the context of *conic Hilbert–Finsler metric on an anchored Banach bundle* and show that Arguillère's results in [1] can be adapted to this framework.

2. Conic sub-Hilbert–Finsler structure on a Banach manifold

2.1. Conic Minkowski norm on a Banach space

Let $\mathbb{E}$ be a Banach space. A *conic domain* $\mathbb{K}$ in $\mathbb{E}$ is an open subset of $\mathbb{E}$ such that if $u \in \mathbb{K}$, then λu belongs to $\mathbb{K}$ for any $\lambda > 0$ and any $u \in \mathbb{K}$.

A *weak conic Minkowski norm* on $\mathbb{E}$ is a map F from a conic domain $\mathbb{K}$ in $\mathbb{E}$ into $[0, \infty[$ with the following properties:

(i) $\forall u \in \mathbb{K} \setminus \{0\},\ F(u) > 0$;
(ii) $\forall \lambda > 0, \forall u \in \mathbb{K},\ F(\lambda u) = \lambda F(u)$;

(iii) F is at least of class C^2 on $\mathbb{K}\setminus\{0\}$ so that the Hessian of $\dfrac{1}{2}F^2$ is well defined.

Note that if 0 belongs to $\mathbb{K}$, then $\mathbb{K} = \mathbb{E}$. Now if Σ is a hypersurface of $\mathbb{E}$ which does not contain 0, then $\mathbb{K} = \{\lambda v,\ v \in \Sigma, \lambda > 0\}$ is a conic domain in $\mathbb{E}$.

If g denotes the Hessian of $\frac{1}{2}F^2$, it is defined by

$$g_u(v, w) = \frac{\partial^2}{\partial s \partial t}\left(\frac{1}{2}F^2(u + sv + tw)\right)_{|s=t=0} \quad \text{for any } u \in \mathbb{K} \text{ and } v, w \text{ in } \mathbb{E}.$$

Therefore g_u is a positive definite bilinear map on $\mathbb{E}$. In particular, g is a (weak) Riemannian metric on the Banach manifold $\mathbb{K}$ and each g_u defines an inner product on $\mathbb{E}$.

From the properties of F, we have the following properties:

- $\forall\lambda > 0,\ g_{\lambda u} = g_u$
- $g_u(u, w) = \dfrac{\partial}{\partial s}\left(\dfrac{1}{2}F^2(u + sw)\right)_{s=0}$.

Remark 1. In finite-dimensional Finsler geometry, the quantity $\sqrt{g_u(u, u)}$ is classically denoted by $||u||$. **This notation as a "norm" is not really appropriate** because, even if $\mathbb{K} = \mathbb{E}$, the definition does not imply that a conic Minkowski norm F is a norm (in the usual sense) on $\mathbb{E}$, since in general we will have $F(u) \neq F(-u)$ (cf. Example 5 in the next subsection). **However, in this paper, we will use this abusive notation.**

As in finite-dimensional Finsler geometry, we introduce:

Definition 2. A **conic Minkowski norm on** $\mathbb{E}$ is a weak conic Minkowski norm such that its hessian g is a strong Riemannian metric on $\mathbb{K}$.

When the domain $\mathbb{K}$ contains 0, then a conic Minkowski norm will be simply called a *Minkowski norm*, even it is not a norm on $\mathbb{E}$ in the classical sense (cf. Remark 1).

If we have a conic Minkowski norm on $\mathbb{E}$, this implies that each inner product g_u on E provides $\mathbb{E}$ with a Hilbert structure. Conversely, if $\mathbb{E}$ is Hilbertizable, consider such an inner product $< \ , \ >$ on $\mathbb{E}$ and denote F its associated norm. For any conic domain in $\mathbb{K}$, the restriction F to $\mathbb{K}$ is a conic Minkowski metric. More generally, given a non-zero vector ξ in $\mathbb{E}$ such that $F(\xi) < 1$, then $\bar{F}(u) = F(u) + < \xi, u >$ in restriction to $\mathbb{K}$ is also a conic Minkowski. We will give more original examples in Section 2.2.

Remark 3. Consider $v, w \in \mathbb{K}$ such that the set $\{tv + (1 - t)w,\ \forall t \in]0, 1[\}$ is contained in $\mathbb{K}$. Then we have the strict triangular inequality $||v+w|| \leq ||v||+||w||$ with equality if and only if $v = \lambda w$. In particular, if $\mathbb{K}$ is convex then F is a convex map. This situation occurs in particular for any Minkowski norm.

2.2. Conic sub-Hilbert–Finsler structure

Consider a Banach bundle $\pi : E \to M$. The fiber over x will be denoted by E_x, and the zero section of E will be denoted by 0_E. An open submanifold $\mathcal{K}$ of E will be called a *conic domain* if the restriction of π to $\mathcal{K}$ is a fibered manifold on M whose typical fiber $\mathbb{K}$ is a convex conic domain in $\mathbb{E}$. For example, consider a hypersurface Σ in E such that Σ does not intersect the zero section 0_E and $\pi_{|\Sigma}$ is a surjective fibration. Then $\mathcal{K} = \{(x, \lambda u), (x, u) \in \Sigma,\ \lambda > 0\}$ is a conic domain in E.

Definition 4. A **conic Hilbert–Finsler metric** on a bundle $\pi : E \to M$ is a continuous map $\mathcal{F}$ from a conic domain $\mathcal{K}$ of E in $[0, \infty[$ which is smooth on $\mathcal{K} \setminus \{0_E\}$ and such that the restriction of $\mathcal{F}_x$ to $\mathcal{K}_x$ is a conic Minkowski norm.

If VE denotes the vertical bundle of E, the Hessian of $\dfrac{1}{2}\mathcal{F}_x^2$ gives rise to a smooth field g of symmetric bilinear forms on $\mathcal{K} \setminus \{0_E\}$. In fact, g is a strong Riemannian metric on VE in restriction to $\mathcal{K} \setminus \{0_E\}$

Example 5. Any strong Riemannian metric g on E is a conic Hilbert–Finsler metric on E with $\mathcal{F}(x, u) = \sqrt{g_x(u, u)}$ and whose domain is $\mathcal{K} = E$. Moreover, assume that there exists a global section X of E such that $g(X, X) < 1$. Then $\mathcal{F}_x(x, u) = \sqrt{g_x(u, u)} - g_x(X, u)$ is also a conic Hilbert–Finsler metric whose domain is E which is not a norm in restriction to each fiber.

Example 6. A Hilbert–Finsler metric on E is a conic Hilbert–Finsler on $\mathcal{K} = E$ such that $\mathcal{F}_x$ is Minkowski norm on E_x for any $x \in M$. Thus it is a particular case of conic Hilbert–Finsler metric.

Example 7. Consider a Hilbert space $\mathbb{H}$ and denote its inner product by $\langle\ ,\ \rangle$. On $E = M \times \mathbb{H}$, we get a strong Riemannian metric $g_x(u, v) = \langle u, v\rangle$. Fix some non zero $\xi \in \mathbb{H}$. For $0 < \alpha < \beta < 1$ we consider

$$\mathcal{K} = \left\{ (x, u) \in M \times \mathbb{H} \setminus \{0\}\ : \alpha < \frac{|\langle \xi, u\rangle|}{\langle \xi, \xi\rangle^{1/2} \langle u, u\rangle^{1/2}} < \beta \right\}.$$

Then $\mathcal{F}(x, u) = \sqrt{g_x(u, u)}$ in restriction to $\mathcal{K}$ is a conic Hilbert–Finsler metric.

For more general examples in finite dimension, the reader can see [5]. It is clear that these examples can be also defined in our context.

An *anchored bundle on a Banach manifold M* is the data (M, E, ρ) of a Banach vector bundle $\pi : E \to M$ over M provided with a bundle morphism $\rho : E \to TM$ called the *anchor*. In this case, we get a weak distribution $\mathcal{D} = \rho(E)$ on M which is not closed in general.

Definition 8. A **conic sub-Hilbert–Finsler structure** on M is the data of an anchored bundle (E, M, ρ) and a conic Hilbert–Finsler metric $\mathcal{F}$ on the bundle $E \to M$ whose domain is a fibered open sub-manifold $\mathcal{K}$ of E. When $\mathcal{K} = E$, such a structure is called a **sub-Hilbert–Finsler structure**.

3. Extremals and local geodesics for admissible curves in a sub-Hilbert–Finsler structure

3.1. Set of admissible curves for an anchored bundle

Consider an anchored bundle on a Banach manifold (M, E, ρ) and denote the associated weak distribution on M by $\mathcal{D} = \rho(E)$.

Definition 9.

1. A curve $c : I = [a, b] \to E$ is called *regulated* if, for any $t \in [a, b[$, the limits $\lim\limits_{s \to t^+} c(s)$ and, for $t \in]a, b]$, $\lim\limits_{s \to t^-} c(s)$ exists.
2. A regulated curve $c : [a, b] \to E$ is called *admissible* (cf. [7]) if we have:
$$\frac{d}{dt}(\pi \circ c) = \rho(c) \ \ a.e.$$
3. Any continuous curve $\gamma : [a, b] \to M$ is called *horizontal*, if there exists an admissible curve $c : [a, b] \to E$ such that $\pi \circ c = \gamma$ and $\dot{\gamma} = \rho(c)$ $a.e.$. In this case, c is called a lift of γ.

According to [7], recall that the set $\mathcal{A}(E)$ of regulated admissible curves defined on $[0, 1]$[1] has a structure of Banach manifold and the set $\mathcal{A}_x(E)$ of admissible curves $c \in \mathcal{A}(E)$ such that $\pi \circ c(0) = x$ is a Banach sub-manifold of $\mathcal{A}(E)$. Moreover, the map $\mathrm{End}_x : \mathcal{A}_x(E) \to M$ defined by $\mathrm{End}_x(c) = \pi \circ c(1)$ is smooth.

Fix some conic domain $\mathcal{K}$ in E. A curve $c : [a, b] \to E$ is called $\mathcal{K}$-*admissible* if c is an admissible curve such that $c([a, b])$ is contained in $\mathcal{K}$. A curve $\gamma : [a, b] \to M$ is called $\mathcal{K}$-horizontal if there exists a $\mathcal{K}$-admissible curve c such that $\gamma = \pi \circ c$. Since $\mathcal{K}$ is an open set in E, it follows that the set $\mathcal{A}(\mathcal{K})$ of $\mathcal{K}$-admissible curves is an open Banach submanifold of $\mathcal{A}(E)$; the set $\mathcal{A}_x(\mathcal{K})$ of $\mathcal{K}$-admissible curves whose origin is in the fiber $\mathcal{K}_x$ is also a Banach manifold.

The $\mathcal{K}$-orbit of x is the set $\mathcal{O}_{\mathcal{K}}(x)$ of points $y \in M$ such that there exists a $\mathcal{K}$-admissible horizontal curve $\gamma : [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

3.2. Energy, length and semi-distance

We consider a conic sub-Hilbert Finsler metric $\mathcal{F}$ on an anchored bundle (E, M, ρ) defined on a conic domain $\mathcal{K}$.

Definition 10.

1. Let $c : [a, b] \to \mathcal{K}$ be a regulated $\mathcal{K}$-admissible curve.
   (i) The **energy** of c is $\mathcal{E}(c) = \displaystyle\int_a^b \frac{1}{2} \mathcal{F}^2(c(t))dt$.
   (ii) The **length** of c is $L(c) = \displaystyle\int_a^b \mathcal{F}(c(t))dt$

[1] Note that any admissible curve $c : [a, b] \to E$ can be reparametrized as an admissible curve $\tilde{c}$ defined on $[0, 1]$.

2. Let $\gamma : [a,b] \to M$ be a $\mathcal{K}$-horizontal curve.
   (i) The **energy** of γ is $\mathcal{E}(\gamma) = \inf\{\mathcal{E}(c), c\ \ \mathcal{K}\text{-admissible lift of } \gamma\}$.
   (ii) The **length** of γ is $L(\gamma) = \inf\{L(c), c\ \ \mathcal{K}\text{-admissible lift of } \gamma\}$.

Note that, if $\gamma : [a,b] \to M$ is a $\mathcal{K}$-horizontal curve, the quantity $F_{\gamma(t)}(\dot{\gamma}(t))$ is well defined outside a countable subset of $[a,b]$. Given any $\mathcal{K}$-lift $c : [a,b] \to \mathcal{K}$, since we have $F_{\gamma(t)}(\dot{\gamma}(t)) \leq \mathcal{F}_{c(t)}(c(t))$ outside a countable of subset of $[a,b]$, the integrals $\displaystyle\int_a^b F_{\gamma(t)}(\dot{\gamma}(t))dt$ and $\displaystyle\frac{1}{2}\int_a^b F^2_{\gamma(t)}(\dot{\gamma}(t))dt$ are well defined. However, there is no reason that this value is exactly $L(c)$ and $\mathcal{E}(c)$ respectively for some $\mathcal{K}$-lift $c : [a,b] \to \mathcal{K}$.
If x and y belong to the same $\mathcal{K}$-orbit, we define a *pseudo distance* $d_{\mathcal{F}}$ by:

$$d_{\mathcal{F}}(x,y) = \inf\{L(c),\ c : [a,b] \to E \text{ is } \mathcal{K}\text{-admissible},\ \pi \circ c(a) = x,\ \pi \circ c(b) = y\};$$

Otherwise, we set $d_{\mathcal{F}}(x,y) = \infty$.

From this definition, we always have the following properties:

- $d_{\mathcal{F}}(x,y) \geq 0$ and $d_{\mathcal{F}}(x,y) = 0$ if $x = y$;
- the triangular inequality $: d_{\mathcal{F}}(x,y) \leq d_{\mathcal{F}}(x,z) + d_{\mathcal{F}}(z,y)$;
- y belongs to $\mathcal{O}_{\mathcal{K}}(x)$ if and only if $d_{\mathcal{F}}(x,y) < \infty$.

In general, $d_{\mathcal{F}}(x,y) = 0$ does not imply $x = y$.
Now, according to [1], we introduce

Definition 11.

1. A $\mathcal{K}$-admissible curve $c : [a,b] \to E$ is called a **local minimizing geodesic** if, for any $t_0 \in [a,b[$, there exists $t_1 > t_0$ with $t_1 - t_0$ small enough, there exists a neighborhood U of $\pi \circ c([t_0,t_1])$ such for any $\mathcal{K}$-admissible $c' : [t_0,t_1] \to \mathcal{K}_U$ with $\pi \circ c'(t_i) = \pi \circ c(t_i)$, $i = 0,1$, we have $L(c_{|[t_0,t_1]}) \leq L(c')$.
2. A $\mathcal{K}$-admissible curve $c : [a,b] \to E$ is called a **geodesic** if for $t_0, t_1 \in [a,b]$ closed enough, we have
$$L(c_{|[t_0,t_1]}) = d_{\mathcal{F}}(\pi \circ c(t_0), \pi \circ c(t_1)).$$
3. A $\mathcal{K}$-admissible curve $c : [a,b] \to E$ is called a **minimizing geodesic** if we have
$$L(c) = d_{\mathcal{F}}(\pi \circ c(a), \pi \circ c(b)).$$

As classically in Finsler geometry, a curve is a minimizing geodesic if and only if it is a minimum of the map $\mathcal{E} : \mathcal{A}_x(\mathcal{K}) \to \mathbb{R}$ with the constraint $\mathrm{End}_x(c) = y$ for some $y \in M$ such that $\mathrm{End}_x^{-1}(y) \cap \mathcal{A}_x(\mathcal{K}) \neq \emptyset$.

Finally, looking for geodesics is a classic problem of minimum with constraint on a Banach manifold as previously described. Assume that such a minimum c exists. A necessary condition is that the differential of the map $\mathfrak{E} = (\mathcal{E}, \mathrm{End}_x) : \mathcal{A}_x(\mathcal{K}) \to \mathbb{R} \times T_{\pi \circ c(1)}M$ is not onto at the point c. More generally, such a point where $T_c\mathfrak{E}$ is not onto is called an **extremal** of $\mathcal{A}_x(\mathcal{K})$. Following the terminology introduced in [1], we have three types of extremals:

(i) The range of the differential $T_c\mathfrak{E}$ is a dense subspace of $\mathbb{R}\times T_{\pi\circ c(1)}M$: c is called an **elusive extremal**. Note that this situation only occurs when M is infinite-dimensional;

(ii) The range of the differential $T_c\mathfrak{E}$ is a closed codimension 1 subspace of $\mathbb{R}\times T_{\pi\circ c(1)}M$: c is called **a normal extremal**;

(iii) the range of the differential $T_c\mathfrak{E}$ is a subspace whose closure is of codimension at least 2 of $\mathbb{R}\times T_{\pi\circ c(1)}M$: c is called an **abnormal extremal**.

If $T_c^*\mathfrak{E}:\mathbb{R}\times T^*_{\pi\circ c(1)}M\to T_c^*\mathcal{A}_x(E)$ is the adjoint map of $T_c\mathfrak{E}$, as in finite dimension, the case (ii) and (iii) are respectively equivalent to

(ii′) There exists $\xi\neq 0$ in $T^*_{\pi\circ c(1)}M$ such that $T_c^*\,\mathrm{End}(\xi)=d_c\mathcal{E}$ where $T_c^*\,\mathrm{End}$ is the adjoint map of $T_c\,\mathrm{End}$;

(iii′) There exists $\xi\neq 0$ in $T^*_{\pi\circ c(1)}M$ such that $T_c^*\,\mathrm{End}(\xi)=0$ which means that in fact c is a singular point of End and so this situation does not depend on the map $\mathcal{E}$.

Note that this co-adjoint version does not work in case (i) since, in this situation, the map $T_c^*\mathfrak{E}$ is injective, but from this view point, there is no difference between the case of an elusive extremal and the case where the range of the differential $T_c\mathfrak{E}$ is equal to $\mathbb{R}\times T_{\pi\circ c(1)}M$.

3.3. Hamiltonian characterization of normal extremal

Consider a conic Hilbert–Finsler metric $\mathcal{F}$ on an anchored bundle (E,M,ρ) defined on a conic domain $\mathcal{K}$. We denote simply $\pi:\mathcal{K}\to M$ the restriction of π to the open manifold $\mathcal{K}$. After restriction if necessary, we assume that $\mathcal{K}$ does not meet the zero section of E. The restriction $\mathbb{L}\mathcal{F}$ of the differential $d(\frac{1}{2}\mathcal{F})$ to the vertical bundle of E defines a local diffeomorphism around any point $(x,u)\in\mathcal{K}$ into the dual bundle E^* of E. Since $\mathbb{L}\mathcal{F}$ is an injective map, it follows that $\mathbb{L}\mathcal{F}$ is an injective (weak) immersion from $\mathcal{K}$ into E^*.

Consider the pull-back $\mathcal{K}\times_M T^*M$ of T^*M over $\mathcal{K}$. We denote by $\tilde{\pi}:\mathcal{K}\times_M T^*M\to T^*M$ the canonical bundle morphism over $\pi:\mathcal{K}\to M$ and by $\tilde{q}$ the natural projection of $\mathcal{K}\times_M T^*M$ on $\mathcal{K}$. If $\rho^*:T^*M\to E^*$ is the adjoint map of ρ, we set

$$\tilde{\mathcal{K}}=\{(x,u,\xi)\in\mathcal{K}\times_M T^*M\ :\ \rho^*(x,\xi)=\mathbb{L}\mathcal{F}(x,\xi)\}\,.$$

With these notations, we have at first:

Proposition 12. *The set $\tilde{\mathcal{K}}$ is a Banach submanifold of $\mathcal{K}\times_M T^*M$ modeled on $\mathbb{M}^*\times\mathbb{M}$. The map $\tilde{\kappa}:T^*M\to\tilde{\mathcal{K}}$ which maps (x,ξ) to $(x,\mathbb{L}\mathcal{F}^{-1}(x,\rho_x^*\xi),\xi)$ is a diffeomorphism from T^*M onto $\tilde{\mathcal{K}}$. Let Ω be the (weak) symplectic canonical form on T^*M. Then there exists a symplectic form $\tilde{\Omega}$ on $\tilde{\mathcal{K}}$ such that $\tilde{\kappa}^*(\Omega)=\tilde{\Omega}$.*

On $\mathcal{K}\times_M T^*M$, consider the Hamiltonian

$$H(x,u,\xi)=\langle\xi,\rho_x(u)\rangle-\frac{1}{2}\mathcal{F}^2(x,u).$$

The link between the Hamiltonian H and the normal extremals is given by the following result:

Theorem 13. *Let $\tilde{H}$ be the restriction of H to $\tilde{\mathcal{K}}$. Then the Hamiltonian field $X_{\tilde{H}}$ associated to $\tilde{H}$ is well defined (relatively to $\tilde{\Omega}$) and a $\mathcal{K}$-admissible curve $c : [0,1] \to \mathcal{K}$ is a normal extremal if and only if there exists a curve $\tilde{c} : [0,1] \to \tilde{\mathcal{K}}$ which is an integral curve of $X_{\tilde{H}}$ and such that $\tilde{q} \circ \tilde{c} = c$. In particular, any normal extremal is smooth. Moreover any normal extremal $c : [0,1] \to E$ is locally minimizing.*

The proof of this result is an adaptation, step by step, of the proof of the corresponding result of Theorem 7 in [1]. A complete proof can be found in [8].

References

[1] S. Arguillère: *Sub-Riemannian Geometry and Geodesics in Banach Manifolds*, arXiv: 1601.00827v1(2016).

[2] R.L. Bryant: *Some remarks on Finsler manifolds with constant flag curvature*, Houston J. Math. **28** (2) (2002) 22–262.

[3] E. Grong, I. Markina, A. Vasil'ev: *Sub-Riemannian geometry on infinite-dimensional manifolds*, arXiv preprint arXiv:1201.2251, (2012) published in Journal of Geometric Analysis, **25**, No. 4 (2015) 2474–2515.

[4] D. Hrimiuc, L. Popescu: *Geodesic of sub-Finslerian Geometry*, Proc. Conf. Prague (2005).

[5] M.A. Javaloyes and M. Sanchez, *On the definition and examples of Finsler metrics*, arXiv. 1111.5066v1 (2011).

[6] R. Montgomery: *A tour of subriemannian geometries, their geodesics and applications*, vol **91** of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, (2002).

[7] F. Pelletier: *Banach structure on the set of regulated admissible paths of a Banach anchored bundle and singularities of endpoint map*, Preprint LAMA (Mars 2017).

[8] F. Pelletier: *Conic sub Hilbert–Finsler structure on a Banach manifold and normal extremals*, Preprint LAMA (June 2017).

[9] F. Pelletier, L. Valère Bouche: *Abnormality of trajectory in sub-Riemannian structure*, Geometry in Nonlinear Control and Differential Inclusions, Banach Center Publ. **32**, Warsaw, (1995), 301–317.

F. Pelletier
UMR 5127 du CNRS & Université de Savoie Mont Blanc
Laboratoire de Mathématique, Campus scientifique
F-73376 Le Bourget-du-Lac Cedex, France
e-mail: fernand.pelletier@univ-smb.fr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 245–250

On Spherically Symmetric Finsler Metrics

Nasrin Sadeghzadeh

Abstract. In this paper, we study spherically symmetric Finsler metrics in R^n. We find equations that characterize the metrics of R-quadratic and Ricci quadratic types.

Mathematics Subject Classification (2010). 53B40, 53C60.

Keywords. Spherically symmetric Finsler metric in R^n, R-quadratic Finsler metric, Ricci quadratic Finsler metric.

1. Introduction

Spherically symmetric metrics form a rich and important class of metrics. In Riemannian geometry these special spaces have been deeply studied by many authors for example, [6] and [14]. The base of general relativity is (pseudo-) Riemannian geometry, it is natural to consider its generalizations based on Finsler geometry. In fact Finsler geometry has applications in physics, too [1]. These metrics are considered in physics. In fact, symmetries of the background geometry of space-time are associated to different physical situations.

Similarly with the definition in general relativity, a spherically symmetric Finsler metric is invariant under any rotations in R^n. In other words, the vector fields generated by rotations are the Killing fields of the Finsler metric. From calculation point of view the Finsler metrics with certain symmetry would greatly simplify the computation. Recently many papers have been published investigating the properties of these metrics, for example [16, 17] and [5].

Riemann curvature is a central concept in Riemannian geometry and was introduced by Riemann in 1854. Berwald generalized it to Finsler metrics. A Finsler metric is said to be R-quadratic if its Riemann curvature is quadratic [4]. R-quadratic metrics were first introduced by Báscó and Matsumoto [2]. They form a rich class in Finsler geometry. There are many interesting works related to this subject [12, 15]. In this paper we are going to study R-quadratic spherically symmetric Finsler metrics in R^n. The necessary and sufficient conditions which the metrics be R-quadratic are considered.

2. Preliminaries

Let F be a Finsler metric defined on a convex domain $\Omega \subset R^n$. F is called spherically symmetric if it is invariant under any rotations in R^n. According to the equation of Killing fields, there exists a positive function ϕ depending on two variables so that $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ where x is a point in the domain Ω, y is a tangent vector at the point x and $\langle\ ,\ \rangle$ and $|\ .\ |$ are standard inner product and norm in Euclidean space. One can see the details in [16]. F has the expression $F = u\phi(r, s)$ where $r = |x|$, $u = |y|$, $v = \langle x, y\rangle$, $s = \frac{\langle x,y\rangle}{|y|}$. Projectively equivalent Finsler metrics on a manifold, namely, geodesics are same up to a parameterization are studied in projective Finsler geometry. A spray on a Finsler manifold is locally defined as

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where G^i is called the Geodesic coefficient of the spray. In local coordinate system, a Finsler space F is projective to another Finsler space $\overline{F}$ if and only if there exists a one-positive homogeneous scalar field P on M satisfying

$$G^i = \overline{G}^i + Py^i,$$

where G^i and $\overline{G}^i$ are the Geodesic coefficients of F and $\overline{F}$ [13].

The Riemann curvature R^i_k of G is defined by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k}y^j - \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} 2\frac{\partial G^i}{\partial y^j} - \frac{\partial G^j}{\partial y^k}. \tag{1}$$

The Riemann curvatures of two projectively related Finsler metrics F and $\overline{F}$ are related by the following equation [13]

$$R^i_k = \overline{R}^i_k + E\delta^i_k + \tau_k y^i, \tag{2}$$

where $E = P^2 - P_{|k}y^K$ and $\tau_k = 3(P_{|k} - PP_{.k}) + E_{.k}$, $E_{.k} = \frac{\partial E}{\partial y^k}$ and $P_{|k}$ is the horizontal derivative of P with respect to $\overline{F}$.

With the Riemann curvature, we define the flag curvature $K = K(P, y)$ by

$$K(P, y) := \frac{g_{ij}R^i_k u^j u^k}{(g_{il}y^i y^l)(g_{jk}u^j u^k) - [g_{ij}y^i u^j]^2},$$

where $y = y^i \frac{\partial}{\partial y^i}$ and $u = u^j \frac{\partial}{\partial y^j}$ with $P = \mathrm{span}\{y, u\}$.

A Finsler metric is of scalar flag curvature, $K = K(x, y)$, if and only if [13]

$$R^i_k = K\{F^2\delta^i_k - FF_{y^k}y^i\}.$$

A Finsler metric is said to be R-quadratic if its Riemann curvature is quadratic in $y \in T_xM$ [12]. There are many non-Riemannian R-quadratic Finsler metrics. For example, all Berwald metrics are R-quadratic. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of Bácsó–Matsumoto [2]. The notion of R-quadratic Finsler metrics was introduced by Z. Shen [12].

In the next section we try to calculate the Riemann curvature of the spherically symmetric Finsler metric and find the condition(s) that the metric is R-quadratic.

Ricci-quadratic are the weaker notions than R-quadratic metrics. In the following we obtain some theorems that characterizes Ricci-quadratic spherically symmetric Finsler metrics in R^n.

3. R-quadratic Spherically Symmetric Finsler Metrics in R^n

In this section we study R-quadratic spherically symmetric Finsler metric in R^n. First we calculate its Riemann curvature.

Lemma 1. *The Riemann curvature of every spherically symmetric Finsler metric in R^n is as follows*

$$R^i_k = Ax^ix_k + Bx^iy_k + Hx_ky^i + Fy_ky^i + u^2\eta\delta^i_k, \tag{3}$$

where

$$A = -\frac{u}{s}B = u^2\alpha, \quad H = uh = u(t - Q_s), \quad \eta = 2Q + e, \quad F = -sh - \eta \tag{4}$$

such that

$$\begin{aligned}
\alpha &:= \left\{\frac{2}{r}Q_r - \frac{s}{r}Q_{rs} - Q_{ss} + (r^2 - s^2)(2QQ_{ss} - Q_s^2) - 2sQQ_s + 4Q^2\right\},\\
e &= p^2 - \frac{s}{r}p_r - p_s + 2sQp + 2(r^2 - s^2)Qp_s,\\
t &= 3(\frac{1}{r}p_r - pp_s - (r^2 - s^2)Q_sp_s - sQ_sp) + e_s.
\end{aligned}$$

Proof. In [16] it is shown that the geodesic coefficient of the spherically symmetric Finsler metric is as follows

$$G^i = upy^i + u^2Qx^i, \tag{5}$$

where

$$p = -\frac{1}{\phi}(s\phi + (r^2 - s^2)\phi_s)Q + \frac{1}{2r\phi}(s\phi_r + r\phi_s), \tag{6}$$

and

$$Q = \frac{1}{2r}\frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}. \tag{7}$$

Putting $\overline{G}^i = u^2Qx^i$ and using (2) one gets after some calculations

$$R^i_k = \overline{R}^i_k + E\delta^i_k + \tau_ky^i,$$

where

$$E = u^2e,$$

and

$$\tau_k = \tau_1x_k + \tau_2y_k,$$

and we have

$$e = p^2 - \frac{s}{r}p_r - p_s + 2sQp + 2(r^2 - s^2)Qp_s, \tag{8}$$

$$\tau_1 = u.t = 3u(\frac{1}{r}p_r - pp_s - (r^2 - s^2)Q_sp_s - sQ_sp) + ue_s, \tag{9}$$

$$\begin{aligned}\tau_2 = 3\big(&p_s - p^2 + spp_s + s^2Q_sp - 2sQp \\ &+ s(r^2 - s^2)Q_sp_s - 2(r^2 - s^2)Qp_s\big) + 2e - se_s.\end{aligned} \tag{10}$$

One can easily check that

$$\tau_2 + st = -e. \tag{11}$$

Using (1) one can calculate $\overline{R}^i_k$ as follows

$$\overline{R}^i_k = Ax^ix_k + Bx^iy_k + Cy^ix_k + Dy^iy_k + G\delta^i_k, \tag{12}$$

where

$$A = u^2\alpha, \quad B = -v\alpha, \quad C = -uQ_s, \quad D = sQ_s - 2Q, \quad G = 2u^2Q, \tag{13}$$

and

$$\alpha := \left\{ \frac{2}{r}Q_r - \frac{s}{r}Q_{rs} - Q_{ss} + (r^2 - s^2)(2QQ_{ss} - Q_s^2) - 2sQQ_s + 4Q^2 \right\}. \tag{14}$$

Then the Riemann curvature tensor of the metric can be calculated as stated as (3). $\square$

Now we calculate $R^i_{k.j.l.p} = \frac{\partial^3 R^i_k}{\partial y^j \partial y^l \partial y^p}$ to investigate the condition(s) for the metrics to be R-quadratic.

Theorem 1. *Every spherically symmetric Finsler metric in R^n is R-quadratic if and only if there are the functions $\alpha(r)$, $h_1(r)$ and $h_2(r)$ such that*

1. $\alpha = \frac{2}{r}Q_r - \frac{s}{r}Q_{rs} - Q_{ss} + (r^2 - s^2)(2QQ_{ss} - Q_s^2) - 2sQQ_s + 4Q^2 = \alpha(r)$,
2. $t_1 = \frac{1}{r}p_r - pp_s - (r^2 - s^2)Q_sp_s - sQ_sp = Q_s + h_1(r)s$,
3. $e = p^2 - \frac{s}{r}p_r - p_s + 2sQp + 2(r^2 - s^2)Qp_s = -2Q - h_1(r)s^2 + h_2(r)$.

Proof. In the previous lemma the Riemann curvature of the spherically symmetric Finsler metric $F = u\varphi(r, s)$ is calculated. Then

$$R^i_{k.j} = \alpha_{kj}x^i + \beta_{kj}y^i + {}^1\gamma_k\delta^i_j + {}^2\gamma_j\delta^i_k. \tag{15}$$

where

$$\alpha_{kj} = u\alpha_sx_kx_j + (2\alpha - s\alpha_s)x_ky_j - (\alpha + s\alpha_s)x_jy_k + \frac{s^2}{u}\alpha_sy_jy_k - v\alpha\delta_{kj}, \tag{16}$$

$$\begin{aligned}\beta_{kj} = h_sx_kx_j &+ \frac{1}{u}(h - sh_s)x_ky_j - \frac{1}{u}(h + sh_s + \eta_s)\left(x_j - \frac{s}{u}y_j\right)y_k \\ &- (sh + \eta)\delta_{kj},\end{aligned} \tag{17}$$

$${}^1\gamma_k = uhx_k - (sh + \eta)y_k, \tag{18}$$

$${}^2\gamma_j = u\eta_sx_j + (2\eta - s\eta_s)y_j. \tag{19}$$

Therefore we have

$$R^i_{k.j.l} = \alpha_{kj.l}x^i + \beta_{kj.l}y^i + \beta_{kj}\delta^i_l + {}^1\gamma_{k.l}\delta^i_j + {}^2\gamma_{j.l}\delta^i_k. \tag{20}$$

and

$$R^i_{k.j.l.p} = \alpha_{kj.l.p}x^i + \beta_{kj.l.p}y^i + \beta_{kj.p}\delta^i_l + \beta_{kj.l}\delta^i_p + {}^1\gamma_{k.l.p}\delta^i_j + {}^2\gamma_{j.l.p}\delta^i_k. \tag{21}$$

Note that $\alpha_{kj.l.p} = \frac{\partial^2\alpha_{kj}}{\partial y^l \partial y^p}$, $\beta_{kj.l.p} = \frac{\partial^2\beta_{kj}}{\partial y^l \partial y^p}$ and so on.

For every R-quadratic Finsler metric we have $R^i_k = R_j{}^i{}_{kl}(x)y^jy^l$, i.e., $R^i_{k.j.l.p} = 0$ and $R^i_{k.j.l}$ is a function of x only. Then equation (20) yields that if the metric be R-quadratic then we would have $\beta_{kj.l} = 0$, i.e., $\beta_{kj} = \beta_{kj}(x)$. Again noting (20) yields

$$\alpha_{kj.l} = \alpha_{kjl}(x), \quad {}^1\gamma_{k.l} = {}^1\gamma_{kl}(x), \quad {}^2\gamma_{k.l} = {}^2\gamma_{kl}(x)$$

From (17) and $\beta_{kj} = \beta_{kj}(x)$ one gets that $h - sh_s = 0$, i.e., there is a function $h_(r)$ such that $h = h_1(r)s$ and $h + sh_s + \eta_s = 0$, i.e., $\eta_s = -2h_1(r)s$. Then $\eta = -h_1(r)s^2 + h_2(r)$, for the scalar function $h_2(r)$. Then taking into account (4) one gets

$$t = Q_s + h_1(r)s, \quad e = -2Q - h_1(r)s^2 + h_2(r). \tag{22}$$

But from (9) and above equation one easily gets $t_1 = Q_s + h_1$. Also $\alpha_{kj.l} = \alpha_{kjl}(x)$ which noting (16) one concludes that $\alpha_s = 0$. Putting the above conditions in (3) yields

$$R^i_k = \{(\alpha(r)(\delta_{pq}x_k - \delta_{kq}x_p)x^i + (h_1x_px_k - h_2\delta_{kp}) - (h_1x_px_q - h_2\delta_{pq})\delta^i_k)\}y^py^q.$$

□

4. Ricci-quadratic spherically symmetric Finsler metrics in R^n

In this section we characterize the Ricci-quadratic spherically symmetric Finsler metrics in R^n.

Theorem 2. *Every spherically symmetric Finsler metric in R^n is Ricci-quadratic if and only if*

$$(r^2 - s^2)(s\alpha_{ss} - \alpha_s) - 4s^2\alpha_s + (n-1)[2(sQ_{ss} - Q_s) + (se_{ss} - e_s)] = 0. \tag{23}$$

Proof. Noting 3 one can easily find the Ricci curvature of every spherically symmetric Finsler metric $F = u\varphi(r,s)$ on $\Omega \subseteq R^n$ as follows

$$Ric = u^2\{(r^2 - s^2)\alpha + (n-1)(2Q + e)\} = u^2R = u^2R,$$

where $R := R(r,s) = (r^2 - s^2)\alpha + (n-1)(2Q + e)$.

One gets that the metric is Ricci-quadratic if and only if $sR_{ss} - R_s = 0$. It completes the proof. □

References

[1] M. Anastasiei, P.L. Antonelli, *Lagrangian which generate sprays in vol. Lagrange and Finsler Geometry*, Applications to Physics and Biology, edited by P.L. Antonelli and R. Miron. Kluwer Academic Publishers, FTPH 76, 1–14, 1996.

[2] S. Bacso, M. Matsumoto, *Randers spaces with the h-curvature tensor H dependent on position alone*, Publications Mathematicae Debrecen, 57, 185–192, 2000.

[3] D. Bao, C. Robles, Z. Shen, *Zermelo Navigation on Riemannian manifolds*, Journal of Differential Geometry 66, 391–449, 2004.

[4] S.S. Chern, Z. Shen, *Riemann–Finsler Geometry*, World Scientic Publishers, 2004.

[5] L. Huang, X. Mo, *On spherically symmetric Finsler metrics of scalar curvature*, Journal of Geometry and Physics, Vol. 62, issue 11, 2279–2287, 2012.

[6] S.S. Lin, C.H. Hsu, T. Makino, *On spherically symmetric solution of the relativistic Euler equation*, Journal of Differential Equations, Vol. 201, 1–24, 2004.

[7] X. Mo, L. Zhou, *The curvatures of the spherically symmetric Finsler metrics*, arXiv:1202.4543

[8] X. Mo, N.M. Solórzano, K. Tenenblatb, *On spherically symmetric Finsler metrics with vanishing Douglas curvature*, Differential Geometry and its Applications, Vol. 31, issue 6, 746–758, 2013.

[9] Y. Pinchover, J. Rubinstein, *An Introduction to Partial Differential Equations*, Cambridge University Press, Cambridge, xii+371 pp., 2005.

[10] N. Sadeghzadeh, M. Hesamfar, *On Projective Invariants of Spherically Symmetric Finsler Spaces in R^n*, International Journal of Geometric Methods in Modern Physics (IJGMMP), Vol. 12, No. 07, 15500741–155007411, 2015.

[11] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2, 2001.

[12] Z. Shen, *R-quadratic Finsler metrics*, Publication Mathematicae Debrecen, 58, 263–274, 2001.

[13] Z. Shen, *Lectures on Finsler Geometry*, World Scientific, 2001.

[14] L.L. Smalley, J.P. Krisch, *On spherically symmetric cosmology in Riemann–Cartan Spin density*, Classical and Quantum Gravity: IOP Journal, Vol. 11, 2517–2522, 1994.

[15] B. Najafi, B. Bidabad, A. Tayebi, *On R-quadratic Finsler metrics*, Iranian Journal of Science and Technology A, 31 (A4), 439–443, 2007.

[16] L. Zhou, *Spherically symmetric Finsler metrics in R^n*, Publication Mathematicae Debrecen, Vol. 80, 1–2(4), 2012.

[17] L. Zhou, *Projective spherically symmetric Finsler metrics with constant flag curvature in R^n*, Vol. 158, Issue 1, 353–364, 2012.

Nasrin Sadeghzadeh
University Of Qom
Alghadir Bld.
P.O. Box 3716146611
Qom, Iran
e-mail: nsadeghzadeh@qom.ac.ir

Part VI

Topics in Spectral Theory

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 253–264

Homogeneous Rank One Perturbations and Inverse Square Potentials

Jan Dereziński

Abstract. Following [2, 3, 5], I describe several exactly solvable families of closed operators on $L^2[0,\infty[$. Some of these families are defined by the theory of singular rank one perturbations. The remaining families are Schrödinger operators with inverse square potentials and various boundary conditions. I describe a close relationship between these families. In all of them one can observe interesting "renormalization group flows" (action of the group of dilations).

Mathematics Subject Classification (2010). 34L40, 33C10.

Keywords. Closed operators, rank one perturbations, one-dimensional Schrödinger operators, Bessel functions, renormalization group.

1. Introduction

My contribution consists of an introduction and 3 sections, each describing interesting families of exactly solvable closed operators on $L^2[0,\infty[$.

The first two sections seem at first unrelated. Only in the third section the reader will see a relationship.

Section 2 is based on [3]. It is devoted to two families of operators, $H_{m,\lambda}$ and H_0^ρ, obtained by a rank one perturbation of a certain generic self-adjoint operator. The operators can be viewed as an elementary toy model illustrating some properties of the renormalization group. Note that in this section we do not use special functions. However we use a relatively sophisticated technique to define an operator, called sometimes *singular perturbation theory* or the *Aronszajn–Donoghue method*, see, e.g., [1, 4, 9].

Section 3 is based on my joint work with Bruneau and Georgescu [2], and also with Richard [5]. It is devoted to Schrödinger operators with potentials proportional to $\frac{1}{x^2}$. Both $-\partial_x^2$ and $\frac{1}{x^2}$ are homogeneous of degree -2. With appropriate homogeneous boundary conditions, we obtain a family of operators H_m, which we call *homogeneous Schrödinger operators*. They are also homogeneous of degree -2. One can compute all basic quantities for these operators using special functions – more precisely, *Bessel-type functions* and the *Gamma function*.

The operators H_m are defined only for $\mathrm{Re}\, m > -1$. We conjecture that they cannot be extended to the left of the line $\mathrm{Re}\, m = -1$ in the sense described in our paper. This conjecture was stated in [2]. It has not been proven or disproved so far.

Finally, Section 4 is based on my joint work with Richard [5], and also on [3]. It describes more general Schrödinger operators with the inverse square potentials. They are obtained by mixing the boundary conditions. These operators in general are no longer homogeneous, because their homogeneity is (weakly) broken by their boundary condition – hence the name *almost homogeneous Schrödinger operators*. They can be organized in two families $H_{m,\kappa}$ and H_0^ν.

It turns out that there exists a close relationship between the operators from Section 4 and from Section 2: they are similar to one another. In particular, they have the same point spectrum.

Almost homogeneous Schrödinger operators in the self-adjoint case have been described in the literature before, see, e.g., [7]. However, the non-self-adjoint case seems to have been first described in [5]. A number of new exact formulas about these operators is contained in [2, 5, 10] and [3].

Let us also mention one amusing observation, which seems to be original, about self-adjoint extensions of

$$-\partial_x^2 + \Big(-\frac{1}{4} + \alpha\Big)\frac{1}{x^2}.$$

The "renormalization group" acts on the set of these extensions, as described in Table 1 after Proposition 15. Depending on $\alpha \in \mathbb{R}$, we obtain 4 "phases" of the problem. Some analogies to the condensed matter physics are suggested.

2. Toy model of renormalization group

Consider the Hilbert space $\mathcal{H} = L^2[0,\infty[$ and the operator X

$$Xf(x) := xf(x).$$

Let $m \in \mathbb{C}$, $\lambda \in \mathbb{C} \cup \{\infty\}$. Following [3], we consider a family of operators formally given by

$$H_{m,\lambda} := X + \lambda |x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|. \tag{1}$$

In the perturbation $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ we use the Dirac ket-bra notation, hopefully self-explanatory. Unfortunately, the function $x \mapsto x^{\frac{m}{2}}$ is never square integrable. Therefore, this perturbation is never an operator. It can be however understood as a quadratic form. We will see below how to interpret (1) as an operator.

If $-1 < \mathrm{Re}\, m < 0$, the perturbation $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is form bounded relatively to X, and then $H_{m,\lambda}$ can be defined by the form boundedness technique. The perturbation is formally rank one. Therefore,

$$\begin{aligned}
&(z - H_{m,\lambda})^{-1} \\
&= (z-X)^{-1} + \sum_{n=0}^{\infty} (z-X)^{-1}|x^{\frac{m}{2}}\rangle(-\lambda)^{n+1}\langle x^{\frac{m}{2}}|(z-X)^{-1}|x^{\frac{m}{2}}\rangle^{n}\langle x^{\frac{m}{2}}|(z-X)^{-1} \\
&= (z-X)^{-1} + \Big(\lambda^{-1} - \langle x^{\frac{m}{2}}|(z-X)^{-1}|x^{\frac{m}{2}}\rangle\Big)^{-1}(z-X)^{-1}|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|(z-X)^{-1}.
\end{aligned}$$

It is an easy exercise in complex analysis to compute

$$\langle x^{\frac{m}{2}}|(z-X)^{-1}|x^{\frac{m}{2}}\rangle = \int_0^\infty x^m (z-x)^{-1}\mathrm{d}x = (-z)^m \frac{\pi}{\sin \pi m}.$$

Therefore, the resolvent of $H_{m,\lambda}$ can be given in a closed form:

$$(z-H_{m,\lambda})^{-1} = (z-X)^{-1} + \Big(\lambda^{-1} - (-z)^m \frac{\pi}{\sin \pi m}\Big)^{-1}(z-X)^{-1}|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|(z-X)^{-1}.$$

The rhs of the above formula defines a function with values in bounded operators satisfying the resolvent equation for all $-1 < \operatorname{Re} m < 1$ and $\lambda \in \mathbb{C}\cup\{\infty\}$. Therefore, the method of pseudoresolvent [8] allows us to define a holomorphic family of closed operators $H_{m,\lambda}$. Note that $H_{m,0} = X$.

The case $m = 0$ is special: $H_{0,\lambda} = X$ for all λ. One can however introduce another holomorphic family of operators H_0^ρ for any $\rho \in \mathbb{C}\cup\{\infty\}$ by

$$(z - H_0^\rho)^{-1} = (z-X)^{-1} - \big(\rho + \ln(-z)\big)^{-1}(z-X)^{-1}|x^0\rangle\langle x^0|(z-X)^{-1}.$$

In particular, $H_0^\infty = X$.

Let $\mathbb{R} \ni \tau \mapsto U_\tau$ be the group of dilations on $L^2[0,\infty[$, that is

$$(U_\tau f)(x) = \mathrm{e}^{\tau/2} f(\mathrm{e}^\tau x).$$

We say that B is homogeneous of degree ν if

$$U_\tau B U_\tau^{-1} = \mathrm{e}^{\nu\tau} B.$$

E.g., X is homogeneous of degree 1 and $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is homogeneous of degree $1+m$.

The group of dilations ("the renormalization group") acts on our operators in a simple way:

$$\begin{aligned}
U_\tau H_{m,\lambda} U_\tau^{-1} &= \mathrm{e}^\tau H_{m,\mathrm{e}^{\tau m}\lambda}, \\
U_\tau H_0^\rho U_\tau^{-1} &= \mathrm{e}^\tau H_0^{\rho+\tau}.
\end{aligned}$$

The essential spectrum of $H_{m,\lambda}$ and H_0^ν is $[0,\infty[$. The point spectrum is more intricate, and is described by the following theorem:

Theorem 1.

1. $z \in \mathbb{C}\backslash[0,\infty[$ *belongs to the point spectrum of* $H_{m,\lambda}$ *iff*
$$(-z)^{-m} = \lambda \frac{\pi}{\sin \pi m}.$$
2. H_0^ρ *possesses an eigenvalue iff* $-\pi < \operatorname{Im}\rho < \pi$, *and then it is* $z = -\mathrm{e}^\rho$.

For a given pair (m, λ) all eigenvalues form a geometric sequence that lies on a logarithmic spiral, which should be viewed as a curve on the Riemann surface of the logarithm. Only its "physical sheet" gives rise to eigenvalues. For m which are not purely imaginary, only a finite piece of the spiral is on the "physical sheet" and therefore the number of eigenvalues is finite.

If m is purely imaginary, this spiral degenerates to a half-line starting at the origin.

If m is real, the spiral degenerates to a circle. But then the operator has at most one eigenvalue.

The following theorem about the number of eigenvalues of $H_{m,\lambda}$ is proven in [5]. It describes an interesting pattern of "phase transitions" when we vary the parameter m. In this theorem, we denote by $\mathrm{spec_p}(A)$ the set of eigenvalues of an operator A and by $\#X$ the number of elements of the set X.

Theorem 2. *Let* $m = m_\mathrm{r} + \mathrm{i}m_\mathrm{i} \in \mathbb{C}\backslash\{0\}$ *with* $|m_\mathrm{r}| < 1$.

(i) *Let* $m_\mathrm{r} = 0$.
   (a) *If* $\frac{\ln(|\varsigma|)}{m_\mathrm{i}} \in]-\pi, \pi[$, *then* $\#\,\mathrm{spec_p}(H_{m,\lambda}) = \infty$,
   (a) *if* $\frac{\ln(|\lambda\frac{\pi}{\sin \pi m}|)}{m_\mathrm{i}} \notin]-\pi, \pi[$ *then* $\#\,\mathrm{spec_p}(H_{m,\lambda}) = 0$.

(ii) *If* $m_\mathrm{r} \neq 0$ *and if* $N \in \mathbb{N}$ *satisfies* $N < \frac{m_\mathrm{r}^2 + m_\mathrm{i}^2}{|m_\mathrm{r}|} \leq N + 1$, *then*
$$\#\,\mathrm{spec_p}(H_{m,\lambda}) \in \{N, N+1\}.$$

3. Homogeneous Schrödinger operators

Let $\alpha \in \mathbb{C}$. Consider the differential expression
$$L_\alpha = -\partial_x^2 + \Big(-\frac{1}{4} + \alpha\Big)\frac{1}{x^2}.$$

L_α is homogeneous of degree -2. Following [2], we would like to interpret L_α as a closed operator on $L^2[0,\infty[$ homogeneous of degree -2.

L_α, and closely related operators H_m that we introduce shortly, are interesting for many reasons.

- They appear as the radial part of the Laplacian in all dimensions, in the decomposition of Aharonov–Bohm Hamiltonian, in the membranes with conical singularities, in the theory of many body systems with contact interactions and in the Efimov effect.

- They have rather subtle and rich properties illustrating various concepts of the operator theory in Hilbert spaces (e.g., the Friedrichs and Krein extensions, holomorphic families of closed operators).
- Essentially all basic objects related to H_m, such as their resolvents, spectral projections, Møller and scattering operators, can be explicitly computed.
- A number of nontrivial identities involving special functions, especially from the Bessel family, find an appealing operator-theoretical interpretation in terms of the operators H_m. E.g., the Barnes identity leads to the formula for Møller operators.

We start the Hilbert space theory of the operator L_α by defining its two naive interpretations on $L^2[0,\infty[$:

1. The minimal operator $L_\alpha^{\min}$: We start from L_α on $C_{\mathrm{c}}^\infty]0,\infty[$, and then we take its closure.
2. The maximal operator $L_\alpha^{\max}$: We consider the domain consisting of all $f \in L^2[0,\infty[$ such that $L_\alpha f \in L^2[0,\infty[$.

We will see that it is often natural to write $\alpha = m^2$. Let us describe basic properties of $L_{m^2}^{\max}$ and $L_{m^2}^{\min}$:

Theorem 3.

1. *For* $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = L_{m^2}^{\max}$.
2. *For* $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq L_{m^2}^{\max}$, *and the codimension of their domains is* 2.
3. $(L_\alpha^{\min})^* = L_{\bar{\alpha}}^{\max}$. *Hence, for* $\alpha \in \mathbb{R}$, $L_\alpha^{\min}$ *is Hermitian.*
4. $L_\alpha^{\min}$ *and* $L_\alpha^{\max}$ *are homogeneous of degree* -2.

Let ξ be a compactly supported cutoff equal 1 around 0.

Let $-1 \leq \operatorname{Re} m$. It is easy to check that $x^{\frac{1}{2}+m}\xi$ belongs to $\operatorname{Dom} L_{m^2}^{\max}$. We define the operator H_m to be the restriction of $L_{m^2}^{\max}$ to

$$\operatorname{Dom} L_{m^2}^{\min} + \mathbb{C}x^{\frac{1}{2}+m}\xi.$$

The operators H_m are in a sense more interesting than $L_{m^2}^{\max}$ and $L_{m^2}^{\min}$:

Theorem 4.

1. *For* $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = H_m = L_{m^2}^{\max}$.
2. *For* $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq H_m \subsetneq L_{m^2}^{\max}$ *and the codimension of the domains is* 1.
3. $H_m^* = H_{\bar{m}}$. *Hence, for* $m \in]-1,\infty[$, H_m *is self-adjoint.*
4. H_m *is homogeneous of degree* -2.
5. $\operatorname{spec} H_m = [0,\infty[$.
6. $\{\operatorname{Re} m > -1\} \ni m \mapsto H_m$ *is a holomorphic family of closed operators.*

The theorem below is devoted to self-adjoint operators within the family H_m.

Theorem 5.

1. *For* $\alpha \geq 1$, $L_\alpha^{\min} = H_{\sqrt{\alpha}}$ *is essentially self-adjoint on* $C_{\mathrm{c}}^\infty]0,\infty[$.

2. *For $\alpha < 1$, $L_\alpha^{\min}$ is Hermitian but not essentially self-adjoint on $C_{\rm c}^\infty]0,\infty[$. It has deficiency indices $1, 1$.*
3. *For $0 \le \alpha < 1$, the operator $H_{\sqrt{\alpha}}$ is the Friedrichs extension and $H_{-\sqrt{\alpha}}$ is the Krein extension of $L_\alpha^{\min}$.*
4. *$H_{\frac12}$ is the Dirichlet Laplacian and $H_{-\frac12}$ is the Neumann Laplacian on a half-line.*
5. *For $\alpha < 0$, $L_\alpha^{\min}$ has no homogeneous selfadjoint extensions.*

Various objects related to H_m can be computed with help of functions from the Bessel family. Indeed, we have the following identity

$$x^{-\frac12}\Big(-\partial_x^2 + \big(-\frac14+\alpha\big)\frac{1}{x^2} \pm 1\Big)x^{\frac12} = -\partial_x^2 - \frac1x\partial_x + \big(-\frac14+\alpha\big)\frac{1}{x^2} \pm 1,$$

where the rhs defines the well-known (modified) Bessel equation.

One can compute explicitly the resolvent of H_m:

Theorem 6. *Denote by $R_m(-k^2;x,y)$ the integral kernel of the operator $(k^2 + H_m)^{-1}$. Then for $\operatorname{Re} k > 0$ we have*

$$R_m(-k^2;x,y) = \begin{cases} \sqrt{xy} I_m(kx)K_m(ky) & \text{if } x<y, \\ \sqrt{xy} I_m(ky)K_m(kx) & \text{if } x>y, \end{cases}$$

where I_m is the modified Bessel function and K_m is the MacDonald function.

The operators H_m are similar to self-adjoint operators. Therefore, they possess the spectral projection onto any Borel subset of their spectrum $[0,\infty[$. In particular, below we give a formula for the spectral projection of H_m onto the interval $[a, b]$:

Proposition 7. *For $0 < a < b < \infty$, the integral kernel of $\mathbb{1}_{[a,b]}(H_m)$ is*

$$\mathbb{1}_{[a,b]}(H_m)(x,y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \, \mathrm{d}k,$$

where J_m is the Bessel function.

One can diagonalize the operators H_m in a natural way, using the so-called Hankel transformation $\mathcal{F}_m$, which is the operator on $L^2[0,\infty[$ given by

$$\big(\mathcal{F}_m f\big)(x) := \int_0^\infty J_m(kx)\sqrt{kx} f(x)\mathrm{d}x \,. \tag{2}$$

Theorem 8. *$\mathcal{F}_m$ is a bounded invertible involution on $L^2[0,\infty[$ diagonalizing H_m, more precisely*

$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = X^2.$$

It satisfies $\mathcal{F}_m A = -A\mathcal{F}_m$, where

$$A = \frac{1}{2\mathrm{i}}(x\partial_x + \partial_x x)$$

is the self-adjoint generator of dilations.

It turns out that the Hankel transformation can be expressed in terms of the generator of dilations. This expression, together with the Stirling formula for the asymptotics of the Gamma function, proves the boundedness of $\mathcal{F}_m$.

Theorem 9. *Set* $\mathcal{I}f(x) = x^{-1}f(x^{-1}), \quad \Xi_m(t) = \mathrm{e}^{\mathrm{i}\ln(2)t}\frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}.$

Then

$$\mathcal{F}_m = \Xi_m(A)\mathcal{I}.$$

Therefore, we have the identity

$$H_m := \Xi_m^{-1}(A)X^{-2}\Xi_m(A). \tag{3}$$

(Result obtained independently by Bruneau, Georgescu, and myself in [2], and by Richard and Pankrashkin in [10].)

The operators H_m generate 1-parameter groups of bounded operators. They possess scattering theory and one can explicitly compute their Møller (wave) operators and the scattering operator.

Theorem 10. *The Møller operators associated to the pair H_m, H_k exist and*

$$\Omega_{m,k}^{\pm} := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_m}\mathrm{e}^{-\mathrm{i}tH_k} = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\mathcal{F}_m\mathcal{F}_k = \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\frac{\Xi_k(A)}{\Xi_m(A)}.$$

The formula (3) valid for $\mathrm{Re}\, m > -1$ can be used as an alternative definition of the family H_m also beyond this domain. It defines a family of closed operators for the parameter m that belongs to

$$\{m \mid \mathrm{Re}\, m \neq -1, -2, \dots\} \cup \mathbb{R}. \tag{4}$$

Their spectrum is always equal to $[0, \infty[$ and they are analytic in the interior of (4).

In fact, $\Xi_m(A)$ is a unitary operator for all real values of m. Therefore, for $m \in \mathbb{R}$, (3) is well defined and self-adjoint.

$\Xi_m(A)$ is bounded and invertible also for all m such that $\mathrm{Re}\, m \neq -1, -2, \dots$. Therefore, formula (3) defines an operator for all such m.

One can then pose various questions:

- What happens with these operators along the lines $\mathrm{Re}\, m = -1, -2, \dots$?
- What is the meaning of these operators to the left of $\mathrm{Re} = -1$? (They are not differential operators!)

Let us describe a certain precise conjecture about the family H_m. In order to state it we need to define the concept of a holomorphic family of closed operators.

The definition (or actually a number of equivalent definitions) of a holomorphic family of bounded operators is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle, and is described, e.g., in [8], see also [6].

Suppose that Θ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a holomorphic family of closed operators if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0, a Banach space $\mathcal{K}$ and a holomorphic family of injective bounded operators $\Theta_0 \ni z \mapsto B(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\operatorname{Ran} B(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)B(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 11. *Suppose that $\{H(z)\}_{z\in\Theta}$ is a function whose values are closed operators on $\mathcal{H}$. Suppose in addition that for any $z \in \Theta$ the resolvent set of $H(z)$ is nonempty. Then $z \mapsto H(z)$ is a holomorphic family of closed operators if and only if for any $z_0 \in \Theta$ there exists $\lambda \in \mathbb{C}$ and a neighborhood Θ_0 of z_0 such that λ belongs to the resolvent set of $H(z)$ for $z \in \Theta_0$ and $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ is holomorphic on Θ_0.*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set. We have the following conjecture (formulated as an open question in [2]), so far unproven:

Conjecture 12. *It is impossible to extend*

$$\{\operatorname{Re} m > -1\} \ni m \mapsto H_m$$

to a holomorphic family of closed operators on a larger connected open subset of $\mathbb{C}$.

4. Almost homogeneous Schrödinger operators

For $-1 < \operatorname{Re} m < 1$ the codimension of $\operatorname{Dom}(L_{m^2}^{\min})$ in $\operatorname{Dom}(L_{m^2}^{\max})$ is two. Therefore, following [5], one can fit a 1-parameter family of closed operators in between $L_{m^2}^{\min}$ in $L_{m^2}^{\max}$, mixing the boundary condition $x^{\frac{1}{2}+m}$ and $x^{\frac{1}{2}-m}$. These operators in general are no longer homogeneous – their homogeneity is broken by the boundary condition. We will say that they are *almost homogeneous*.

More precisely, for any $\kappa \in \mathbb{C} \cup \{\infty\}$ let $H_{m,\kappa}$ be the restriction of $L_{m^2}^{\max}$ to the domain

$$\begin{aligned}\operatorname{Dom}(H_{m,\kappa}) = \big\{ f \in \operatorname{Dom}(L_{m^2}^{\max}) \mid \ &\text{for some } c \in \mathbb{C}, \\ &f(x) - c\big(x^{1/2-m} + \kappa x^{1/2+m}\big) \in \operatorname{Dom}(L_{m^2}^{\min}) \text{ around } 0\big\}, \quad \kappa \neq \infty;\end{aligned}$$

$$\begin{aligned}\operatorname{Dom}(H_{m,\infty}) = \big\{ f \in \operatorname{Dom}(L_{m^2}^{\max}) \mid \ &\text{for some } c \in \mathbb{C}, \\ &f(x) - cx^{1/2+m} \in \operatorname{Dom}(L_{m^2}^{\min}) \text{ around } 0\big\}.\end{aligned}$$

The case $m = 0$ needs a special treatment. For $\nu \in \mathbb{C} \cup \{\infty\}$, let H_0^ν be the restriction of $L_0^{\max}$ to

$$\begin{aligned}\operatorname{Dom}(H_0^\nu) = \big\{ f \in \operatorname{Dom}(L_0^{\max}) \mid \ &\text{for some } c \in \mathbb{C}, \\ &f(x) - c\big(x^{1/2} \ln x + \nu x^{1/2}\big) \in \operatorname{Dom}(L_0^{\min}) \text{ around } 0\big\}, \quad \nu \neq \infty;\end{aligned}$$

$$\mathrm{Dom}(H_0^\infty) = \big\{ f \in \mathrm{Dom}(L_0^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\ f(x) - cx^{1/2} \in \mathrm{Dom}(L_0^{\min}) \text{ around } 0 \big\}.$$

Here are the basic properties of almost homogeneous Schrödinger operators.

Proposition 13.

1. *For any* $|\operatorname{Re}(m)| < 1$, $\kappa \in \mathbb{C} \cup \{\infty\}$
$$H_{m,\kappa} = H_{-m,\kappa^{-1}}.$$
2. $H_{0,\kappa}$ *does not depend on* κ*, and these operators coincide with* H_0^∞*.*
3. *We have*
$$U_\tau H_{m,\kappa} U_{-\tau} = \mathrm{e}^{-2\tau} H_{m,\mathrm{e}^{-2\tau m}\kappa},$$
$$U_\tau H_0^\nu U_{-\tau} = \mathrm{e}^{-2\tau} H_0^{\nu+\tau}.$$
In particular, only
$$H_{m,0} = H_{-m}, \quad H_{m,\infty} = H_m, \quad H_0^\infty = H_0$$
are homogeneous.

The following proposition describes self-adjoint cases among these operators.

Proposition 14.
$$H_{m,\kappa}^* = H_{\bar{m},\bar{\kappa}} \qquad \text{and} \qquad H_0^{\nu *} = H_0^{\bar{\nu}}.$$
In particular,

(i) $H_{m,\kappa}$ *is self-adjoint for* $m \in]-1,1[$ *and* $\kappa \in \mathbb{R} \cup \{\infty\}$*, and for* $m \in \mathrm{i}\mathbb{R}$ *and* $|\kappa| = 1$*.*

(ii) H_0^ν *is self-adjoint for* $\nu \in \mathbb{R} \cup \{\infty\}$*.*

The essential spectrum of $H_{m,\kappa}$ and H_0^ν is always $[0,\infty[$. The following proposition describes the point spectrum in the self-adjoint case.

Proposition 15.

1. *If* $m \in]-1,1[$ *and* $\kappa \geq 0$ *or* $\kappa = \infty$*, then* $H_{m,\kappa}$ *has no eigenvalues.*
2. *If* $m \in]-1,1[$ *and* $\kappa < 0$*, then* $H_{m,\kappa}$ *has a single eigenvalue at* $-4\big(\frac{\Gamma(m)}{\kappa\Gamma(-m)}\big)^{\frac{1}{m}}$*.*
3. *If* $m \in \mathrm{i}\mathbb{R}$ *and* $|\kappa| = 1$*, then* $H_{m,\kappa}$ *has an infinite sequence of eigenvalues accumulating at* $-\infty$ *and* 0*. If* $m = \mathrm{i}m_{\mathrm{I}}$ *and* $\mathrm{e}^{\mathrm{i}\alpha} = \frac{\kappa\Gamma(-\mathrm{i}m_{\mathrm{I}})}{\Gamma(\mathrm{i}m_{\mathrm{I}})}$*, then these eigenvalues are* $-4\exp(-\frac{\alpha+2\pi n}{m_{\mathrm{I}}})$*,* $n \in \mathbb{Z}$*.*

It is interesting to analyze how the set of self-adjoint extensions of the Hermitian operator
$$L_\alpha^{\min} = -\partial_x^2 + \Big(-\frac{1}{4} + \alpha\Big)\frac{1}{x^2}$$
depends on the real parameter α. Self-adjoint extensions form a set isomorphic either to a point or to a circle. The "renormalization group" acts on this set by a continuous flow, as described by Proposition 13. This flow may have fixed points.

The following table describes the various "phases" of the theory of self-adjoint extensions of $L_\alpha^{\min}$. To each phase I give a name inspired by condensed matter physics. The reader does not have to take these names very seriously, however I suspect that they have some deeper meaning.

$1 \leq \alpha$	"gas"	point	Unique fixed point: Friedrichs extension = Krein extension.
$0 < \alpha < 1$	"liquid"	circle	Two fixed points: Friedrichs and Krein extension. Ren. group flows from Krein to Friedrichs. On one semicircle of non-fixed points all have one bound state; on the other all have no bound states.
$\alpha = 0$	"liquid-solid phase transition"	circle	Unique fixed point: Friedrichs extension = Krein extension. Ren. group flows from Krein to Friedrichs. Non-fixed points have one bound state.
$\alpha < 0$	"solid"	circle	No fixed points. Ren. group rotates the circle. All have infinitely many bound states.

TABLE 1

Table 1 can be represented by the picture shown in Figure 1, which is self-explanatory.

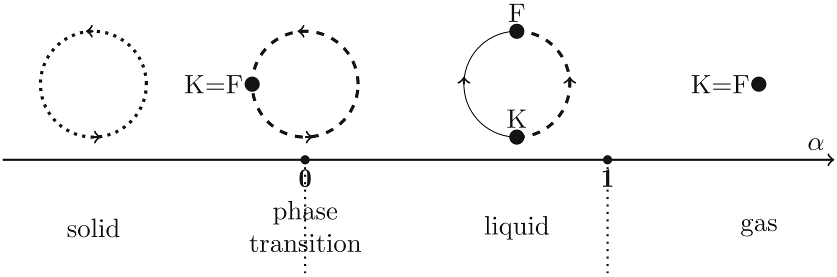


FIGURE 1

There exists a close link between almost homogeneous Schrödinger operators described in this section and the "toy model of renormalization group" described in Section 2. It turns out that the corresponding operators are similar to one another.

Define the unitary operator

$$(If)(x) := x^{-\frac{1}{4}} f(2\sqrt{x}).$$

Its inverse is

$$(I^{-1}f)(x) := \Big(\frac{y}{2}\Big)^{\frac{1}{2}} f\Big(\frac{y^2}{4}\Big).$$

Note that

$$I^{-1}XI = \frac{X^2}{4}, \quad I^{-1}AI = \frac{A}{2}.$$

We change slightly notation: the operators H_m, $H_{m,\kappa}$ and H_0^ν of this section will be denoted $\tilde{H}_m$, $\tilde{H}_{m,\kappa}$ and $\tilde{H}_0^\nu$. Recall that in (2) we introduced the Hankel transformation $\mathcal{F}_m$, which is a bounded invertible involution satisfying

$$\mathcal{F}_m \tilde{H}_m \mathcal{F}_m^{-1} = X^2,$$
$$\mathcal{F}_m A \mathcal{F}_m^{-1} = -A.$$

Recall also that in Section 2 we introduced the operators $H_{m,\lambda}$ and H_0^ρ.

The following theorem is proven in [3]:

Theorem 16.

1. *If* $\lambda \frac{\pi}{\sin(\pi m)} = \kappa \frac{\Gamma(m)}{\Gamma(-m)}$, *then the operators* $H_{m,\lambda}$ *are similar to* $\tilde{H}_{m,\kappa}$*:*

$$\mathcal{F}_m^{-1} I^{-1} H_{m,\lambda} I \mathcal{F}_m = \frac{1}{4} \tilde{H}_{m,\kappa},$$

2. *If* $\rho = -2\nu$, *then the operators* H_0^ρ *are similar to* $\tilde{H}_0^\nu$*:*

$$\mathcal{F}_m^{-1} I^{-1} H_0^\rho I \mathcal{F}_m = \frac{1}{4} \tilde{H}_0^\nu,$$

Acknowledgment

The financial support of the National Science Center, Poland, under the grant UMO-2014/15/B/ST1/00126, is gratefully acknowledged. The author is grateful to Serge Richard, Vladimir Georgescu and Laurent Bruneau for discussions and collaboration.

References

[1] S. Albeverio and P. Kurasov, *Singular Perturbations of Differential Operators*, Cambridge Univ. Press, 2000.

[2] L. Bruneau, J. Dereziński, and V. Georgescu, *Homogeneous Schrödinger operators on half-line*, Ann. Henri Poincaré **12** (2011), 547–590.

[3] Jan Dereziński, *Homogeneous Rank One Perturbations*, Ann. Henri Poincaré (2017), 3249–3268.

[4] J. Dereziński and R. Früboes, *Renormalization of Friedrichs Hamiltonians*, Rep. Math. Phys **50** (2002), 433–438.

[5] J. Dereziński and S. Richard, *On Schrödinger operators with inverse square potentials on the half-line*, Ann. Henri Poincaré **18** (2017), 869–928.

[6] J. Dereziński and M. Wrochna, *Continuous and holomorphic functions with values in closed operators*, Journ. Math. Phys **55** (2014), 083512.

[7] D.M. Gitman, I.V. Tyutin, and B.L. Voronov, *Self-adjoint extensions in quantum mechanics: general theory and applications to Schrödinger and Dirac equations with singular potentials*, Progress in Mathematical Physics, vol. 62, Birkhäuser, 2012.

[8] Tosio Kato, *Perturbation theory for linear operators. reprint of the corr. print. of the* 2*nd ed.* 1980*. Classics in Mathematics*, 1995.

[9] A. Kiselev and B. Simon, *Rank one perturbations with infinitesimal coupling*, J. Funct. Anal **130** (1995), no. 2, 345–356.

[10] K. Pankrashkin and S. Richard, *Spectral and scattering theory for the Aharonov–Bohm operators*, Rev. Math. Phys **23** (2011), no. 1, 53–81.

Jan Dereziński
Department of Mathematical Methods in Physics
Faculty of Physics
University of Warsaw
Pasteura 5
PL-02-093 Warszawa, Poland
e-mail: jan.derezinski@fuw.edu.pl

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 265–272

Generalized Unitarity Relation for Linear Scattering Systems in One Dimension

Ali Mostafazadeh

Abstract. We derive a generalized unitarity relation for an arbitrary linear scattering system that may violate unitarity, time-reversal invariance, $\mathcal{PT}$-symmetry, and transmission reciprocity.

Mathematics Subject Classification (2010). 34L25, 47A40, 78A60.

Keywords. Scattering, unitarity, reciprocity, transfer matrix, scattering matrix, time-reversal invariance, $\mathcal{PT}$-symmetry, coherent perfect absorption.

1. Introduction

The scattering phenomenon defined by a real scattering potential $v(x)$ through the time-independent Schrödinger equation,

$$-\psi''(x) + v(x)\psi(x) = k^2\psi(x), \tag{1}$$

satisfies the unitarity relation:

$$|R^{l/r}(k)|^2 + |T^{l/r}(k)|^2 = 1, \tag{2}$$

where $R^{l/r}(k)$ and $T^{l/r}(k)$ are respectively left/right reflection and transmission amplitudes. The latter determine the asymptotic behavior of the scattering solutions of (1) according to

$$\psi_l(k,x) \to \begin{cases} \mathcal{N}_+(k)\left[e^{ikx} + R^l(k)\,e^{-ikx}\right] & \text{for } x \to -\infty, \\ \mathcal{N}_+(k)T^l(k)\,e^{ikx} & \text{for } x \to +\infty, \end{cases} \tag{3}$$

$$\psi_r(k,x) \to \begin{cases} \mathcal{N}_-(k)T^r(k)\,e^{-ikx} & \text{for } x \to -\infty, \\ \mathcal{N}_-(k)\left[e^{-ikx} + R^r(k)\,e^{ikx}\right] & \text{for } x \to +\infty. \end{cases} \tag{4}$$

These respectively correspond to scattering setups where left-/right-incident waves of amplitude $\mathcal{N}_{+/-}(k)$ are scattered by the potential $v(x)$.

For a real scattering potential, one can show that [1]

$$|R^l(k)| = |R^r(k)|, \tag{5}$$

$$T^l(k) = T^r(k). \tag{6}$$

Therefore the unitarity relation takes the form

$$|R^{l/r}(k)|^2 + |T(k)|^2 = 1, \tag{7}$$

where $T(k)$ stands for the common value of $T^l(k)$ and $T^r(k)$.

Reciprocity in transmission (6) turns out to be a universal feature of all real and complex scattering potentials [2, 3]. To see this we first recall that the Wronskian of any pair of solutions $\psi_{1,2}(x)$ of (1), i.e., $W[\psi_1(x), \psi_2(x)] := \psi_1(x)\psi_2'(x) - \psi_1'(x)\psi_2(x)$, is independent of x. If we compute $W[\psi_l(x), \psi_r(x)]$ for $x \to -\infty$ and $x \to +\infty$ we respectively find $2ik/T^l(k)$ and $2ik/T^r(k)$. The fact that these must be equal to the same constant implies (6) for $k \neq 0$. This is actually the one-dimensional realization of the celebrated reciprocity theorem which is for example proven for real potentials in Ref. [4].

Unlike (6), (5) is violated by generic complex scattering potentials. A striking demonstration of this fact is the existence of unidirectionally reflectionless complex potentials [5]. These are potentials whose reflection amplitudes fulfill either $R^l(k) = 0 \neq R^r(k)$ or $R^r(k) = 0 \neq R^l(k)$ for some $k \in \mathbb{R}^+$. It turns out that these conditions are invariant under the combined action of parity and time-reversal transformation ($\mathcal{PT}$), where $\mathcal{T}\psi(x) := \psi(x)^*$ and $\mathcal{P}\psi(x) := \psi(-x)$ respectively define the parity and time-reversal transformations [6]. This in turn makes $\mathcal{PT}$-symmetric potentials[1] the principal examples of unidirectionally reflectionless potentials. This together with the interesting properties of their spectral singularities [7] have made $\mathcal{PT}$-symmetric scattering potentials a focus of intensive research activity during the past decade [8].

Among the outcomes of the research done in this subject is the discovery of the following generalization of the unitarity relation (7) for $\mathcal{PT}$-symmetric potentials [9]:

$$|T(k)|^2 \pm |R^l(k)R^r(k)| = 1. \tag{8}$$

Another curious observation is that reflection and transmission amplitudes of $\mathcal{PT}$-symmetric scattering potentials satisfy

$$|R^l(-k)| = |R^r(k)|, \qquad |T(-k)| = |T(k)|. \tag{9}$$

These were initially conjectured in [10] based on evidence provided by the study of a complexified Scarf II potential. They were subsequently proven as immediate consequences of the following identities that hold for $\mathcal{PT}$-symmetric scattering potentials [1].

$$R^{l/r}(-k) = -e^{2i\tau(k)}R^{r/l}(k), \qquad T(-k) = T(k)^*, \tag{10}$$

[1]These are potentials that satisfy $v(-x)^* = v(x)$.

where $e^{i\tau(k)} := T(k)/|T(k)|$. In view of the second of these equations, we can write the first in the form

$$R^{l/r}(-k)T(-k) + R^{r/l}(k)T(k) = 0. \tag{11}$$

The analysis leading to the proof of (10) also reveals that the reflection and transmission amplitudes of both real and $\mathcal{PT}$-symmetric scattering potentials fulfill [1]

$$R^{l/r}(k)R^{l/r}(-k) + |T(k)|^2 = 1. \tag{12}$$

It is not difficult to see that this reduces to (7) and (8) for real and $\mathcal{PT}$-symmetric potentials, respectively.

The purpose of the present article is to establish a generalization of (12) that holds for every linear scattering system, even those that are not defined by a local potential [11].

2. General scattering systems in one dimension

Consider a wave equation in $1+1$ dimensions that admits time-harmonic solutions: $e^{-i\omega t}\psi(x)$, where $\psi : \mathbb{R} \to \mathbb{C}$ solves a time-independent wave equation,

$$\mathscr{W}[\psi, x] = 0. \tag{13}$$

This equation, which may be nonlocal or even nonlinear, defines a meaningful scattering phenomenon if for $x \to \pm\infty$ its solutions tend to those of

$$-\psi''(x) = k^2\psi(x). \tag{14}$$

In other words, solutions of (13) satisfy the asymptotic boundary conditions:

$$\psi(x) \to A_-(k)e^{ikx} + B_-(k)e^{-ikx} \text{ for } x \to -\infty, \tag{15}$$

$$\psi(x) \to A_+(k)e^{ikx} + B_+(k)e^{-ikx} \text{ for } x \to +\infty, \tag{16}$$

where $A_\pm$ and $B_\pm$ are complex-valued coefficient functions. We call the 2×2 matrices $\mathbf{M}(k)$ and $\mathbf{S}(k)$ satisfying

$$\mathbf{M}(k)\begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix} = \begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix}, \tag{17}$$

$$\mathbf{S}(k)\begin{bmatrix} A_-(k) \\ B_+(k) \end{bmatrix} = \begin{bmatrix} A_+(k) \\ B_-(k) \end{bmatrix}, \tag{18}$$

the transfer and scattering matrices of the scattering system. If (13) is nonlinear, their entries, $M_{ij}(k)$ and $S_{ij}(k)$, are respectively nonlinear functions of (A_-, B_-) and (A_-, B_+). In the following we focus our attention to scattering phenomena defined by linear wave equations.[2]

Because (A_-, B_-) and (A_+, B_+) determine the behavior of the solutions $\psi(x)$ at $x = -\infty$ and $x = +\infty$, the global existence and uniqueness of the solution of the

[2] A linear wave equation is an equation of the form (13) such that the linear combinations of its solutions are also solutions of this equation.

initial-value problem defined by (13) and (15) implies that $\mathbf{M}(k)$ is an invertible matrix, i.e.,

$$\det \mathbf{M}(k) \neq 0. \tag{19}$$

Under this condition the scattering problem for the wave equation (13) is well posed. We therefore assume that it holds true. The inverse of $\mathbf{M}(k)$ allows us to specify the asymptotic expression for the solutions of (13) at $x = -\infty$ in terms of their asymptotic expression at $x = +\infty$.

Let $\psi_\pm(k,x)$ be the solutions of (13) that satisfy

$$\psi_\pm(k,x) = e^{\pm ikx} \quad \text{for} \quad x \to \pm\infty. \tag{20}$$

Then Eq. (17) implies

$$\psi_-(k,x) \to M_{22} e^{ikx} + M_{12}(k) e^{-ikx} \text{ for } x \to +\infty, \tag{21}$$

$$\psi_+(k,x) \to \frac{-M_{21}(k)e^{ikx} + M_{22}e^{-ikx}}{\det \mathbf{M}(k)} \text{ for } x \to -\infty. \tag{22}$$

$\psi_\pm$ are called the Jost solutions of the wave equation (13). Comparing (20)–(22) with (3) and (4) and using the linearity of (13), we can respectively identify $\psi_l(k,x)$ and $\psi_r(k,x)$ with $\mathcal{N}_+(k)T^l(k)\psi_+(k,x)$ and $\mathcal{N}_-(k)T^r(k)\psi_-(k,x)$. Furthermore, this identification implies

$$\begin{aligned} M_{11}(k) &= \frac{D(k)}{T^r(k)}, & M_{12}(k) &= \frac{R^r(k)}{T^r(k)}, \\ M_{21}(k) &= -\frac{R^l(k)}{T^r(k)}, & M_{22}(k) &= \frac{1}{T^r(k)}, \end{aligned} \tag{23}$$

$$\begin{aligned} R^l(k) &= -\frac{M_{21}(k)}{M_{22}(k)}, & T^l(k) &= \frac{\det \mathbf{M}(k)}{M_{22}(k)}, \\ R^r(k) &= \frac{M_{12}(k)}{M_{22}(k)}, & T^r(k) &= \frac{1}{M_{22}(k)}, \end{aligned} \tag{24}$$

where

$$D(k) := T^l(k)T^r(k) - R^l(k)R^r(k) = \frac{M_{11}(k)}{M_{22}(k)}. \tag{25}$$

We can similarly relate the entries of the scattering matrix to the reflection and transmission coefficients by enforcing (18) for the coefficient functions of the Jost solutions $\psi_\pm(k,x)$. In view of (20)–(22), this gives

$$S_{11}(k) = T^l(k), \quad S_{12}(k) = R^r(k), \quad S_{21}(k) = R^l(k), \quad S_{22}(k) = T^l(k). \tag{26}$$

In particular,

$$\det \mathbf{S}(k) = D(k). \tag{27}$$

The above-mentioned requirements on the global existence of the solutions of (13) that satisfy asymptotic boundary conditions (15), (16), and (20) restrict the wave operator $\mathscr{W}$. For example if $\mathscr{W}$ is the Schrödinger operator $-\partial_x^2 + v(x)$ for a potential $v : \mathbb{R} \to \mathbb{C}$, we can satisfy these requirements provided that $v(x)$ fulfills the Faddeev condition: $\int_{-\infty}^{\infty}(1+|x|)|v(x)|dx < \infty$, [12].

3. Generalized unitarity relation

Let us make the k-dependence of the solutions of the wave equation (13) explicit by using $\psi(k,x)$ in place of $\psi(x)$ in (15) and (16). Consider the implications of the transformations:

$$\psi(k,x) \xrightarrow{\mathcal{R}} \breve{\psi}(k,x) := (\mathcal{R}\psi)(k,x) \quad := \psi(-k,x), \tag{28}$$

$$\psi(k,x) \xrightarrow{\mathcal{P}} \widetilde{\psi}(k,x) := (\mathcal{P}\psi)(k,x) \quad := \psi(k,-x), \tag{29}$$

$$\psi(k,x) \xrightarrow{\mathcal{T}} \overline{\psi}(k,x) := (\mathcal{T}\psi)(k,x) \quad := \psi(k,x)^*, \tag{30}$$

$$\psi(k,x) \xrightarrow{\mathcal{PT}} \widetilde{\overline{\psi}}(k,x) := (\mathcal{PT}\psi)(k,x) := \psi(k,-x)^*. \tag{31}$$

It is not difficult to see that the transformed wave functions, $\breve{\psi}(k,x), \widetilde{\psi}(k,x)$, $\overline{\psi}(k,x)$, and $\widetilde{\overline{\psi}}(k,x)$ also tend to plane waves at spatial infinities. Therefore they determine scattering phenomena. By analogy to the definition of the transfer matrix $\mathbf{M}(k)$ for $\psi(k,x)$, i.e., (17), we can introduce the transfer matrices for $\breve{\psi}(k,x), \widetilde{\psi}(k,x), \overline{\psi}(k,x)$, and $\widetilde{\overline{\psi}}(k,x)$. We respectively label them by $\mathbf{M}(-k), \widetilde{\mathbf{M}}(k)$, $\overline{\mathbf{M}}(k)$, and $\widetilde{\overline{\mathbf{M}}}(k)$. In view of (28)–(30), we can show that

$$\mathbf{M}(-k) = \boldsymbol{\sigma}_1\mathbf{M}(k)\boldsymbol{\sigma}_1, \quad \widetilde{\mathbf{M}}(k) = \boldsymbol{\sigma}_1\mathbf{M}(k)^{-1}\boldsymbol{\sigma}_1, \tag{32}$$

$$\overline{\mathbf{M}}(k) = \boldsymbol{\sigma}_1\mathbf{M}(k)^*\boldsymbol{\sigma}_1, \quad \widetilde{\overline{\mathbf{M}}}(k) = \mathbf{M}(k)^{-1*}, \tag{33}$$

where $\boldsymbol{\sigma}_1 := \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ is the first Pauli matrix.

Similarly we can introduce the reflection and transmission amplitudes for $\breve{\psi}(k,x), \widetilde{\psi}(k,x)$, $\overline{\psi}(k,x)$, and $\widetilde{\overline{\psi}}(k,x)$, which by virtue of their relationship to $\mathbf{M}(-k), \widetilde{\mathbf{M}}(k)$, $\overline{\mathbf{M}}(k)$, and $\widetilde{\overline{\mathbf{M}}}(k)$ and Eqs. (32) and (33), take the form:

$$\begin{aligned} R^l(-k) &= -\frac{R^r(k)}{D(k)}, & T^l(-k) &= \frac{T^l(k)}{D(k)}, \\ R^r(-k) &= -\frac{R^l(k)}{D(k)}, & T^r(-k) &= \frac{T^r(k)}{D(k)}, \end{aligned} \tag{34}$$

$$\begin{aligned} \widetilde{R}^l(k) &= R^r(k), & \widetilde{T}^l(k) &= T^r(k), \\ \widetilde{R}^r(k) &= R^l(k), & \widetilde{T}^r(k) &= T^l(k), \end{aligned} \tag{35}$$

$$\begin{aligned} \overline{R}^l(k) &= -\frac{R^r(k)^*}{D(k)^*}, & \overline{T}^l(k) &= \frac{T^l(k)^*}{D(k)^*}, \\ \overline{R}^r(k) &= -\frac{R^l(k)^*}{D(k)^*}, & \overline{T}^r(k) &= \frac{T^r(k)^*}{D(k)^*}, \end{aligned} \tag{36}$$

$$\begin{aligned}\tilde{\tilde{R}}^{l}(k) &= -\frac{R^l(k)^*}{D(k)^*}, & \tilde{\tilde{T}}^{l}(k) &= \frac{T^r(k)^*}{D(k)^*},\\ \tilde{\tilde{R}}^{r}(k) &= -\frac{R^r(k)^*}{D(k)^*}, & \tilde{\tilde{T}}^{r}(k) &= \frac{T^l(k)^*}{D(k)^*},\end{aligned} \tag{37}$$

respectively.

Next, we invert (34) to express $R^r(k)$ and $T^r(k)$ in terms of $R^l(-k)$, $T^r(-k)$, and $D(k)$. Substituting the result in (25), we find

$$D(k)\left[T^r(-k)T^l(k) + R^l(-k)R^l(k) - 1\right] = 0. \tag{38}$$

Similarly, we can solve (34) for $R^l(k)$ and $T^l(k)$ in terms of $R^r(-k)$, $T^l(-k)$, and $D(k)$, and use (25) to establish:

$$D(k)\left[T^l(-k)T^r(k) + R^r(-k)R^r(k) - 1\right] = 0. \tag{39}$$

Equations (38) and (39) imply that whenever $D(k) \neq 0$,

$$T^{l/r}(-k)T^{r/l}(k) + R^{l/r}(-k)R^{l/r}(k) = 1. \tag{40}$$

This is a generalized unitarity relation that reduces to (12) whenever the scattering system has reciprocal transmission and $D(k) \neq 0$ for all $k \in \mathbb{R}^+$. Both of these conditions are satisfied for scattering systems determined by the Schrödinger equation for a local time-reversal invariant (real) or $\mathcal{PT}$-symmetric potential. According to the reciprocity theorem they have reciprocal transmission, and as we show in the sequel they satisfy $|D(k)| = 1$. To see this, first we note that according to (33) the transfer matrix for time-reversal-invariant and $\mathcal{PT}$-symmetric systems[3] respectively fulfill

$$\mathbf{M}(k)^* = \boldsymbol{\sigma}_1 \mathbf{M}\, \boldsymbol{\sigma}_1, \tag{41}$$

$$\mathbf{M}(k)^* = \mathbf{M}(k)^{-1}. \tag{42}$$

We can use these equations to show that

$$\mathcal{T}\text{-symmetry} \quad \Rightarrow \quad M_{11}(k)^* = M_{22}(k), \tag{43}$$

$$\mathcal{PT}\text{-symmetry} \quad \Rightarrow \quad M_{11}(k)^* = \frac{M_{22}(k)}{\det \mathbf{M}(k)}. \tag{44}$$

For time-reversal-invariant systems, Eqs. (25) and (43) imply:

$$|D(k)| = \left|\frac{M_{11}(k)}{M_{22}(k)}\right| = \left|\frac{M_{11}(k)^*}{M_{22}(k)}\right| = 1. \tag{45}$$

[3] By definition, time-reversal invariance and $\mathcal{PT}$-symmetry of a scattering system respectively mean that its reflection and transmission amplitudes, and consequently its transfer and scattering matrices are invariant under time-reversal and $\mathcal{PT}$ transformations.

In light of (25) and (44), we also find the following result for $\mathcal{PT}$-symmetric scattering systems.

$$|D(k)| = \left|\frac{M_{11}(k)}{M_{22}(k)}\right| = \left|\frac{M_{11}(k)}{\det \mathbf{M}(k)}\right| \left|\frac{\det \mathbf{M}(k)}{M_{22}(k)}\right| = \left|\frac{M_{22}(k)^*}{M_{11}(k)^*}\right| = \frac{1}{|D(k)|}, \tag{46}$$

which means $|D(k)| = 1$.

Note that the proof of the identity $|D(k)| = 1$ we have just presented does not make use of the transmission reciprocity. Therefore it holds for every scattering system possessing time-reversal invariance or $\mathcal{PT}$-symmetry. In view of (39), it implies that the reflection and transmission amplitudes of these systems fulfill (40) for all $k \in \mathbb{R}^+$.

For scattering systems that are neither time-reversal-invariant nor $\mathcal{PT}$-symmetric, there may exist values of k for which $D(k) = 0$, in which case (40) may be violated for these values of k. According to (27), these are the real and positive zeros k_0 of $\det \mathbf{S}(k)$. Clearly $\det \mathbf{S}(k_0) = 0$ means that $\mathbf{S}(k_0)$ has a vanishing eigenvalue, i.e., there are complex numbers A_{0-} and B_{0+} such that

$$\mathbf{S}(k_0) \begin{bmatrix} A_{0-} \\ B_{0+} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{47}$$

In light of (15), (16), and (18), this equation proves the existence of a solution $\psi_{\rm in}(k, x)$ of the wave equation that satisfies purely incoming asymptotic boundary conditions for $k = k_0$, i.e.,

$$\psi_{\rm in}(k_0, x) \to \begin{cases} A_{0-} e^{ik_0 x} & \text{for } x \to -\infty, \\ B_{0+} e^{-ik_0 x} & \text{for } x \to +\infty. \end{cases}$$

This solution describes a rather remarkable situation where the system absorbs a pair of incident waves traveling towards it in opposite directions. This phenomenon is called coherent perfect absorption or antilasing [13–17].

The above analysis shows that for every scattering system and $k \in \mathbb{R}^+$, either k is a wavenumber at which the system acts as a coherent perfect absorber or its reflection and transmission amplitudes satisfy the generalized unitarity relation (40).

Let us conclude by noting that the term 'generalized unitarity relation' refers to the fact that for a real scattering potential where the wave operator is a Hermitian Schrödinger operator, this relation reduces to the unitarity relation (7). This follows from the reciprocity theorem and Eqs. (34) and (36), which for time-reversal-invariant systems imply

$$R^{l/r}(-k) = R^{l/r}(k)^*, \qquad T^{l/r}(-k) = T^{l/r}(k)^*.$$

Acknowledgment

This work has been supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) in the framework of the project no: 114F357 and by Turkish Academy of Sciences (TÜBA).

References

[1] A. Mostafazadeh, Generalized unitarity and reciprocity relations for $\mathcal{PT}$-symmetric scattering potentials, J. Phys. A: Math. Theor. **47**, 505303 (2014).

[2] Z. Ahmed, Schrödinger transmission through one-dimensional complex potentials, Phys. Rev. A **64**, 042716 (2001)

[3] A. Mostafazadeh and H. Mehri-Dehnavi, Spectral singularities, biorthonormal systems and a two-parameter family of complex point interactions, J. Phys. A: Math. Theor. **42**, 125303 (2009).

[4] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Pergamon, Oxford, 1977.

[5] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, D.N. Christodoulides, Unidirectional invisibility induced by PT-symmetric periodic structures, Phys. Rev. Lett. **106**, 213901 (2011).

[6] A. Mostafazadeh, Invisibility and $\mathcal{PT}$-symmetry, Phys. Rev. A **87**, 012103 (2013).

[7] A. Mostafazadeh, Physics of Spectral Singularities, in Proceedings of XXXIII Workshop on Geometric Methods in Physics, held in Białowieża, Poland, June 29-July 5, 2014, Trends in Mathematics, pp. 145–165, Springer International Publishing, Switzerland, 2015; preprint arXiv:1412.0454.

[8] V.V. Konotop, J. Yang, and D.A. Zezyulin, Nonlinear waves in PT-symmetric systems, Rev. Mod. Phys. **88**, 035002 (2016).

[9] L. Ge, Y.D. Chong, and A.D. Stone, Conservation relations and anisotropic transmission resonances in one-dimensional $\mathcal{PT}$-symmetric photonic heterostructures, Phys. Rev. A **85** 023802 (2012).

[10] Z. Ahmed, New features of scattering from a one-dimensional non-Hermitian (complex) potential, J. Phys. A: Math. Theor. **45**, 032004 (2012).

[11] J.G. Muga, J.P. Palao, B. Navarro, and I.L. Egusquiza, Complex absorbing potentials, Phys. Rep. **395**, 357–426 (2004).

[12] R.R. D. Kemp, A singular boundary value problem for a non-self-adjoint differential opeartor, Canadian J. Math. **10**, 447–462 (1958).

[13] Y.D. Chong, L. Ge, H. Cao, and A.D. Stone, Coherent perfect absorbers: Time-reversed lasers, Phys. Rev. Lett. **105**, 053901 (2010).

[14] S. Longhi, Backward lasing yields a perfect absorber, Physics **3**, 61 (2010).

[15] S. Longhi, $\mathcal{PT}$-symmetric laser absorber, Phys. Rev. A **82**, 031801 (2010).

[16] W. Wan, Y. Chong, L. Ge, H. Noh, A.D. Stone, and H. Cao, Time-reversed lasing and interferometric control of absorption, Science **331**, 889–892 (2011).

[17] A. Mostafazadeh, Self-dual spectral singularities and coherent perfect absorbing lasers without $\mathcal{PT}$-symmetry, J. Phys. A: Math. Gen. **45**, 444024 (2012).

Ali Mostafazadeh
Departments of Mathematics and Physics
Koç University
Sarıyer 34450
Istanbul, Turkey
e-mail: amostafazadeh@ku.edu.tr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 273–278

Differential Equations on Polytopes: Laplacians and Lagrangian Manifolds, Corresponding to Semiclassical Motion

Andrei Shafarevich

Abstract. The aim of this work is to describe certain constructions and results concerning differential operators on polyhedral surfaces. In particular, we study properties of Laplacians as well as behavior of localized solutions of wave equations.

Mathematics Subject Classification (2010). Primary 37K25; Secondary 53C22, 57R45, 11P21, 11P21.

Keywords. Polyhedral surfaces, Laplacians, trace formulas, wave equations.

1. Introduction

Differential operators on polyhedral surfaces were intensively studied during last decades (see, e.g., [1] and references therein). Many papers are devoted to such topics as spectral theory, determinants, trace formulas etc. Nice properties of such operators are due to the fact that polyhedra are almost everywhere flat; from the other hand, there appear interesting effects, caused by singularities (vertices). Further we announce certain results concerning properties of Laplacians and behavior of solutions to wave equations.

2. Laplacians on polyhedra

2.1. Polyhedral surfaces

We will consider polyhedral surfaces – compact 2D oriented surfaces M, glued from a finite number of flat polygons in a usual manner. The surfaces will be not necessary embedded in $\mathbb{R}^3$; the total angles $\beta_1, \dots, \beta_M$ at the vertices can be less

The work was supported by the Russian Scientific Foundation (grant 16-11-10069).

or greater than 2π – the unique condition is the Gauss-Bonnet relation

$$\sum_{j=1}^{M}\left(1-\frac{\beta_j}{2\pi}\right)=\chi(M),$$

where χ denotes Euler characteristics.

In the last part of the paper we will consider wave equations on the simplest noncompact polyherdon – infinite pyramid.

Remark 1. Each polyhedron admits a natural complex structure. Namely, if P is a point of a face, then the natural complex coordinate is $z = x_1 + ix_2$, where (x_1, x_2) are standard Euclidean coordinate on $\mathbb{R}^2$. The same states for the points, lying on edges – one can unfold the vicinity of such a point to the plane and then introduce the same coordinate. If P is a vertex with total angle β then the vicinity of P can be unfolded to the plane angle of volume β; the natural coordinate on M near P is $\zeta = z^{2\pi/\beta}$, where $z = x_1 + ix_2$ is a standard coordinate on the plane. This complex structure, in particular, induces the smooth structure of each polyhedral surface.

Remark 2. The natural metric on a polyhedral surface has the form $ds^2 = dzd\bar{z}$ outside the vertices; near the vertex it has the form

$$ds^2=\left(\frac{\beta}{2\pi}\right)^2|\zeta|^{2\left(\frac{\beta}{2\pi}-1\right)}d\zeta d\bar{\zeta}$$

and has singularities at vertices. In particular, the wave equation in coordinates (y_1, y_2), $\zeta = y_1 + iy_2$ has the form

$$\frac{\partial^2 u}{\partial t^2}=\frac{2\pi}{\beta}(y_1^2+y_2^2)^{1-\frac{\beta}{2\pi}}\left(\frac{\partial^2 u}{\partial y_1^2}+\frac{\partial^2 u}{\partial y_2^2}\right).$$

The velocity of waves vanishes (if $\beta < 2\pi$) or becomes infinite (if $\beta > 2\pi$) at vertices; such a situation appear, in particular, when long waves meet small obstacles (islands or narrow hollows).

2.2. Definitions of Laplacians

Further we discuss properties of Laplacians and wave equations on polyhedral surfaces; in order to define the corresponding operators, one has to state boundary conditions in singular points (vertices of polyhedra). These conditions can be defined by the following natural arguments.

1. The Laplacian must be self-adjoint.
2. On the "regular" part of the surface the Laplacian must coincide with the usual one.

The formal definition has the following form. Consider the non-compact smooth Riemannian manifold

$M_0 = M\backslash\{P_1, \dots, P_M\}$, where P_j are vertices. Consider the usual Laplace–Beltrami operator $\tilde{\Delta}$ on $C_0^\infty(M_0)$ and let Δ_0 denote the closure of this operator with respect to the graph norm $\|\circ\|_\Delta$: $\|u\|_\Delta^2 = \|u\|^2 + \|\tilde{\Delta}u\|^2$, where $\|\circ\|$ denotes the L^2-norm. Clearly, Δ_0 is a symmetric operator in $L^2(M)$.

Definition 3. The Laplace operator on a polyhedral surface M is a self-adjoint extension of Δ_0.

Remark 4. The Laplacian is not unique; different operators are defined by different boundary conditions at vertices. Namely, the explicit description of the corresponding domains has the following form. For each vertex P with total angle β consider the set of functions

$$F_0^+ = 1, \quad F_0^- = \log r,$$

$$F_k^\pm = \left(\frac{2\pi|k|}{\beta}\right)^{-1/2} r^{\pm\frac{2\pi|k|}{\beta}} \mathrm{e}^{\frac{2\pi i k\theta}{\beta}}, \quad k \in \mathbb{Z}\backslash\{0\}, \quad |k| < \frac{\beta}{2\pi}.$$

Here r, θ are polar coordinates near P (r is a geodesic distance, $\theta \bmod \beta$ is angle coordinate on the unfolding). Functions u from the domain of Δ have the following asymptotics near each vertex:

$$u = \sum_k \alpha_k^+ F_k^+ + \alpha_k^- F_k^- + O(r), \quad k \in \mathbb{Z}, \quad |k| < \frac{\beta}{2\pi}.$$

Now we collect all the coefficients α_j^+ and α_j^- for all vertices and form an even-dimensional vector $\alpha = (\alpha^+, \alpha^-) \in \mathbb{C}^M \oplus \mathbb{C}^M$. Let us fix in the latter space a plane L, Lagrangian with respect to the standard skew-Hermitian form

$$[\alpha, \lambda] = \sum_{j=1}^{M} (\alpha_j^+ \bar{\lambda}_j^- - \alpha_j^- \bar{\lambda}_j^+).$$

The boundary conditions have the form $\alpha \in L$; they can be written explicitly in terms of unitary matrix U, defining L:

$$i(E+U)\alpha^- + (E-U)\alpha^+ = 0,$$

where E is the $M \times M$ unit matrix.

Remark 5. General boundary conditions match all the vertices together; sometimes it is more natural to consider local boundary conditions which deal with each vertex separately; formally it means that the plane L is a direct sum of planes, corresponding to vertices (the matrix U is formed by the corresponding diagonal blocks).

3. Spaces of harmonic functions

Now we describe the kernel of the Laplacian Δ^L, corresponding to the Lagrangian plane L.

Theorem 6. *The kernel of the operator Δ^L is isomorphic to the intersection $L \cap L_0$ where Lagrangian plane L_0 is defined by the polyhedron itself.*

Remark 7. In general position the intersection is zero, so there are no nontrivial harmonic functions.

Remark 8. The plane L_0 can be expressed in terms of the Mittag-Leffler problem, corresponding to the Riemannian surface M.

Remark 9. The kernel of the Friedrichs extension ($\alpha^- = 0$) is one-dimensional and is formed by constants.

Remark 10. For convex polyhedra in $\mathbb{R}^3$ with N vertices and Laplacians with local boundary conditions, harmonic functions can be described explicitly. Namely, in this case the polyhedron is isomorphic to the Riemann sphere; let z be a global coordinate on this sphere and let $\mathrm{e}^{i\varphi_j}$ be 1×1 unitary matrices, defining boundary conditions. Arbitrary harmonic function has the form

$$f = c_0 + \sum_{j=1}^{N} c_j \log|z - z_j|,$$

where constants c_j satisfy the following linear system of equations

$$\cos\theta_j (c_0 + \sum_{i \neq j} c_i \log|z_i - z_j|) + \sin\theta_j c_j = 0, \quad \sum_{j=1}^{N} c_j = 0.$$

4. Trace formulas

Recall the classical McKean–Singer formula for smooth compact closed d-dimensional Rimannian manifold M

$$(4\pi t)^{d/2} \operatorname{tr}(\mathrm{e}^{t\Delta}) = \operatorname{vol}(M) + \frac{t}{3}\int_M R d\sigma + \frac{\pi t^2}{180}\int_M P_2(R) d\sigma + \dots .$$

Here $\operatorname{vol}(M)$ is the Riemannian volume of M, R is the scalar curvature, P_2 is a polynomial in the derivatives of a Riemann tensor. For a smooth compact 2D surface M this formula has the form

$$\operatorname{tr}(\mathrm{e}^{t\Delta}) = \frac{\operatorname{Area}(M)}{4\pi t} + \frac{1}{6}\chi(M) + O(t).$$

Here we obtain the analogous formulas for a compact polyhedron.

Theorem 11. *Let $\alpha^- = 0$ (Friedrichs Laplacian). Then*

$$\operatorname{tr}(\mathrm{e}^{t\Delta}) = \frac{\operatorname{Area}(M)}{4\pi t} + \frac{1}{12}\sum_k \left(\frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi}\right) + O(\mathrm{e}^{-c/t}).$$

For arbitrary Laplacian

$$\operatorname{tr}(\mathrm{e}^{t\Delta}) = \frac{\operatorname{Area}(M)}{4\pi t} + \frac{1}{12}\sum_k \left(\frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi}\right) + \sum_{j=1}^{\infty} t^{j/2} q_j(\log t),$$

$$q_j(\log t) = \sum_{s=0}^{\infty} \frac{g_{js}}{(\log t)^s}.$$

Here all the series are asymptotical.

5. Localized solutions of the wave equation

Finally, let us consider the Cauchy problem for the wave equation on a polyhedron with δ-type initial function.

$$u_{tt} = \Delta u, \quad u|_{t=0} = \frac{1}{\varepsilon^2} u_0 \left(\frac{z - z_0}{\varepsilon} \right), \quad u_t|_{t=0} = 0, \quad \varepsilon \to 0. \tag{1}$$

Here z_0 is a point of a face Q, $u_0(y) \in C_0^\infty(Q)$. For small ε the initial function has the form of narrow peak, concentrated near the point z_0; the weak limit of this function is the delta-function supported at z_0. We will consider the simplest noncompact polyhedron – an infinite pyramid with one vertex (evidently, such a polyhedron is isometric to an infinite cone). Further we describe the set which is a natural analog of the support of singularities for distributions.

Definition 12. The asymptotic support of the solution u is the set Q_t:

$$u(x,t) = O(1), \quad x \notin Q_t.$$

The following assertion is almost evident.

Proposition 13. *For sufficiently small t*

$$Q_t : |z - z_0| = t.$$

Remark 14. This assertion means that for small times the initial localized perturbation propagates along geodesics – straight lines, starting from z_0. Now we describe scattering on the vertex of the pyramid.

Theorem 15. *Let M be an infinite pyramid. Then for sufficiently large t*

$$Q_t = Q_1 \cup Q_2,$$

where Q_1 is the geodesic sphere with the center at z_0 and radius equal to t, while Q_2 is the geodesic sphere with the center at the vertex and radius $t - d$, where d is the distance between z_0 and the vertex.

6. Lagrangian manifolds, corresponding to localized solutions

It is well known, that propagation of singularities of solutions for hyperbolic equations on a smooth Riemannian manifold M is connected with Lagrangian submanifolds in T^*M – solutions can be represented via Maslov canonic operators on these submanifolds. The same situation is valid for polyhedra; namely, for the Cauchy problem under consideration the following proposition holds.

Proposition 16. *Main term of asymptotic solution of the Cauchy problem* (1) *can be expressed in terms of Maslov canonic operator on the union of two Lagrangian submanifolds in T^*M:*

$$\Lambda = \Lambda_1 \cup \Lambda_2,$$

where

$$\Lambda_1: \quad \zeta = \lambda^{\frac{2\pi}{\beta}}, \quad p = \frac{\beta(\zeta - z_0)}{2\pi(\bar{\lambda})^{\frac{2\pi}{\beta}-1}|\lambda - z_0|}, \quad \lambda \in \mathbb{C},$$

$$\Lambda_2: \quad \zeta = \lambda^{\frac{2\pi}{\beta}}, \quad p = \frac{\beta|\lambda|}{2\pi(\bar{\lambda})^{\frac{2\pi}{\beta}}}, \quad \lambda \in \mathbb{C}.$$

Here ζ is a global complex coordinate on $M \cong \mathbb{C}$, (ζ, p) are the corresponding coordinates on $T^*M \cong \mathbb{C}^2$.

Remark 17. For the simplest case $\beta = \pi$ the previous formulas have the form

$$\Lambda_1: \quad \zeta = \lambda^2, \quad p = \frac{\lambda - z_0}{2|\lambda - z_0|\bar{\lambda}},$$

$$\Lambda_2: \quad \zeta = \lambda^2, \quad p = \frac{\lambda^2}{2|\lambda|^3}.$$

References

[1] L. Hillairet, A. Kokotov. *Krein formula and S-matrix for Euclidean surfaces with conical singularities*, Journal of Geometric Analysis, **23**(3), 1498–1529, (2013)

Andrei Shafarevich
"M.V. Lomonosov" Moscow State University
Leninskie Gory 1
119992 Moscow, Russia
e-mail: shafarev@yahoo.com

Part VII

Representation Theory

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 281–287

Coadjoint Orbits in Representation Theory of pro-Lie Groups

Daniel Beltiţă and Amel Zergane

Abstract. We present a one-to-one correspondence between equivalence classes of unitary irreducible representations and coadjoint orbits for a class of pro-Lie groups including all connected locally compact nilpotent groups and arbitrary infinite direct products of nilpotent Lie groups.

Mathematics Subject Classification (2010). Primary 22A25; Secondary 22A10, 22D10, 22D25.

Keywords. Pro-Lie group, coadjoint orbit.

1. Introduction

In this paper we sketch an approach to unitary representation theory for a class of projective limits of Lie groups, in the spirit of the method of coadjoint orbits from representation theory of Lie groups. (See [2] for more details.) The importance of this method stems from the fact that the groups under consideration here are not locally compact in general, hence they may not have a Haar measure, and therefore it is not possible to model their representation theory in the usual way, using Banach algebras or C^*-algebras.

By way of motivation, we discuss a simple example (cf. [2, Ex. 4.10]), which shows that the usual C^*-algebraic approach to group representation theory does not work for topological groups which are not locally compact. Let $G = (\mathbb{R}^{\mathbb{N}}, +)$ be the abelian group which is the underlying additive group of the vector space of all sequences of real numbers. Since the linear dual space $(\mathbb{R}^{\mathbb{N}})^* = \mathbb{R}^{(\mathbb{N})}$ is the vector space of all finitely supported sequences of real numbers, it easily follows that there exists a bijection $\Psi_G\colon \widehat{G} \to \mathbb{R}^{(\mathbb{N})}$ (compare also Corollary 9). Specifically, for every

The work of the first-named author was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS–UEFISCDI, project number PN-II-RU-TE-2014-4-0370.

$\lambda = (\lambda_j)_{j\in\mathbb{N}} \in \mathbb{R}^{(\mathbb{N})}$, $\Psi_G^{-1}(\lambda) \in \widehat{G}$ is the equivalence class of the one-dimensional representation

$$\chi_\lambda \colon G \to \mathrm{U}(1), \quad \chi_\lambda((x_j)_{j\in\mathbb{N}}) := \exp\Big(\mathrm{i}\sum_{j\in\mathbb{N}} \lambda_j x_j\Big)$$

where $\mathrm{U}(1) := \{z \in \mathbb{C} \mid |z| = 1\}$. However, as the vector space $\mathbb{R}^{(\mathbb{N})}$ is infinite-dimensional, it is not locally compact, hence it is not homeomorphic to the spectrum of any C^*-algebra. Consequently, the irreducible representation theory of G cannot be exhaustively described via any C^*-algebra.

2. Preliminaries

Lie theory

We use upper case Roman letters to denote Lie groups, and their corresponding lower case Gothic letters to denote the Lie algebras. We will also use the notation $\mathbf{L}$ for the Lie functor which associates to each Lie group its Lie algebra, hence for any Lie group G one has $\mathbf{L}(G) = \mathfrak{g}$. We denote the exponential map of a Lie group G by $\exp_G \colon \mathfrak{g} \to G$, and if this map is bijective, then we denote its inverse by $\log_G \colon G \to \mathfrak{g}$. For any morphism of Lie groups $q \colon G \to H$, its corresponding morphism of Lie algebras is denoted by $\mathbf{L}(q) \colon \mathfrak{g} \to \mathfrak{h}$, hence one has the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathbf{L}(q)} & \mathfrak{h} \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow{q} & H \end{array}$$

as is well known. The coadjoint action of a Lie group is denoted by $\mathrm{Ad}_G^* \colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$, and its corresponding set of coadjoint orbits is denoted by $\mathfrak{g}^*/G$ or $\mathbf{L}(G)^*/G$. If $q \colon G \to H$ is a surjective morphism of Lie groups, then one has a map $\mathbf{L}(q)^* : \mathfrak{h}^* \to \mathfrak{g}^*$ such that for every coadjoint H-orbit $\mathcal{O} \in \mathfrak{h}^*/H$ its image $\mathbf{L}(q)^*(\mathcal{O})$ is a coadjoint G-orbit, and one thus obtains a map

$$\mathbf{L}(q)^*_{\mathrm{Ad}^*} \colon \mathfrak{h}^*/H \to \mathfrak{g}^*/G, \quad \mathcal{O} \mapsto \mathbf{L}(q)^*(\mathcal{O}).$$

Representation theory

For any topological group G we denote by $\widehat{G}$ its unitary dual, that is, its set of unitary equivalence classes $[\pi]$ of unitary irreducible representations $\pi \colon G \to \mathcal{B}(\mathcal{H})$. If $q \colon G \to H$ is a continuous surjective morphism of topological groups, then we define

$$\widehat{q} \colon \widehat{H} \to \widehat{G}, \quad [\pi] \mapsto [\pi \circ q].$$

Proposition 1. *Let G be any connected nilpotent Lie group with its universal covering $p \colon \widetilde{G} \to G$, and denote $\Gamma := \mathrm{Ker}\, p \subseteq \widetilde{G}$. We define*

$$\mathfrak{g}^*_{\mathbb{Z}} := \{\xi \in \mathfrak{g}^* \mid (\xi \circ \mathbf{L}(p) \circ \log_{\widetilde{G}})(\Gamma) \subseteq \mathbb{Z}\}.$$

Then the following assertions hold:

1. *The set* Γ *is a discrete subgroup of the center of* $\widetilde{G}$.
2. *The set* $\mathfrak{g}^*_{\mathbb{Z}}$ *is invariant with respect to the coadjoint action of* G.
3. *There exists an injective correspondence* $\Psi_G\colon \widehat{G} \to \mathfrak{g}^*/G$, *whose image is exactly the set of all coadjoint* G*-orbits contained in* $\mathfrak{g}^*_{\mathbb{Z}}$, *such that if* H *is any other connected nilpotent Lie group with a surjective morphism of Lie groups* $q\colon G \to H$, *then the diagram*

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\Psi_G} & \mathfrak{g}^*/G \\ {\scriptstyle\widehat{q}}\big\uparrow & & \big\uparrow{\scriptstyle \mathbf{L}(q)^*_{\mathrm{Ad}^*}} \\ \widehat{H} & \xrightarrow{\Psi_H} & \mathfrak{h}^*/H \end{array}$$

is commutative.

Proof. See [2, Prop. A.3]. □

3. Pro-Lie groups and their Lie algebras

The main results that we give below (see Theorem 7 and its corollaries) are applicable to pro-Lie groups and are stated in terms of Lie algebras and coadjoint orbits of these groups. Therefore we discuss these notions in this section. Our general reference for pro-Lie groups is the monograph [6], and we also refer to the paper [7] for the relation between pro-Lie groups and infinite-dimensional Lie groups.

Any topological group in this paper is assumed to be Hausdorff by definition. A Cauchy net in a topological group G is a net $\{g_j\}_{j\in J}$ in G with the property that for every neighborhood V of $\mathbf{1} \in G$ there exists $j_V \in J$ such that for all $i, k \in J$ with $i \geq j_V$ and $k \geq j_V$ one has $g_i g_k^{-1} \in V$. A topological group G is called *complete* if every Cauchy net in G is convergent. Every locally compact group is complete by [6, Rem. 1.31].

For any topological group G we denote by $\mathcal{N}(G)$ the set of its *co-Lie subgroups*, that is, the closed normal subgroups $N \subseteq G$ for which G/N is a finite-dimensional Lie group. We say that G is a *pro-Lie group* if it is complete and for every neighborhood V of $\mathbf{1} \in G$ there exists $N \in \mathcal{N}(G)$ with $N \subseteq V$ (cf. [6, Def. 3.25]). If this is the case, then $\mathcal{N}(G)$ is closed under finite intersections, hence it is a filter basis (cf. [6, page 148]).

Pro-Lie groups can be equivalently defined as the limits of projective systems of Lie groups, by [6, Th. 3.39].

Definition 2. For any pro-Lie group G, its set of continuous 1-parameter subgroups

$$\mathbf{L}(G) := \{X \in \mathcal{C}(\mathbb{R}, G) \mid (\forall t, s \in \mathbb{R}) \quad X(t+s) = X(t)X(s)\}$$

is endowed with its topology of uniform convergence on the compact subsets of $\mathbb{R}$. Then the topological space $\mathbf{L}(G)$ has the structure of a locally convex Lie algebra over $\mathbb{R}$, whose scalar multiplication, vector addition and bracket satisfy the following conditions for all $t, s \in \mathbb{R}$ and $X_1, X_2 \in \mathbf{L}(G)$:

$$\begin{aligned}(t \cdot X_1)(s) &= X_1(ts);\\ (X_1 + X_2)(t) &= \lim_{n\to\infty} (X_1(t/n)X_2(t/n))^n;\\ [X_1, X_2](t^2) &= \lim_{n\to\infty} (X_1(t/n)X_2(t/n)X_1(-t/n)X_2(-t/n))^{n^2},\end{aligned}$$

where the convergence is uniform on the compact subsets of $\mathbb{R}$. (See, for instance, [1, Ex. 2.7(4.)].) One also has the dual vector space

$$\mathbf{L}(G)^* := \{\xi\colon \mathbf{L}(G) \to \mathbb{R} \mid \xi \text{ is linear and continuous}\}$$

endowed with its locally convex topology of pointwise convergence on $\mathbf{L}(G)$. The *adjoint action* is $\mathrm{Ad}_G\colon G \times \mathbf{L}(G) \to \mathbf{L}(G)$, $(g, X) \mapsto \mathrm{Ad}_G(g)X := gX(\cdot)g^{-1}$, and this defines by duality the *coadjoint action*

$$\mathrm{Ad}_G^*\colon G \times \mathbf{L}(G)^* \to \mathbf{L}(G)^*, \quad (g, \xi) \mapsto \mathrm{Ad}_G^*(g)\xi := \xi \circ \mathrm{Ad}_G(g^{-1}).$$

We denote by $\mathbf{L}(G)^*/G$ the set of all coadjoint orbits, that is, the orbits of the above coadjoint action.

In the following proposition we summarize a few basic properties of Lie algebras of connected locally compact groups. A pro-Lie group G is called *pronilpotent* if for every $N \in \mathcal{N}(G)$ the finite-dimensional Lie group G/N is nilpotent. (See [6, Def. 10.12].)

Proposition 3. *If G is a connected locally compact group, then the following assertions hold:*

1. *G is a pro-Lie group and its Lie algebra $\mathbf{L}(G)$ is the direct product of a finite-dimensional Lie algebra, an abelian (possibly infinite-dimensional) Lie algebra, and a (possibly infinite) product of simple compact Lie algebras.*
2. *The following conditions are equivalent:*
   (a) *The group G is pronilpotent.*
   (b) *The Lie algebra $\mathbf{L}(G)$ is the product of a finite-dimensional nilpotent Lie algebra and an abelian (possibly infinite-dimensional) Lie algebra.*
   (c) *The Lie algebra $\mathbf{L}(G)$ is nilpotent (possibly infinite-dimensional).*
   (d) *The group G is nilpotent.*

Proof. The first assertion follows by [5, Th. 4] or [8, Cor. 4.24]. See also [4, Th. 2.1.2.2].

For the second assertion, we first recall from [6, Th. 10.36 and Def. 7.42] that the group G is pronilpotent if and only if its Lie algebra $\mathbf{L}(G)$ is pronilpotent, that is, every finite-dimensional quotient algebra of $\mathbf{L}(G)$ is nilpotent. Therefore, in view of Assertion 1, one has

$$\text{(2a)} \iff \text{(2b)} \iff \text{(2c)}.$$

Moreover, one clearly has (2d) $\implies$ (2a).

We now prove (2b) $\implies$ (2d). To this end let $\pi_G\colon \widetilde{G} \to G$ be the universal morphism defined in [8, Def. 4.20] and [6, page 259]. Then $\mathbf{L}(\pi_G)\colon \mathbf{L}(\widetilde{G}) \to \mathbf{L}(G)$

is an isomorphism of Lie algebras and the image of π_G is dense in G by [6, Th. 6.6 (i) and (iv)]. It follows at once by condition (2b) and [8, Th. 4.23] that the group $\widetilde{G}$ is nilpotent. Then, as the image of π_G is dense in G, we obtain (2d), and this completes the proof. □

4. Main results

Theorem 7 below provides an exhaustive description of the unitary dual of a class of topological groups that are not locally compact. As we discussed in the introduction, unitary dual spaces of non-locally compact groups in general cannot be described in terms of representation theory of C^*-algebras.

For the following definition we recall that if X is an arbitrary nonempty set, then a *filter basis* on X is a nonempty set B whose elements are nonempty subsets of X having the property that for any $X_1, X_2 \in B$ there exists $X_0 \in B$ with $X_0 \subseteq X_1 \cap X_2$. If X is moreover endowed with a topology, then one says that the filter basis B *converges* to a point $x_0 \in X$ if for every neighborhood V of x_0 there exists $X \in B$ with $X \subseteq V$.

Example 4. Here are some basic examples of filter bases.

1. Every neighborhood basis at any point of a topological space is a filter basis converging to that point.
2. If G is a group endowed with the discrete topology and B is a set of subgroups of G such that the trivial subgroup $G_0 := \{\mathbf{1}\}$ is an element of B, then B is a filter basis on G converging to $\mathbf{1} \in G$ since for any $G_1, G_2 \in B$ one has $G_0 \subseteq G_1 \cap G_2$ and on the other hand G_0 is contained in any neighborhood of $\mathbf{1} \in G$.
3. If G is a topological group with the property that for every neighborhood V of $\mathbf{1} \in G$ there exists a co-Lie subgroup $N \in \mathcal{N}(G)$ with $N \subseteq V$, then $\mathcal{N}(G)$ is a filter basis on G converging to $\mathbf{1} \in G$ since in fact for every $N_1, N_2 \in \mathcal{N}(G)$ one has $N_1 \cap N_2 \in \mathcal{N}(G)$. (See [6, page 148].) In particular, this holds true for pro-Lie groups.

Definition 5. An *amenable filter basis* on a topological group G is a filter basis $\mathcal{N} \subseteq \mathcal{N}(G)$ converging to $\mathbf{1} \in G$ such that every topological group $N \in \mathcal{N}$ is amenable.

Example 6. Here are two examples of amenable filter basis that are needed in Corollaries 8–9:

1. If G is a connected locally compact group, then $\mathcal{N}(G)$ is an amenable filter basis. In fact, every $N \in \mathcal{N}(G)$ is compact hence amenable, and on the other hand $\mathcal{N}(G)$ converges to $\mathbf{1} \in G$ by the theorem of Yamabe. (See for instance [4, Th. 0.1.5].)
2. Let $\{G_j\}_{j\in J}$ be an infinite family of nilpotent Lie groups with their direct product topological group $G := \prod_{j\in J} G_j$. Denote by $\mathcal{N}$ the set of all subgroups of G of the form $N_F := \prod_{j\in J} N_j$ associated to any finite subset $F \subseteq J$,

with $N_j = \{\mathbf{1}\} \subseteq G_j$ if $j \in F$ and $N_j = G_j$ if $j \in J \setminus F$. It is clear that every N_F of this form has the following properties: N_F is a closed normal subgroup of G that is isomorphic to $\prod_{j\in J\setminus F} G_j$ hence N_F is amenable by [2, Prop. 3.8], and moreover G/N_F is isomorphic to $\prod_{j\in F} G_j$, which is a Lie group since F is a finite set, hence $N_F \in \mathcal{N}(G)$. For any finite subsets $F_1, F_2 \subseteq J$ one clearly has $N_{F_1} \cap N_{F_2} = N_{F_1\cup F_2}$, where $F_1 \cup F_2$ is again a finite subset of J, hence $\mathcal{N}$ is a filter basis on G. Moreover, by the definition of an infinite direct product of topologies, it follows that the filter basis $\mathcal{N}$ converges to $\mathbf{1} \in G$. Consequently, $\mathcal{N}$ is an amenable filter basis on G.

Theorem 7. *Let G be a complete topological group with an amenable filter basis $\mathcal{N}$ for which G/N is a connected nilpotent Lie group for every $N \in \mathcal{N}$. Then there exists a well-defined bijective correspondence*

$$\Psi_G \colon \widehat{G} \to \mathbf{L}(G)^*/G, \quad [\pi] \mapsto \mathcal{O}^\pi$$

between the equivalence classes of unitary irreducible representations of G and the set of all coadjoint G-orbits contained in the G-invariant set

$$\mathbf{L}(G)^*_{\mathbb{Z}} := \{\xi \in \mathbf{L}(G)^* \mid (\exists N \in \mathcal{N})(\exists \eta \in \mathbf{L}(G/N)^*_{\mathbb{Z}}) \quad \xi = \eta \circ \mathbf{L}(p_N)\}.$$

Every unitary irreducible representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is thus associated to the coadjoint G-orbit $\mathcal{O}^\pi := \mathbf{L}(p_N)^*(\mathcal{O}_0) \subseteq \mathbf{L}(G)^*_{\mathbb{Z}}$, where $N \in \mathcal{N}$ and $\mathcal{O}_0 \subseteq \mathbf{L}(G/N)^*_{\mathbb{Z}}$ is the coadjoint (G/N)-orbit associated with a unitary irreducible representation $\pi_0 \colon G/N \to \mathcal{B}(\mathcal{H})$ satisfying $\pi_0 \circ p_N = \pi$.

Proof. See [2, Th. 4.6]. □

In connection with the following corollary we note that the Lie algebras of connected locally compact nilpotent groups can be described as in Proposition 3.

Corollary 8. *If G is a connected locally compact nilpotent group, then there is a bijective correspondence $\Psi_G \colon \widehat{G} \to \mathbf{L}(G)^*/G$ onto the set of all coadjoint G-orbits contained in a certain G-invariant subset $\mathbf{L}(G)^*_{\mathbb{Z}} \subseteq \mathbf{L}(G)$. For any filter basis $\mathcal{N} \subseteq \mathcal{N}(G)$ converging to the identity one has*

$$\mathbf{L}(G)^*_{\mathbb{Z}} := \{\xi \in \mathbf{L}(G)^* \mid (\exists N \in \mathcal{N})(\exists \eta \in \mathbf{L}(G/N)^*_{\mathbb{Z}}) \quad \xi = \eta \circ \mathbf{L}(p_N)\}.$$

Proof. See [2, Cor. 4.7]. □

We now draw a corollary of Theorem 7 that applies to pro-Lie groups which are not locally compact.

Corollary 9. *If $\{G_j\}_{j\in J}$ is a family of connected nilpotent Lie groups, with their direct product topological group $G := \prod_{j\in J} G_j$, then there is a bijective correspondence $\Psi_G \colon \widehat{G} \to \mathbf{L}(G)^*/G$ onto the set of all coadjoint G-orbits contained in the G-invariant subset $\mathbf{L}(G)^*_{\mathbb{Z}} \subseteq \mathbf{L}(G)$. Here we define*

$$\mathbf{L}(G)^*_{\mathbb{Z}} := \{\xi \in \mathbf{L}(G)^* \mid (\exists F \in \mathcal{F})(\exists \eta \in \mathbf{L}(G_F)^*_{\mathbb{Z}}) \quad \xi = \eta \circ \mathbf{L}(p_F)\}$$

where $\mathcal{F}$ is the set of all finite subsets $F \subseteq J$, and for every $F \in \mathcal{F}$ we define $G_F := \prod_{j\in F} G_j$ and $p_F \colon G \to G_F$ is the natural projection.

Proof. See [2, Cor. 4.9]. □

Remark 10. The amenability hypotheses of Theorem 7 may actually be removed, using some results of [9].

Acknowledgment

We are grateful to Karl-Hermann Neeb for Remark 10.

References

[1] I. Beltiţă, D. Beltiţă, *On differentiability of vectors in Lie group representations*. J. Lie Theory **21** (2011), no. 4, 771–785.

[2] D. Beltiţă, A. Zergane, *Amenability and representation theory of pro-Lie groups*. Math. Z. **286** (2017), no. 1–2, 701–722.

[3] H. Boseck, *Über Darstellungen lokal-kompakter topologischer Gruppen*. Math. Nachr. **74** (1976), 233–251.

[4] H. Boseck, G. Czichowski and K.P. Rudolph, *Analysis on Topological Groups – General Lie Theory*. Teubner-Texte zur Mathematik, 37. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1981.

[5] V.M. Glushkov, *Lie algebras of locally bicompact groups*. (Russian) Uspehi Mat. Nauk (N.S.) **12** (1957) no. 2 (74), 137–142.

[6] K.H. Hofmann, S.A. Morris, *The Lie theory of connected pro-Lie groups*. EMS Tracts in Mathematics, 2. European Mathematical Society (EMS), Zürich, 2007.

[7] K.H. Hofmann, K.-H. Neeb, *Pro-Lie groups which are infinite-dimensional Lie groups*. Math. Proc. Cambridge Philos. Soc. **146** (2009), no. 2, 351–378.

[8] R.K. Lashof, *Lie algebras of locally compact groups*. Pacific J. Math. **7** (1957), 1145–1162.

[9] K.-H. Neeb, *On differentiable vectors for representations of infinite dimensional Lie groups*. J. Funct. Anal. **259** (2010), no. 11, 2814–2855.

Daniel Beltiţă
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O. Box 1-764
Bucharest, Romania
e-mail: beltita@gmail.com

Amel Zergane
Higher Institute of Applied Sciences and Technology of Sousse
Mathematical Physics Laboratory, Special Functions and Applications
City Ibn Khaldoun 4003
Sousse, Tunisia
e-mail: amel.zergane@yahoo.fr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 289–308

Conformal Symmetry Breaking on Differential Forms and Some Applications

Toshiyuki Kobayashi

Abstract. Rapid progress has been made recently on symmetry breaking operators for real reductive groups. Based on Program A–C for branching problems (T. Kobayashi [Progr. Math. 2015]), we illustrate a scheme of the classification of (local and nonlocal) symmetry breaking operators by an example of conformal representations on differential forms on the model space $(X, Y) = (S^n, S^{n-1})$, which generalizes the scalar case (Kobayashi–Speh [Mem. Amer. Math. Soc. 2015]) and the case of local operators (Kobayashi–Kubo–Pevzner [Lect. Notes Math. 2016]). Some applications to automorphic form theory, motivations from conformal geometry, and the methods of proofs are also discussed.

Mathematics Subject Classification (2010). Primary 22E46; Secondary 11F70, 53A30.

Keywords. Branching rule, conformal geometry, reductive group, symmetry breaking.

1. Branching problems – Stages A to C

Suppose Π is an irreducible representation of a group G. We may regard Π as a representation of its subgroup G' by restriction, which we denote by $\Pi|_{G'}$. The restriction $\Pi|_{G'}$ is not irreducible in general. In case it can be given as the direct sum of irreducible G'-modules, the decomposition is called the *branching law* of the restriction $\Pi|_{G'}$.

Example 1 (fusion rule). *Let π_1 and π_2 be representations of a group H. The outer tensor product $\Pi := \pi_1 \boxtimes \pi_2$ is a representation of the product group $G := H \times H$, and its restriction $\Pi|_{G'}$ to the subgroup $G' := \mathrm{diag}(H)$ is nothing but the tensor product representation $\pi_1 \otimes \pi_2$. In this case, the branching law is called the fusion rule.*

For real reductive Lie groups such as $G = GL(n, \mathbb{R})$ or $O(p, q)$, irreducible representations Π are usually infinite-dimensional and do not always possess highest weight vectors, consequently, the restriction $\Pi|_{G'}$ to subgroups G' may involve various (sometimes "wild") aspects.

Example 2. *The fusion rule of two irreducible unitary principal series representations of* $GL(n, \mathbb{R})$ $(n \geq 3)$ *involve continuous spectrum and infinite multiplicities in the direct integral of irreducible unitary representations.*

By the *branching problem* (in a wider sense than the usual), we mean the problem of understanding how the restriction $\Pi|_{G'}$ behaves as a representation of the subgroup G'. We treat non-unitary representations Π as well. In this case, instead of considering the irreducible decomposition of the restriction $\Pi|_{G'}$, we may investigate continuous G'-homomorphisms

$$T \colon \Pi|_{G'} \to \pi$$

to irreducible representations π of the subgroup G'. We call T a *symmetry breaking operator* (SBO, for short). The dimension of the space of symmetry breaking operators

$$m(\Pi, \pi) := \dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\Pi|_{G'}, \pi)$$

may be thought of as a variant of the "multiplicity". Finding a formula of $m(\Pi, \pi)$ is a substitute of the branching law $\Pi|_{G'}$ when Π is not a unitary representation.

The author proposed in [19] a program for branching problems in the following three stages:

Stage A. Abstract feature of the restriction $\Pi|_{G'}$.
Stage B. Branching laws.
Stage C. Construction of symmetry breaking operators.

Loosely speaking, Stage B concerns a decomposition of representations, whereas Stage C asks for a decomposition of vectors.

For "abstract features" of the restriction in Stage A, we may think of the following aspects:

A.1. Spectrum of the restriction $\Pi|_{G'}$:

- (discretely decomposable case, [12, 14, 15]) branching problems could be studied purely algebraic and combinatorial approaches;
- (continuous spectrum) branching problems may be of analytic feature (*e.g.*, Example 2).

A.2. Estimate of multiplicities for the restriction $\Pi|_{G'}$:

- multiplicities may be infinite (see Example 2);
- multiplicities may be at most one in special settings (*e.g.*, theta correspondence [7], Gross–Prasad conjecture [6], real forms of strong Gelfand pairs [35], visible actions [17], *etc.*).

The goal of Stage A in branching problems is to analyze aspects such as A.1 and A.2 in complete generality. If multiplicities of the restriction $\Pi|_{G'}$ are known *a priori* to be bounded in Stage A, one might be tempted to find irreducible decompositions (Stage B), and moreover to construct explicit symmetry breaking operators (Stage C). Thus, results in Stage A might also serve as a foundation for further detailed study of the restriction $\Pi|_{G'}$ (Stages B and C).

This article is divided into three parts. First, we discuss Stage A in Section 3 with focus on multiplicities in both regular representations on homogeneous spaces and branching problems based on a joint work [26] with T. Oshima, and give some perspectives of the subject through the classification theory [23] joint with T. Matsuki about the pairs (G, G') for which multiplicities in branching laws are always finite.

Second, we take (G, G') to be $(O(n+1, 1), O(n, 1))$ as an example of such pairs, and explain the first test case for the classification problem of symmetry breaking operators (Stages B and C). The choice of our setting is motivated by conformal geometry, and is also related to the local Gross–Prasad conjecture [6, 31]. We survey the classification theory of conformally covariant SBO for differential forms on the model space $(X, Y) = (S^n, S^{n-1})$: for local operators based on a recent book [21] with T. Kubo and M. Pevzner in Section 5 and for nonlocal operators based on a recent monograph [29] with B. Speh and its generalization [30] in Section 6.

In Section 7, we discuss an ongoing work with Speh on some applications of these results to a question from automorphic form theory, in particular, about the periods of irreducible representations with nonzero $(\mathfrak{g}, K)$-cohomologies. The resulting condition to admit periods is compared with a recent L^2-theory [1] joint with Y. Benoist.

Detailed proofs of the new results in Sections 6 and 7 will be given in separate papers [20, 30].

Notation. $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2. Preliminaries: smooth representations

We would like to treat non-unitary representations as well for the study of branching problems. For this we recall some standard concepts of continuous representations of Lie groups.

Suppose Π is a continuous representation of G on a Banach space V. A vector $v \in V$ is said to be *smooth* if the map $G \to V$, $g \mapsto \Pi(g)v$ is of C^∞-class. Let V^∞ denote the space of smooth vectors of the representation (Π, V). Then V^∞ is a G-invariant dense subspace of V, and V^∞ carries a Fréchet topology with a family of semi-norms $\|v\|_{i_1 \cdots i_k} := \|d\Pi(X_{i_1}) \cdots d\Pi(X_{i_k})v\|$, where $\{X_1, \ldots, X_n\}$ is a basis of the Lie algebra $\mathfrak{g}_0$ of G. Thus we obtain a continuous Fréchet representation (Π^∞, V^∞) of G.

Suppose now that G is a real reductive linear Lie group, K a maximal compact subgroup of G, and $\mathfrak{g}$ the complexification of the Lie algebra $\mathfrak{g}_0$ of G. Let $\mathcal{HC}$ denote the category of Harish-Chandra modules whose objects and morphisms are $(\mathfrak{g}, K)$-modules of finite length and $(\mathfrak{g}, K)$-homomorphisms, respectively. Let Π be a continuous representation of G on a complete locally convex topological vector space V. Assume that the G-module Π is of finite length. We say Π is *admissible* if

$$\dim_{\mathbb{C}} \operatorname{Hom}_K(\tau, \Pi|_K) < \infty$$

for all irreducible finite-dimensional representations τ of K. We denote by V_K the space of K-finite vectors. Then $V_K \subset V^\infty$ and the Lie algebra $\mathfrak{g}$ leaves V_K invariant. The resulting $(\mathfrak{g}, K)$-module on V_K is called the underlying $(\mathfrak{g}, K)$-module of Π, and will be denoted by Π_K.

For any admissible representation Π on a Banach space V, the smooth representation (Π^∞, V^∞) depends only on the underlying $(\mathfrak{g}, K)$-module. We say (Π^∞, V^∞) is an *admissible smooth representation*. By the Casselman–Wallach globalization theory, (Π^∞, V^∞) has moderate growth, and there is a canonical equivalence of categories between the category $\mathcal{HC}$ of Harish-Chandra modules and the category of admissible smooth representations of G ([37, Chap. 11]). In particular, the Fréchet representation Π^∞ is uniquely determined by its underlying $(\mathfrak{g}, K)$-module. We say Π^∞ is the *smooth globalization* of $\Pi_K \in \mathcal{HC}$.

For simplicity, by an *irreducible smooth representation*, we shall mean an irreducible admissible smooth representation of G. We denote by $\widehat{G}_{\mathrm{smooth}}$ the set of equivalence classes of irreducible smooth representations of G. Via the underlying $(\mathfrak{g}, K)$-modules, we may regard the unitary dual $\widehat{G}$ as a subset of $\widehat{G}_{\mathrm{smooth}}$.

3. Multiplicities in symmetry breaking

Let $G \supset G'$ be a pair of real reductive groups. For $\Pi \in \widehat{G}_{\mathrm{smooth}}$ and $\pi \in \widehat{G'}_{\mathrm{smooth}}$, we denote by $\operatorname{Hom}_{G'}(\Pi|_{G'}, \pi)$ the space of symmetry breaking operators, and define the *multiplicity* (for smooth representations) by

$$m(\Pi, \pi) := \dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}. \tag{1}$$

Note that $m(\Pi, \pi)$ is well defined without the unitarity assumption on Π and π.

We established a geometric criterion for multiplicities to be finite (more strongly, to be bounded) as follows:

Theorem 3 ([26], see also [13, 18]). *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups.*

(1) *The following two conditions on the pair (G, G') are equivalent:*

(FM) (finite multiplicities) $m(\Pi, \pi) < \infty$ *for all*

$$\Pi \in \widehat{G}_{\mathrm{smooth}} \quad \textit{and} \quad \pi \in \widehat{G'}_{\mathrm{smooth}};$$

(PP) (geometry) $(G \times G')/\operatorname{diag}(G')$ *is real spherical.*

(2) *The following two conditions on the pair* (G, G') *are equivalent:*

(BM) (bounded multiplicities) *There exists* $C > 0$ *such that*

$$m(\Pi, \pi) \le C \quad \textit{for all } \Pi \in \widehat{G}_{\mathrm{smooth}} \textit{ and } \pi \in \widehat{G'}_{\mathrm{smooth}};$$

(BB) (complex geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\mathrm{diag}(G'_{\mathbb{C}})$ *is spherical.*

Here we recall that a connected complex manifold $X_{\mathbb{C}}$ with holomorphic action of a complex reductive group $G_{\mathbb{C}}$ is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. There has been an extensive study of spherical varieties in algebraic geometry and finite-dimensional representation theory. In contrast, concerning the real setting, in search of a good framework for global analysis on homogeneous spaces which are broader than the usual (*e.g.*, reductive symmetric spaces), the author proposed:

Definition 4 ([13]). Let G be a real reductive Lie group. We say a connected smooth manifold X with smooth G-action is *real spherical* if a minimal parabolic subgroup P of G has an open orbit in X.

We discovered in [13, 26] that these geometric properties (spherical/real spherical) are exactly the conditions that a reductive group G has a "strong grip" of the space of functions on X in the context of multiplicities of (infinite-dimensional) irreducible representations occurring in the regular representation of G on $C^\infty(X)$:

Theorem 5 ([26, Thms. A and C]). *Suppose G is a real reductive linear Lie group, H is an algebraic reductive subgroup, and $X = G/H$.*

(1) *The homogeneous space X is real spherical if and only if*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, C^\infty(X)) < \infty \textit{ for all } \pi \in \widehat{G}_{\mathrm{smooth}}.$$

(2) *The complexification $X_{\mathbb{C}}$ is spherical if and only if*

$$\sup_{\pi \in \widehat{G}_{\mathrm{smooth}}} \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, C^\infty(X)) < \infty.$$

Methods of proof. In [26], we obtained not only the equivalences in Theorem 5 but also quantitative estimates of the dimension. The proof for the upper estimate in [26] uses the theory of regular singularities of a system of partial differential equations by taking an appropriate compactification with normal crossing boundaries, whereas the proof for the lower estimate uses the construction of a "generalized Poisson transform". Furthermore, these estimates hold for the representations of G on the space of smooth sections for equivariant vector bundles over $X = G/H$ without assuming that H is reductive. For instance, this applies also to the case where H is a maximal unipotent subgroup of G, giving a Kostant–Lynch estimate to the dimension of the space of Whittaker vectors ([26, Ex. 1.4 (3)]).

Back to Theorem 3 on branching problems, the geometric estimates of multiplicities is proved by applying Theorem 5 to the pair $(G \times G', \mathrm{diag}(G'))$ together with some careful arguments on topological vector spaces ([18, Thm. 4.1]).

Classification theory. Theorem 3 serves Stage A in branching problems, and singles out nice settings in which we could expect to go further on Stages B and C of the detailed study of symmetry breaking.

So it would be useful to develop a classification theory of pairs (G, G') for which the geometric criteria (PP) or (BB) in Theorem 3 are satisfied.

- The geometric criterion (BB) in Theorem 3 appeared in the context of *finite-dimensional* representations already in 1970s, and such pairs $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ were classified infinitesimally, see [32]. The classification of real forms (G, G') satisfying the condition (BB) follows readily from that of complex pairs $(G_{\mathbb{C}}, G'_{\mathbb{C}})$, see [23]. Sun–Zhu [35] proved that the constant C in Theorem 3 can be taken to be one (*multiplicity-free theorem*) in many of real forms (G, G'), see [31, Rem. 2.2] for multiplicity-two results for some other real forms.
- The pairs $({}^{\backprime}G \times {}^{\backprime}G, \operatorname{diag}({}^{\backprime}G))$ for real reductive groups ${}^{\backprime}G$ satisfying the geometric criterion (PP) in Theorem 3 were classified in [13].
- More generally, symmetric pairs (G, G') satisfying the geometric criterion (PP) in Theorem 3 was classified by the author and Matsuki [23]. The methods are a linearization technique and invariants of quivers.

In turn, these classification results give an *a priori* estimate of multiplicities in branching problems by Theorem 3.

Example 6 (finite multiplicities for the fusion rule, [13, Ex. 2.8.6], see also [18, Cor. 4.2]). *Suppose G is a simple Lie group. Then the following two conditions are equivalent:*

(i) $\dim_{\mathbb{C}} \operatorname{Hom}_G(\pi_1 \otimes \pi_2, \pi_3) < \infty$ *for all* $\pi_1, \pi_2, \pi_3 \in \widehat{G}_{\mathrm{smooth}}$;
(ii) *G is either compact or locally isomorphic to $SO(n, 1)$.*

Example 7. *Let $(G, G') = (O(p + r, q), O(r) \times O(p, q))$.*

(1) $m(\Pi, \pi) < \infty$ *for all* $\Pi \in \widehat{G}_{\mathrm{smooth}}$ *and* $\pi \in \widehat{G'}_{\mathrm{smooth}}$.
(2) $m(\Pi, \pi) \leq 1$ *for all* $\Pi \in \widehat{G}_{\mathrm{smooth}}$ *and* $\pi \in \widehat{G'}_{\mathrm{smooth}}$ *if and only if* $p+q+r \leq 4$ *or* $r = 1$.

See [18] for the further classification theory of symmetric pairs (G, G') that guarantee finite multiplicity properties for symmetry breaking.

4. Conformally covariant SBOs

This section discusses a question on symmetry breaking with respect to a pair of conformal manifolds $X \supset Y$.

Let (X, g) be a Riemannian manifold. Suppose that a Lie group G acts conformally on X. This means that there exists a positive-valued function $\Omega \in C^\infty(G \times X)$ (*conformal factor*) such that

$$L_h^* g_{h\cdot x} = \Omega(h, x)^2 g_x \quad \text{for all } h \in G,\ x \in X,$$

where we write $L_h\colon X \to X, x \mapsto h\cdot x$ for the action of G on X. When X is oriented, we define a locally constant function

$$\mathit{or}\colon G\times X \longrightarrow \{\pm 1\}$$

by $\mathit{or}(h)(x) = 1$ if $(L_h)_{*x}\colon T_xX \longrightarrow T_{L_hx}X$ is orientation-preserving, and $= -1$ if it is orientation-reversing.

Since both the conformal factor Ω and the orientation map or satisfy cocycle conditions, we can form a family of representations $\varpi^{(i)}_{\lambda,\delta}$ of G with parameters $\lambda\in\mathbb{C}$ and $\delta\in\mathbb{Z}/2\mathbb{Z}$ on the space $\mathcal{E}^i(X)$ of differential i-forms on X $(0\le i\le \dim X)$ defined by

$$\varpi^{(i)}_{\lambda,\delta}(h)\alpha := \mathit{or}(h)^{\delta}\Omega(h^{-1},\cdot)^{\lambda}L^*_{h^{-1}}\alpha, \quad (h\in G). \tag{2}$$

The representation $\varpi^{(i)}_{\lambda,\delta}$ of the conformal group G on $\mathcal{E}^i(X)$ will be simply denoted by $\mathcal{E}^i(X)_{\lambda,\delta}$, and referred to as the *conformal representation* on differential i-forms.

Suppose that Y is an orientable submanifold. Then Y is endowed with a Riemannian structure $g|_Y$ by restriction, and we can define in a similar way a family of representations $\mathcal{E}^j(Y)_{\nu,\varepsilon}$ $(\nu\in\mathbb{C}, \varepsilon\in\mathbb{Z}/2\mathbb{Z}, 0\le j\le\dim Y)$ of the conformal group of $(Y, g|_Y)$.

We consider the full group of conformal diffeomorphisms and its subgroup defined as

$$\begin{aligned}\mathrm{Conf}(X) &:= \{\text{conformal diffeomorphisms of } (X,g)\},\\ \mathrm{Conf}(X;Y) &:= \{\varphi\in\mathrm{Conf}(X) : \varphi(Y) = Y\}.\end{aligned} \tag{3}$$

Then there is a natural group homomorphism

$$\mathrm{Conf}(X;Y)\to\mathrm{Conf}(Y), \quad \varphi\mapsto\varphi|_Y. \tag{4}$$

Definition 8. A linear map $T\colon \mathcal{E}^i(X)_{\lambda,\delta}\to\mathcal{E}^j(Y)_{\nu,\varepsilon}$ is a *conformally covariant symmetry breaking operator* (conformally covariant SBO, for short) if T intertwines the actions of the group $\mathrm{Conf}(X;Y)$.

We shall write

$$H\left(\begin{array}{c|c} i & j\\ \lambda,\delta & \nu,\varepsilon\end{array}\right) := \mathrm{Hom}_{\mathrm{Conf}(X;Y)}(\mathcal{E}^i(X)_{\lambda,\delta}|_{\mathrm{Conf}(X;Y)}, \mathcal{E}^j(Y)_{\nu,\varepsilon}) \tag{5}$$

$$\cup$$

$$D\left(\begin{array}{c|c} i & j\\ \lambda,\delta & \nu,\varepsilon\end{array}\right) := \mathrm{Diff}_{\mathrm{Conf}(X;Y)}(\mathcal{E}^i(X)_{\lambda,\delta}|_{\mathrm{Conf}(X;Y)}, \mathcal{E}^j(Y)_{\nu,\varepsilon}) \tag{6}$$

for the space of continuous conformally covariant SBOs and its subspace of differential SBOs, namely, those operators T satisfying the local property: $\mathrm{Supp}(T\alpha)\subset\mathrm{Supp}(\alpha)$ for all $\alpha\in\mathcal{E}^i(X)_{\lambda,\delta}$. This support condition is a generalization of Peetre's characterization [34] of differential operators in the $X = Y$ case ([27, Def. 2.1], for instance).

We address a general problem motivated by conformal geometry:

Problem 9 (conformally covariant symmetry breaking operators). *Let $X \supset Y$ are orientable Riemannian manifolds.*

(1) *Determine when* $H\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right) \neq \{0\}$.

(2) *Determine when* $D\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right) \neq \{0\}$.

(3) *Construct an explicit basis of* $H\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right)$ *and* $D\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right)$.

Problem 9 (1) and (2) may be thought of as Stage B of branching problems in Section 1, while Problem 9 (3) as Stage C.

In the case where $X = Y$ and $i = j = 0$, a classical prototype of such operators is a second-order differential operator called the Yamabe operator

$$\Delta + \frac{n-2}{4(n-1)}\kappa \in \mathrm{Diff}_{\mathrm{Conf}(X)}(\mathcal{E}^0(X)_{\frac{n}{2}-1,\delta}, \mathcal{E}^0(X)_{\frac{n}{2}+1,\delta}),$$

where n is the dimension of the manifold X, Δ is the Laplacian, and κ is the scalar curvature, see [24, Thm. A], for instance. Conformally covariant differential operators of higher order are also known: the Paneitz operator (fourth-order) [33], or more generally, the so-called GJMS operators [5] are such operators. Turning to operators acting on differential forms, we observe that the exterior derivative d, the codifferential d^*, and the Hodge $*$ operator are also examples of conformally covariant operators on differential forms, namely, $j = i+1$, $i-1$, and $n-i$, respectively, with an appropriate choice of the parameter $(\lambda, \nu, \delta, \varepsilon)$. As is well known, Maxwell's equations in four-dimension can be expressed in terms of conformally covariant operators on differential forms.

Let us consider the general case where $X \neq Y$. From the viewpoint of conformal geometry, we are interested in "natural operators" T that persist for all pairs of Riemannian manifolds $X \supset Y$ of fixed dimension. We note that Problem 9 is trivial for individual pairs $X \supset Y$ such that $\mathrm{Conf}(X;Y) = \{e\}$, because any linear operator becomes automatically an SBO. In contrast, the larger $\mathrm{Conf}(X;Y)$ is, the more constraints on T will be imposed. Thus we highlight the case of large conformal groups as the first step to attack Problem 9.

In general, the conformal group cannot be so large. We recall from [10, Thms. 6.1 and 6.2] the upper estimate of the dimension of the conformal group:

Fact 10. *Let X be a compact Riemannian manifold of dimension $n \geq 3$. Then* $\dim \mathrm{Conf}(X) \leq \frac{1}{2}(n+1)(n+2)$. *The equality holds if and only if* $(\mathrm{Conf}(X), X)$ *is locally isomorphic to* $(O(n+1,1), S^n)$.

Concerning a pair (X, Y) of Riemannian manifolds, we obtain the following.

Proposition 11. *Let $X \supset Y$ be Riemannian manifolds of dimension n and m, respectively. Then* $\dim \mathrm{Conf}(X;Y) \leq \frac{1}{2}(m+1)(m+2)$. *The equality holds if $X = S^n$ and Y is a totally geodesic submanifold which is isomorphic to S^m.*

Proof. The first inequality follows from Fact 10 via the group homomorphism (4). If $(X,Y) = (S^n, S^m)$, then $\mathrm{Conf}(X)$ and $\mathrm{Conf}(X;Y)$ are locally isomorphic to $O(n+1,1)$ and $O(m+1,1)$, respectively, whence the second assertion. $\square$

From now on, we shall consider the pair

$$(X,Y) = (S^n, S^{n-1}), \tag{7}$$

as a model case with largest symmetries, where $Y = S^{n-1}$ is embedded as a totally geodesic submanifold of $X = S^n$. As mentioned, the pair $(\mathrm{Conf}(X), \mathrm{Conf}(X;Y))$ is locally isomorphic to the pair

$$(G, G') = (O(n+1,1), O(n,1)). \tag{8}$$

We remind that this pair appeared in Section 3 on branching problems, see the case where $r = 1$ in Example 7. As an *a priori* estimate in Stage A, see Theorem 3 (2), Example 7, [21, Thm. 2.6], and [35], we have

$$\dim_{\mathbb{C}} H\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right) \le 4 \quad \text{for any } (i,j,\lambda,\nu,\delta,\varepsilon). \tag{9}$$

In turn, the estimate (9) gives an upper bound for the dimension of the space of "natural" conformal covariant SBOs, $\mathcal{E}^i(X)_{\lambda,\delta} \to \mathcal{E}^j(Y)_{\nu,\varepsilon}$ that persist for all pairs $X \supset Y$ of codimension one. In the next two sections, we explain briefly a solution to Problem 9 (Stages B and C) in the model case (7).

5. Classification theory of conformally covariant differential SBOs

In the case where symmetry breaking operators are given as differential operators, Problem 9 in the model space (7) was solved in a joint work [21] with Kubo and Pevzner. In this section, we introduce its flavors briefly. First of all, the solution to Problem 9 (2), a question in Stage B of branching problems, may be stated as follows.

Theorem 12. *Suppose $n \ge 3$, $0 \le i \le n$, $0 \le j \le n-1$, $\lambda, \nu \in \mathbb{C}$, and $\delta, \varepsilon \in \{\pm\}$. Then the following three conditions on 6-tuple $(i,j,\lambda,\nu,\delta,\varepsilon)$ are equivalent:*

(i) $D\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right) \neq \{0\}$.

(ii) $\dim_{\mathbb{C}} D\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right) = 1$.

(iii) *The parameter $(i,j,\lambda,\nu,\delta,\varepsilon)$ satisfies*

$$\{j, n-j-1\} \cap \{i-2, i-2, i, i+1\} \neq \emptyset, \tag{10}$$

$$\nu - \lambda \in \mathbb{N},$$

$$\text{a certain condition } Q \equiv Q_{i,j} \text{ on } (\lambda,\nu,\delta,\varepsilon). \tag{11}$$

The first condition (10) concerns the degrees i and j of differential forms. Loosely speaking, conformally covariant differential SBOs exist only if the degrees i and j are close to each other or the sum $i + j$ is close to n. The last "additional" condition $Q_{i,j}$ depends on (i, j). We give the condition $Q_{i,j}$ explicitly in the following two cases:

- Case $j = i$. $Q_{i,i}$ amounts to $\nu \in \mathbb{C}$ and $\delta \equiv \varepsilon \equiv \nu - \lambda \mod 2$.
- Case $j = i + 1$. For $1 \leq i \leq n - 2$, $Q_{i,i+1}$ amounts to $(\lambda, \nu) = (0, 0)$ and $\delta \equiv \varepsilon \equiv 0 \mod 2$; for $i = 0$, $Q_{0,1}$ amounts to $\lambda \in -\mathbb{N}$, $\nu = 0$, and $\delta \equiv \varepsilon \equiv \lambda$ mod 2.

See [21, Thm. 1.1] for the precise conditions in the other remaining six cases.

Second, we go on with Problem 9 (3) (Stage C) about the construction of symmetry breaking operators. For this we work with the pair $(\mathbb{R}^n, \mathbb{R}^{n-1})$ of the flat Riemannian manifolds which are conformal to $(S^n \setminus \{\mathrm{pt}\}, S^{n-1} \setminus \{\mathrm{pt}\})$ via the stereographic projection.

We begin with a *scalar-valued* operator (Juhl's operator, [8]). Suppose that our hyperplane $Y = \mathbb{R}^{n-1}$ of $X = \mathbb{R}^n$ is defined by $x_n = 0$ in the coordinates $(x_1, \ldots, x_n)$. For $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$, we define a homogeneous differential operator of order k by

$$\mathcal{D}_k^\mu := \sum_{0 \leq i \leq [\frac{k}{2}]} a_i(\mu)(-\Delta_{\mathbb{R}^{n-1}})^i \frac{\partial^{k-2i}}{\partial x_n^{k-2i}} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n),$$

where $\{a_i(\mu)\}$ are the coefficients of the Gegenbauer polynomial:

$$C_k^\mu(t) = \sum_{0 \leq i \leq [\frac{k}{2}]} a_i(\mu) t^{k-2i}.$$

Building on the scalar-valued operators, we introduced in [21] *matrix-valued* differential symmetry breaking operators

$$\mathcal{D}_{\lambda,k}^{i \to j} : \mathcal{E}^i(\mathbb{R}^n) \to \mathcal{E}^j(\mathbb{R}^{n-1})$$

for each pair (i, j) satisfying (10). We illustrate a concrete formula when $j = i$. We set

$$\mathcal{D}_{\lambda,k}^{i \to i} := \mathrm{Rest}_{x_n=0} \circ (\mathcal{D}_{k-2}^{\mu+1} dd^* + a\mathcal{D}_{k-1}^\mu d\iota_{\frac{\partial}{\partial x_n}} + b\mathcal{D}_k^\mu),$$

where d^* is the codifferential, $\iota_{\frac{\partial}{\partial x_n}} : \mathcal{E}^i(\mathbb{R}^n) \to \mathcal{E}^j(\mathbb{R}^{n-1})$ is the inner multiplication of the vector field $\frac{\partial}{\partial x_n}$, and

$$a := \begin{cases} 1 & (k\text{: odd}) \\ \lambda + i - \frac{n}{2} + k & (k\text{: even}) \end{cases}, \quad b := \frac{\lambda + k}{2}, \quad \mu := \lambda + i - \frac{n-1}{2}.$$

Thus $\mathcal{D}_{\lambda,k}^{i \to i}$ is obtained as the composition of a $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^i(\mathbb{C}^n), \bigwedge^i(\mathbb{C}^{n-1}))$-valued homogeneous differential operator on $\mathbb{R}^n$ of order k with the restriction map to the hyperplane $\mathbb{R}^{n-1}$.

The matrix-valued differential operators $\mathcal{D}_{\lambda,k}^{i\to j}\colon \mathcal{E}^i(\mathbb{R}^n) \to \mathcal{E}^j(\mathbb{R}^{n-1})$ were defined in [21, Chap. 1] also for the other seven cases when the condition (iii) in Theorem 12 is fulfilled.

Methods of proof in finding the formulæ for $\mathcal{D}_{\lambda,k}^{i\to j}$. The approach in [21] is based on the *F-method* [16], which reduces a problem of finding the operators $\mathcal{D}_{\lambda,k}^{i\to j}$ to another problem of finding polynomial solutions to a system of ordinary differential equations (*F-system*). An alternative approach for $j = i-1, i$ is given in [20] by taking the residues of the regular symmetry breaking operators (see also Section 6 below).

With the aforementioned operators $\mathcal{D}_{\lambda,k}^{i\to j}$, Problem 9 (3) for differential operators were solved in [21, Thms. 1.4–1.8], which may be thought of as an answer to Stage C of branching problems. We illustrate the results with the following two theorems in the case where $j = i$ and $i+1$.

Theorem 13 ($j = i$ case). *Suppose $\nu \in \mathbb{C}$, $k := \nu - \lambda \in \mathbb{N}$, and $\delta \equiv \varepsilon \equiv k \mod 2$.*

(1) *The linear map $\mathcal{D}_{\lambda,k}^{i\to i}$ extends to a conformally covariant symmetry breaking operator from $\mathcal{E}^i(S^n)_{\lambda,\delta}$ to $\mathcal{E}^i(S^{n-1})_{\nu,\varepsilon}$.*
(2) *Conversely, any conformally covariant differential symmetry breaking operator from $\mathcal{E}^i(S^n)_{\lambda,\delta}$ to $\mathcal{E}^i(S^{n-1})_{\nu,\varepsilon}$ is proportional to $\mathcal{D}_{\lambda,k}^{i\to i}$, or its renormalization* ([21, (1.10)]).

Theorem 14 ($j = i+1$ case).

(1) *Suppose $1 \le i \le n-2$, $(\lambda,\nu) = (n-2i, n-2i+3)$, and $\delta \equiv \varepsilon \equiv 1 \mod 2$. Then the linear map*

$$\mathrm{Rest} \circ d\colon \mathcal{E}^i(S^n)_{\lambda,\delta} \to \mathcal{E}^{i+1}(S^{n-1})_{\nu,\varepsilon}$$

is a conformally covariant SBO. Conversely, a nonzero conformally covariant differential SBO from $\mathcal{E}^i(S^n)_{\lambda,\delta}$ to $\mathcal{E}^{i+1}(S^{n-1})_{\nu,\varepsilon}$ exists only for the above parameters, and such an operator is proportional to Rest $\circ\, d$.

(2) *Suppose $i = 0$, $\lambda \in \{0,-1,-2,\dots\}$, $\nu = 0$, and $\delta \equiv \varepsilon \equiv \lambda \mod 2$. Then the linear map*

$$\mathrm{Rest}_{x_n=0} \circ \mathcal{D}_{-\lambda}^{\lambda-\frac{n-1}{2}} \circ d\colon \mathcal{E}^0(\mathbb{R}^n) \to \mathcal{E}^1(\mathbb{R}^{n-1})$$

extends to a conformally covariant SBO from $\mathcal{E}^0(S^n)_{\lambda,\delta}$ to $\mathcal{E}^1(S^{n-1})_{0,\varepsilon}$. Conversely, a nonzero conformally covariant differential SBO from $\mathcal{E}^0(S^n)_{\lambda,\delta}$ to $\mathcal{E}^1(S^{n-1})_{\nu,\varepsilon}$ exists only for the above parameters, and such an operator is proportional to the above operator.

Remark 15.

(1) By using the Hodge $*$ operator on X or its submanifold Y, the other six cases can be reduced to either the $j = i$ case (Theorem 13) or the $j = i+1$ case (Theorem 14). The construction and classification of differential symmetry

breaking operators in the model space (7) is thus completed. Its generalization to the pseudo-Riemannian case is proved in [22].

(2) Special cases of Theorem 13 were known earlier. The case $j = i = 0$ (scalar-valued case) was discovered by A. Juhl [8]. Different approaches have been proposed by Fefferman–Graham [4], Kobayashi–Ørsted–Souček–Somberg [25], and Clerc [3] among others. Our approach uses an algebraic Fourier transform of Verma modules (*F-method*), see [16, 27].

(3) The case $n = 2$ is closely related to the celebrated Rankin–Cohen bidifferential operator via holomorphic continuation [28].

6. Classification theory: nonlocal conformally covariant SBOs

In this section we consider nonlocal operators such as integral operators as well, and thus complete the classification problem (Problem 9) for the model space $(X, Y) = (S^n, S^{n-1})$.

Building on the classification results on $D\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right)$ in Section 5, we want to

- find $\dim_{\mathbb{C}} H\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right) / D\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right)$;
- find a basis in $H\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right)$ modulo $D\left(\begin{array}{c|c} i & j \\ \lambda, \delta & \nu, \varepsilon \end{array}\right)$.

This idea fits well with the general strategy to understand the whole space of symmetry breaking operators between principal series representations of a reductive group and its subgroup G' by using the filtration given by the support of distribution kernels [29, Chap. 11, Sec. 2]. Thus we start with the general setting where (G, G') is a pair of real reductive Lie groups. Let $P = MAN$ and $P' = M'A'N'$ be Langlands decompositions of minimal parabolic subgroups of G and G', respectively. For an irreducible representation (σ, V) of M and a one-dimensional representation $\mathbb{C}_\lambda$ of A, we define a principal series representation of G by unnormalized parabolic induction

$$I(\sigma, \lambda) := \operatorname{Ind}_P^G(\sigma \otimes \mathbb{C}_\lambda \otimes \mathbf{1}).$$

Similarly, we define that of the subgroup G', to be denoted by

$$J(\tau, \nu) := \operatorname{Ind}_{P'}^{G'}(\tau \otimes \mathbb{C}_\nu \otimes \mathbf{1})$$

for an irreducible representation (τ, W) of M' and a one-dimensional representation $\mathbb{C}_\nu$ of A'.

By abuse of notation, we identify a representation with its representations space, and set $V_\lambda := V \otimes \mathbb{C}_\lambda$ and $W_\nu := W \otimes \mathbb{C}_\nu$. Let $\mathcal{V}_\lambda^*$ be the dualizing bundle of the G-homogeneous bundle $G \times_P V_\lambda$ over the real flag manifold G/P. Then there is a natural linear bijection between the space of symmetry breaking operators and the space of invariant distributions (see [29, Prop. 3.2]):

$$\mathrm{Hom}_{G'}(I_\delta(\sigma,\lambda)|_{G'}, J_\varepsilon(\tau,\nu)) \xrightarrow{\sim} (\mathcal{D}'(G/P,\mathcal{V}_\lambda^*)\otimes W_\nu)^{\Delta(P')}, \quad T \mapsto K_T, \tag{12}$$

Suppose now that the condition (PP) in Theorem 3 is fulfilled. Then this implies that $\#(P'\backslash G/P) < \infty$, see [26, Rem. 2.5 (4)]. We denote by $\{Z_\alpha\}$ the totality of P'-orbits on G/P. We define a partial order $\alpha \prec \beta$ by $Z_\alpha \subset \overline{Z_\beta}$, the closure of Z_β in G/P. Then there is the unique minimal index $\alpha_{\min}$ corresponding to the closed P'-orbit in G/P, and maximal ones $\beta_1, \ldots, \beta_N$ corresponding to open P'-orbits in G/P.

We observe that the support $\mathrm{Supp}(K_T)$ of the distribution kernel K_T is a closed P'-invariant subset of G/P, and accordingly, define

$$H(\alpha) \equiv H^{\sigma,\lambda}_{\tau,\nu}(\alpha) := \{T \in \mathrm{Hom}_{G'}(I(\sigma,\lambda)|_{G'}, J(\tau,\nu)) : \mathrm{Supp}(K_T) \subset \overline{Z_\alpha}\}$$

via the isomorphism (12). Clearly, $H(\alpha) \subset H(\beta)$ if $\alpha \prec \beta$. It follows from [27, Lem. 2.3] that

$$H(\alpha_{\min}) = \mathrm{Diff}_{G'}(I(\sigma,\lambda)|_{G'}, J(\tau,\nu)).$$

In contrast to the smallest support $Z_{\alpha_{\min}}$, a symmetry breaking operator T is called *regular* ([29, Def. 3.3]) if $\mathrm{Supp}(K_T)$ contains Z_{β_j} for some $1 \le j \le N$.

We now return to the special setting (8). Then the Levi subgroup MA of the minimal parabolic subgroup $P = MAN$ of $G = O(n+1,1)$ is given by $(O(n) \times O(1)) \times \mathbb{R}$. For $0 \le i \le n$, $\delta \in \{\pm\}$, and $\lambda \in \mathbb{C}$, we consider the outer tensor product representation $\bigwedge^i(\mathbb{C}^n) \boxtimes \delta \boxtimes \mathbb{C}_\lambda$ of MA, and extend it to P by letting N act trivially. The resulting P-module is denoted simply by $\bigwedge^i(\mathbb{C}^n) \otimes \delta \otimes \mathbb{C}_\lambda$. We define an unnormalized principal series representation of $G = O(n+1,1)$ by

$$I_\delta(i,\lambda) \equiv I(\textstyle\bigwedge^i(\mathbb{C}^n) \boxtimes \delta, \lambda) := \mathrm{Ind}_P^G(\textstyle\bigwedge^i(\mathbb{C}^n) \otimes \delta \otimes \mathbb{C}_\lambda).$$

Lemma 16. *Let $0 \le i \le n$, $\delta \in \{\pm\}$, $\lambda \in \mathbb{C}$.*

(1) *The G-module $I_\delta(i,\lambda)$ is irreducible if $\lambda \notin \mathbb{Z}$.*
(2) *There is a natural isomorphism $\mathcal{E}^i(S^n)_{\lambda,\delta} \simeq I_{(-1)^i\delta}(i,\lambda+i)$ as G-modules.*

For the proof of Lemma 16 (2), see [21, Prop. 2.3].

Lemma 16 (2) suggests that we can reformulate Problem 9 about differential forms on the pair of conformal manifolds (7) into a question of symmetry breaking operators between principal series representations for the pair (8) of reductive groups. We write $\widetilde{D}$ and $\widetilde{H}$ if we use $I_\delta(i,\lambda)$ and $J_\varepsilon(j,\nu) = \mathrm{Ind}_{P'}^{G'}(\bigwedge^j(\mathbb{C}^{n-1}) \otimes \varepsilon \otimes \mathbb{C}_\nu)$ instead of D and H in (6) and (5), respectively. By Lemma 16 (2), we have

$$H\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right) = \widetilde{H}\left(\begin{array}{c|c} i & j \\ \lambda+i, (-1)^i\delta & \nu+j, (-1)^j\varepsilon \end{array}\right)$$

and similarly for D and $\widetilde{D}$. Thus we want to

- find $\dim_{\mathbb{C}} \widetilde{H}\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right) / \widetilde{D}\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right)$;
- find a basis in $\widetilde{H}\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right)$ modulo $\widetilde{D}\left(\begin{array}{c|c} i & j \\ \lambda,\delta & \nu,\varepsilon \end{array}\right)$.

First, we obtain:

Theorem 17 (localness theorem). *If $j \neq i-1$ or i, then*

$$\widetilde{H}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix} = \widetilde{D}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix}.$$

In the setting (8), there exists a unique open P'-orbit in G/P, and accordingly, there exists at most one family of (generically) regular symmetry breaking operators from the G-modules $I_\delta(i,\lambda)$ to the G'-modules $J_\varepsilon(j,\nu)$. We prove that such a family exists if and only if $j = i-1$ or i, and it plays a crucial role in the classification problem of SBOs modulo the space $\widetilde{D}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix}$ of differential SBOs as follows. We introduce the set of "special parameters" by

$$\Psi_{\mathrm{sp}} := \big\{(\lambda,\nu,\delta,\varepsilon) \in \mathbb{C}^2 \times \{\pm\}^2 : \nu-\lambda \in 2\mathbb{N} \text{ when } \delta\varepsilon = 1 \text{ or } \nu-\lambda \in 2\mathbb{N}+1 \text{ when } \delta\varepsilon = -1\big\}. \tag{13}$$

Theorem 18. *Suppose $j = i-1$ or i, and $\delta,\varepsilon \in \{\pm\}$. Then there exists a family of continuous G'-homomorphism*

$$\widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon} \colon I_\delta(i,\lambda) \to J_\varepsilon(j,\nu)$$

such that $\widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon}$ depends holomorphically on $(\lambda,\nu) \in \mathbb{C}^2$ and that the set of the zeros of $\widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon}$ is discrete in $(\lambda,\nu) \in \mathbb{C}^2$.

(1) *If $(\lambda,\nu,\delta,\varepsilon) \notin \Psi_{\mathrm{sp}}$ then $\widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon} \neq 0$ and*

$$\widetilde{H}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix} = \mathbb{C}\widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon} \supsetneqq \widetilde{D}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix} = \{0\}.$$

(2) *If $(\lambda,\nu,\delta,\varepsilon) \in \Psi_{\mathrm{sp}}$ and $\widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon} \neq 0$, then*

$$\widetilde{H}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix} = \widetilde{D}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix}.$$

(3) *If $(\lambda,\nu,\delta,\varepsilon) \in \Psi_{\mathrm{sp}}$ and $\widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon} = 0$, then*

$$\dim_{\mathbb{C}} \widetilde{H}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix} = \dim_{\mathbb{C}} \widetilde{D}\begin{pmatrix} i & j \\ \lambda,\delta & \nu,\varepsilon \end{pmatrix} + 1.$$

The discrete set $\{(i,j,\lambda,\nu,\delta,\varepsilon) : \widetilde{\mathbb{A}}^{i,j}_{\lambda,\nu,\delta\varepsilon} = 0\}$ has been determined in [20], and thus the classification of conformally covariant symmetry breaking operators

$$\mathcal{E}^i(X)_{\lambda,\delta} \to \mathcal{E}^j(Y)_{\nu,\varepsilon}$$

for the model space $(X,Y) = (S^n, S^{n-1})$ is accomplished. A detailed proof for the classification together with some important properties of symmetry breaking operators (Stage C) such as

- (K, K')-spectrum (a generalized eigenvalue),
- functional equations,
- residue formulæ,

will be given in separate papers (see [20] for the residue formulæ, and [30] for the classification).

7. Application to periods and automorphic form theory

Let G be a reductive group, and H a reductive subgroup.

Definition 19. An irreducible admissible smooth representation Π of G is H-*distinguished* if $\mathrm{Hom}_H(\Pi|_H, \mathbb{C}) \neq \{0\}$. In this case, it is also said that Π has an H-*period*. By the Frobenius reciprocity theorem, the condition is equivalent to $\mathrm{Hom}_G(\Pi, C^\infty(G/H)) \neq \{0\}$.

In this section, we discuss an application of symmetry breaking operators to find periods (Definition 19) of irreducible unitary representations. We highlight the case when Π has nonzero $(\mathfrak{g}, K)$-cohomologies. The motivation comes from automorphic form theory, of which we now recall a prototype.

Fact 20 (Matsushima–Murakami, [2]). *Let Γ be a cocompact discrete subgroup of G. Then we have*

$$H^*(\Gamma\backslash G/K; \mathbb{C}) \simeq \bigoplus_{\Pi \in \widehat{G}} \mathrm{Hom}_G(\Pi, L^2(\Gamma\backslash G)) \otimes H^*(\mathfrak{g}, K; \Pi_K).$$

The left-hand side gives topological invariants of the locally symmetric space $M = \Gamma\backslash G/K$, whereas the right-hand side is described in terms of the representation theory. We note that $\mathrm{Hom}_G(\Pi, L^2(\Gamma\backslash G))$ is finite-dimensional for all $\Pi \in \widehat{G}$ by a theorem of Gelfand–Piateski–Shapiro, and the sum is taken over the following finite set

$$\widehat{G}_{\mathrm{cohom}} := \{\Pi \in \widehat{G} : H^*(\mathfrak{g}, K; \Pi_K) \neq \{0\}\},$$

which was classified by Vogan and Zuckerman [36].

In the case where $G = O(n+1, 1)$, there are $2(n+1)$ elements in $\widehat{G}_{\mathrm{cohom}}$. Following the notation in [21, Thm. 2.6], we label them as

$$\{\Pi_{\ell,\delta} : 0 \leq \ell \leq n+1, \delta \in \{\pm\}\},$$

and we define

$$\begin{aligned} \mathrm{Index} \equiv \mathrm{Index}_G \colon \widehat{G}_{\mathrm{cohom}} &\to \{0, 1, \ldots, n+1\}, & \Pi_{\ell,\delta} &\mapsto \ell, \\ \mathrm{sgn} \equiv \mathrm{sgn}_G \colon \widehat{G}_{\mathrm{cohom}} &\to \{\pm\}, & \Pi_{\ell,\delta} &\mapsto \delta. \end{aligned}$$

We illustrate the labeling by two examples:

Example 21 (one-dimensional representations). *There are four one-dimensional representations of G, which are given as*

$$\{\Pi_{0,+} \simeq \mathbf{1}, \Pi_{0,-}, \Pi_{n+1,+}, \Pi_{n+1,-} \simeq \det\}.$$

Example 22 (tempered representations). *For n odd Π is the smooth representation of a discrete series representation of G iff* $\mathrm{Index}(\Pi) = \frac{1}{2}(n+1)$, *whereas for n even Π is that of tempered representation of G iff* $\mathrm{Index}(\Pi) \in \{\frac{n}{2}, \frac{n}{2}+1\}$.

We give a necessary and sufficient condition for the existence of symmetry breaking operators between irreducible representations of G and those of the subgroup G' with nonzero $(\mathfrak{g}, K)$-cohomologies:

Theorem 23 ([30]). *Let* $(G, G') = (O(n+1,1), O(n,1))$, *and* $(\Pi, \pi) \in \widehat{G}_{\mathrm{cohom}} \times \widehat{G'}_{\mathrm{cohom}}$. *Then the following three conditions on (Π, π) are equivalent.*

(i) $\mathrm{Hom}_{G'}(\Pi^\infty|_{G'}, \pi^\infty) \neq \{0\}$.
(ii) *The outer tensor product representation* $\Pi^\infty \boxtimes \pi^\infty$ *is* $\mathrm{diag}(G')$*-distinguished.*
(iii) $\mathrm{Index}_G(\Pi) - 1 \leq \mathrm{Index}_{G'}(\pi) \leq \mathrm{Index}_G(\Pi)$ *and* $\mathrm{sgn}(\Pi) = \mathrm{sgn}(\pi)$.

The proof uses the symmetry breaking operators that are discussed in Section 6 and the relationship between $\widehat{G}_{\mathrm{cohom}}$ and conformal representations on differential forms on the sphere S^n summarized as below.

Lemma 24 ([21, Thm. 2.6]). *If* $\Pi \in \widehat{G}_{\mathrm{cohom}}$, *then* Π^∞ *can be realized as a subrepresentation of* $\mathcal{E}^i(S^n)_{0,\delta}$ *with* $i = \mathrm{Index}_G(\Pi)$ *and* $\delta = (-1)^i \mathrm{sgn}_G(\Pi)$ *if* $\mathrm{Index}_G(\Pi) \neq n+1$, *and also as a quotient of* $\mathcal{E}^i(S^n)_{0,\delta}$ *with* $i = \mathrm{Index}_G(\Pi) - 1$ *and* $\delta = (-1)^i \mathrm{sgn}_G(\Pi)$ *if* $\mathrm{Index}_G(\Pi) \neq 0$.

To end this section, we consider a tower of subgroups of a reductive group G:

$$\{e\} = G^{(0)} \subset G^{(1)} \subset \cdots \subset G^{(n)} \subset G^{(n+1)} = G.$$

Accordingly, there is a family of homogeneous spaces with G-equivariant quotient maps:

$$G = G/G^{(0)} \to G/G^{(1)} \to \cdots \to G/G^{(n+1)} = \{\mathrm{pt}\}.$$

In turn, we have natural inclusions of G-modules:

$$C^\infty(G) = C^\infty(G/G^{(0)}) \supset C^\infty(G/G^{(1)}) \supset \cdots \supset C^\infty(G/G^{(n+1)}) = \mathbb{C}.$$

A general question is:

Problem 25. *Let* $\Pi \in \widehat{G}_{\mathrm{smooth}}$. *Find k as large as possible such that Π is $G^{(k)}$-distinguished, or equivalently, such that the smooth representation Π^∞ can be realized in* $C^\infty(G/G^{(k)})$.

Any irreducible admissible smooth representation of G can be realized in the regular representation on $C^\infty(G/G^{(0)}) \simeq C^\infty(G)$ via matrix coefficients, whereas irreducible representations that can be realized in $C^\infty(G/G^{(0)}) = \mathbb{C}$ is the trivial one-dimensional representation $\mathbf{1}$.

Suppose that $G = O(n+1,1)$, and consider a chain of subgroups of G by

$$G^{(k)} := O(k,1) \quad (0 \leq k \leq n+1).$$

Then $G^{(n+1)} = G$, however, $G^{(0)}$ is not exactly $\{e\}$ but $G^{(0)} = O(1)$ is a finite group of order two. Accordingly, we consider $\Pi \in \widehat{G}_{\mathrm{cohom}}$ with $\mathrm{sgn}(\Pi) = +$ below.

Theorem 26. *Suppose* $\Pi \in \widehat{G}_{\mathrm{cohom}}$ *with* $\mathrm{sgn}(\Pi) = +$*. Then*

$$\mathrm{Hom}_G(\Pi^\infty, C^\infty(G/G^{(k)})) \neq \{0\} \quad \textit{for all } k \leq n+1-\mathrm{Index}_G(\Pi).$$

Example 27 (one-dimensional representations). *Suppose that* $\Pi \in \widehat{G}_{\mathrm{cohom}}$ *with* $\mathrm{sgn}_G(\Pi) = +$*. We consider two opposite extremal cases, i.e.,* $\mathrm{Index}_G(\Pi) = 0$ *and* $= n+1$*. If* $\mathrm{Index}_G(\Pi) = 0$*, then* Π *is isomorphic to the trivial one-dimensional representation* $\mathbf{1}$*, and can be realized in* $C^\infty(G/G^{(k)})$ *for all* $0 \leq k \leq n+1$ *as in Theorem* 26*. On the other hand, if* $\mathrm{Index}_G(\Pi) = n+1$*, then* Π *is another one-dimensional representation of* G *(*$\Pi_{n+1,+} \simeq \chi_{-+}$ *with the notation [21, (2.9)]). In this case,* Π *can be realized in* $C^\infty(G/G^{(k)})$ *iff* $k = 0$*, namely, iff* $G^{(k)} = O(1)$*.*

Remark 28. The size of an (infinite-dimensional) representation could be measured by its Gelfand–Kirillov dimension, or more precisely, by its associated variety or by the partial flag variety for which its localization can be realized as a $\mathcal{D}$-module. Then one might expect the following assertion:

$$\begin{aligned}&\textit{the larger the isotropy subgroup } G^{(k)} \textit{ is (i.e., the larger } k \textit{ is),}\\ &\textit{the "smaller" irreducible subrepresentations of } C^\infty(G/G^{(k)}) \textit{ become.}\end{aligned} \tag{14}$$

This is reflected partially in Theorem 26, however, Theorem 26 asserts even sharper results. To see this, we set

$$r := \min(\mathrm{Index}_G(\Pi), n+1-\mathrm{Index}_G(\Pi)).$$

Then the underlying $(\mathfrak{g}, K)$-module Π_K can be expressed as a cohomological parabolic induction from a θ-stable parabolic subalgebra $\mathfrak{q}_r$ with Levi subgroup $N_G(\mathfrak{q}_r) \simeq SO(2)^r \times O(n+1-2r, 1)$ ([9], see also [11, Thm. 3]). Theorem 26 tells that if $n+1 \leq 2k$, then the larger k is, the smaller $r = \mathrm{Index}_G(\Pi)$ becomes, namely, the smaller the $(\mathfrak{g}, K)$-modules that are cohomologically parabolic induced modules from $\mathfrak{q}_r$ become. This matches (14). On the other hand, if $2k \leq n+1$, then the constraints in Theorem 26 provide an interesting phenomenon which is opposite to (14) because $r = n+1-\mathrm{Index}_G(\Pi)$, and thus suggest sharper estimates than (14). For instance, the representation $\Pi_{n+1,+} (\simeq \chi_{-+})$ is "small" because it is one-dimensional, but it can be realized in $C^\infty(G/G^{(k)})$ only for $k = 0$ as we saw in Example 27.

Remark 29 (comparison with L^2-theory). Theorem 26 implies that the smooth representation Π^∞ of a tempered representation Π with nonzero $(\mathfrak{g}, K)$-cohomologies (see Example 22) occurs in $C^\infty(G/G^{(k)})$ if $k \leq \frac{n}{2}+1$. On the other hand, for a reductive homogeneous space G/H, a general criterion for the unitary representation $L^2(G/H)$ to be tempered was proved in a joint work [1] with Y. Benoist by a geometric method. In particular, the unitary representation $L^2(G/G^{(k)})$ is tempered if and only if $k \leq \frac{n}{2}+1$, see [1, Ex. 5.10].

Acknowledgement

The author was partially supported by Grant-in-Aid for Scientific Research (A) (25247006), Japan Society for the Promotion of Science. This article is based on the plenary lecture that the author delivered at the XXXVIth Workshop on Geometric Methods in Physics at Białowieża in 2017.

References

[1] Y. Benoist, T. Kobayashi, *Temperedness of reductive homogeneous spaces*. J. Eur. Math. Soc. **17**, (2015), pp. 3015–3036.

[2] A. Borel, N. Wallach, Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups. Math. Surveys Monogr. **67**, Amer. Math. Soc., 2000.

[3] J.-L. Clerc, *Another approach to Juhl's conformally covariant differential operators from S^n to S^{n-1}*. SIGMA, **13**, (2017), Paper No. 026, 18 pp.

[4] C. Fefferman, C.R. Graham, *Juhl's formulae for GJMS operators and Q-curvatures*, J. Amer. Math. Soc. **26**, (2013), 1191–1207.

[5] C.R. Graham, R. Jenne, L.J. Mason, G.A.J. Sparling, *Conformally invariant powers of the Laplacian.* I. *Existence*. J. London Math. Soc., (2) **46**, (1992), 557–565.

[6] B. Gross, D. Prasad, *On the decomposition of a representations of SO_n when restricted to SO_{n-1}*. Canad. J. Math. **44**, (1992), no. 5 974–1002.

[7] R. Howe, *θ-series and invariant theory*. Proc. Symp. Pure Math. **33**, (1979), Amer. Math. Soc., 275–285.

[8] A. Juhl, Families of Conformally Covariant Differential Operators, Q-Curvature and Holography. Progr. Math., **275**, Birkhäuser, Basel, 2009.

[9] A.W. Knapp, D. Vogan, Cohomological induction and unitary representations. Princeton Mathematical Series, **45**. Princeton University Press, 1995. xx+948 pp. ISBN: 0-691-03756-6.

[10] S. Kobayashi, Transformation Groups in Differential Geometry. Reprint of the 1972 edition. Classics in Mathematics (Springer-Verlag, Berlin, 1995).

[11] T. Kobayashi, Singular Unitary Representations and Discrete Series for Indefinite Stiefel Manifolds $U(p,q;\mathbb{F})/U(p-m,q;\mathbb{F})$. Mem. Amer. Math. Soc., **462**, 1992. 106 pp. ISBN:9810210906.

[12] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications*. Invent. Math., **117**, (1994), 181–205.

[13] T. Kobayashi, *Introduction to harmonic analysis on real spherical homogeneous spaces*. Proceedings of the 3rd Summer School on Number Theory "Homogeneous Spaces and Automorphic Forms" in Nagano (F. Sato, ed.), 1995, 22–41 (in Japanese).

[14] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups* II – *micro-local analysis and asymptotic K-support*. Ann. of Math., **147**, (1998), 709–729.

[15] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups* III – *restriction of Harish-Chandra modules and associated varieties*. Invent. Math., **131**, (1998), 229–256.

[16] T. Kobayashi, *F-method for constructing equivariant differential operators.* Contemp. Math. **598**, (2013), pp. 139–146, Amer. Math. Soc., Providence, RI.

[17] T. Kobayashi, *Propagation of multiplicity-freeness property for holomorphic vector bundles.* 113–140, Progr. Math., **306**, Birkhäuser/Springer, New York, 2013.

[18] T. Kobayashi, *Shintani functions, real spherical manifolds, and symmetry breaking operators.* Dev. Math., **37**, (2014), 127–159.

[19] T. Kobayashi, *A program for branching problems in the representation theory of real reductive groups.* In: Representations of Lie Groups, In Honor of David A. Vogan, Jr. on his 60th Birthday, Progr. Math. **312**, pp. 277–322. Birkhäuser, 2015.

[20] T. Kobayashi, *Residue formula for regular symmetry breaking operators for differential forms.* preprint, arXiv:1709.05035.

[21] T. Kobayashi, T. Kubo, M. Pevzner, Conformal Symmetry Breaking Differential Operators for Differential Forms on Spheres. Lecture Notes in Math., vol. **2170**, iv + 192 pages, 2016. ISBN: 978-981-10-2657-7.

[22] T. Kobayashi, T. Kubo, M. Pevzner, *Conformal symmetry breaking operators for anti-de Sitter spaces.* arXiv:1610.09475, to appear in Trends in Math., pp. 69–85.

[23] T. Kobayashi, T. Matsuki, *Classification of finite-multiplicity symmetric pairs.* Transform. Groups, **19**, (2014), pp. 457–493, Special issue in honor of Dynkin for his 90th birthday.

[24] T. Kobayashi, B. Ørsted, *Analysis on the minimal representation of* $O(p,q)$. I. *Realization via conformal geometry.* Adv. Math. **180**, (2003) 486–512.

[25] T. Kobayashi, B. Ørsted, P. Somberg, V. Souček, *Branching laws for Verma modules and applications in parabolic geometry,* Part I. Adv. Math. **285**, (2015), 1796–1852.

[26] T. Kobayashi, T. Oshima, *Finite multiplicity theorems for induction and restriction.* Adv. Math. **248**, (2013), 921–944.

[27] T. Kobayashi, M. Pevzner, *Differential symmetry breaking operators.* I. *General theory and F-method*, Selecta Math. (N.S.) **22**, (2016), pp. 801–845.

[28] T. Kobayashi, M. Pevzner, *Differential symmetry breaking operators.* II. *Rankin–Cohen operators for symmetric pairs.* Selecta Math. (N.S.) **22**, (2016), pp. 847–911.

[29] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups. Mem. Amer. Math. Soc., vol. **238**, No. 1126, (2015), v+112 pages, ISBN: 978-1-4704-1922-6.

[30] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups, II. Lecture Notes in Math. vol. **2234**, Springer 2018, 367 pp., ISBN: 978-981-13-5 (in press). Available also at arXiv:1801.00158.

[31] T. Kobayashi, B. Speh, *Symmetry breaking for orthogonal groups and a conjecture by B. Gross and D. Prasad.* In Geometric Aspects of the Trace Formula, Simons Symposia, Springer, 2018, pp. 245–266; also available at arXiv:1702.00263.

[32] M. Krämer, *Multiplicity free subgroups of compact connected Lie groups.* Arch. Math. (Basel) **27**, (1976), 28–36.

[33] S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds.* SIGMA **4**, (2008), Paper 036.

[34] J. Peetre, *Une caractérisation abstraite des opérateurs différentiels.* Math. Scand. **7**, (1959), pp. 211–218.

[35] B. Sun, C.-B. Zhu, *Multiplicity one theorems: the Archimedean case*. Ann. of Math. (2) **175**, (2012), no. 1, 23–44.

[36] D. Vogan, G. Zuckerman, *Unitary representations with nonzero cohomology*. Compositio Math. **53**, (1984), no. 1, 51–90.

[37] N. Wallach, Real Reductive Groups. II. Pure and Applied Mathematics, **132**. Academic Press, 1992. xiv+454 pp., ISBN: 978-0127329611.

Toshiyuki Kobayashi
Graduate School of Mathematical Sciences
and
Kavli IPMU
The University of Tokyo, Japan
e-mail: toshi@ms.u-tokyo.ac.jp

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 309–327

Representations of the Anyon Commutation Relations

Eugene Lytvynov

Abstract. We discuss some representations of the anyon commutation relations (ACR) both in the discrete and continuous cases. These non-Fock representations yield, in the vacuum state, gauge-invariant quasi-free states on the ACR algebra. In particular, we extend the construction from [20] to the case where the generator of the one-point function is not necessarily a real operator.

Mathematics Subject Classification (2010). 47L10, 47L60, 47L90, 81R10.

Keywords. Anyon commutation relations, creation and annihilation operators, gauge-invariant quasi-free states.

1. Introduction

Let $\mathcal{H}$ and $\mathfrak{F}$ be complex separable Hilbert spaces. (We assume that the scalar product is always linear in the first variable and antilinear in the second.) Let $a^+(f)$, $a^-(f)$ be linear operators in $\mathfrak{F}$ indexed by vectors $f \in \mathcal{H}$. We assume that the operators $a^+(f)$, $a^-(f)$ are defined on a dense subspace $\mathfrak{D}$ of $\mathfrak{F}$ and map $\mathfrak{D}$ into itself. We further assume that the mapping $\mathcal{H} \ni f \mapsto a^+(f)$ is linear and the restriction of $a^+(f)^*$ to $\mathfrak{D}$ is equal to $a^-(f)$. This also implies that the mapping $\mathcal{H} \ni f \mapsto a^-(f)$ is antilinear.

Consider the following commutation relations:

$$a^+(f)a^+(g) = \pm a^+(g)a^+(f), \tag{1}$$

$$a^-(f)a^-(g) = \pm a^-(g)a^-(f), \tag{2}$$

$$a^-(f)a^+(g) = \pm a^+(g)a^-(f) + (g, f)_{\mathcal{H}}. \tag{3}$$

The choice of the sign plus in (1)–(3) gives the *canonical commutation relations* (*CCR*), describing bosons, while the choice of the sign minus gives the *canonical anticommutation relations* (*CAR*), describing fermions. The $a^+(f)$ and $a^-(f)$ are called the *creation and annihilation operators*, respectively.

Note that (2) follows from (1). It is a consequence of (3) that, in the CAR case, the operators $a^+(f)$ and $a^-(f)$ are, in fact, bounded.

Let $\mathbf{A}$ denote the complex algebra generated by the operators $a^+(f)$, $a^-(f)$ satisfying (1)–(3) and the identity operator $\mathbf{1}$. $\mathbf{A}$ is called the *CCR algebra* or the *CAR algebra*, respectively.

Let τ be a state on $\mathbf{A}$, i.e., $\tau : \mathbf{A} \to \mathbb{C}$ is a linear functional satisfying $\tau(A^*A) \geq 0$ for each $A \in \mathbf{A}$ and $\tau(\mathbf{1}) = 1$. Given a state τ on the algebra $\mathbf{A}$, the GNS construction gives the corresponding representation of the commutation relations (1)–(3).

Because of the commutation relation (3), each element of the algebra $\mathbf{A}$ can be represented as a finite sum of *Wick ordered operators*:

$$a^+(g_1)\cdots a^+(g_k)a^-(h_1)\cdots a^-(h_n), \quad k, n \in \mathbb{N}_0,\ k+n \geq 1.$$

Therefore, a state τ on $\mathbf{A}$ is completely determined by the functions $\mathbf{S}^{(k,n)} : \mathcal{H}^{k+n} \to \mathbb{C}$ given by

$$\mathbf{S}^{(k,n)}(g_1, \dots, g_k, h_1, \dots, h_n) := \tau\big(a^+(g_1)\cdots a^+(g_k)a^-(h_1)\cdots a^-(h_n)\big). \tag{4}$$

It was already known in the 1960s (e.g., [7, 8]) that a complete description of irreducible representations of the CCR/CAR (up to unitary equivalence) is not a realistic problem. This is why it was important to single out and study those representations that are physically relevant and have important mathematical properties. The easiest and most standard representation of the CCR/CAR is the Fock representation, which is realized in $\mathfrak{F} = \mathcal{F}_s(\mathcal{H})$ in the case of the CCR and in $\mathfrak{F} = \mathcal{F}_a(\mathcal{H})$ in the case of the CAR. Here $\mathcal{F}_s(\mathcal{H})$ and $\mathcal{F}_a(\mathcal{H})$ are the symmetric and antisymmetric Fock spaces over $\mathcal{H}$. The corresponding vacuum state τ has the property that the functionals $\mathbf{S}^{(k,n)}$ are all equal to zero.

In fact, the above-mentioned vacuum state τ belongs to the class of the *quasi-free states*, which were actively studied since the 1960s, see, e.g., the papers [1–6, 17, 21, 22] and the monographs [11, 12].

Giving the definition of a general quasi-free state requires some technical effort. However, in this paper, we will only be interested in the subclass of gauge-invariant quasi-free states, whose definition will be now recalled.

One says that the state τ is *gauge-invariant* if it is invariant under the group of Bogolyubov transformations

$$\begin{aligned} a^+(h) &\mapsto a^+(e^{i\theta}h) = e^{i\theta}a^+(h), \\ a^-(h) &\mapsto a^-(e^{i\theta}h) = e^{-i\theta}a^-(h), \quad \theta \in [0, 2\pi). \end{aligned}$$

By (4), τ is gauge-invariant if and only if $\mathbf{S}^{(k,n)} \equiv 0$ for $k \neq n$. Thus, a gauge-invariant state is completely determined by the functionals $\mathbf{S}^{(n,n)}$ ($n \in \mathbb{N}$), called the *n-point functions*.

A gauge-invariant state is called *quasi-free* if its n-point functions are determined by the one-point function in the following sense: in the case of the CAR algebra, we have

$$\mathbf{S}^{(n,n)}(g_n,\dots,g_1,h_1,\dots,h_n) = \det\left[\mathbf{S}^{(1,1)}(g_i,h_j)\right]_{i,j=1,\dots,n}$$

and in the case of the CCR algebra, we have

$$\mathbf{S}^{(n,n)}(g_n,\dots,g_1,h_1,\dots,h_n) = \operatorname{per}\left[\mathbf{S}^{(1,1)}(g_i,h_j)\right]_{i,j=1,\dots,n}.$$

Here, for a square matrix A, $\det A$ and $\operatorname{per} A$ denote the determinant and the permanent of A, respectively.

Consider the sesquilinear form $\mathbf{S}^{(1,1)}$. In the CAR case, it is automatically bounded, while in the CCR case it is natural to assume this. Denote by K the bounded linear operator in $\mathcal{H}$ that is the generator of $\mathbf{S}^{(1,1)}$, i.e.,

$$\mathbf{S}^{(1,1)}(f,g) = (Kf,g)_{\mathcal{H}}. \tag{5}$$

Then, for each $K \geq 0$, there exists a corresponding gauge-invariant quasi-free state on the CCR algebra, while for the CAR algebra this holds for $0 \leq K \leq \mathbf{1}$, see, e.g., [11]. (In fact, these assumptions on K are also necessary for the existence of a gauge-invariant quasi-free state.)

In this paper, we will be interested in the representations of the *anyon commutation relations* (*ACR*), which form a continuous bridge between the CCR and CAR. These commutation relations are characterized by a pair $(q,\bar{q})$ of complex conjugate numbers of modulus 1, or equivalently by the real number $\Re(q) \in [-1,1]$.

In the physics literature, in the case where the physical space has dimension two (a plain), intermediate statistics have been discussed since Leinass and Myrheim [18] conjectured their existence in 1977. The first mathematically rigorous prediction of intermediate statistics was done by Goldin, Menikoff and Sharp [14, 15] in 1980, 1981. The name *anyon* was given to such statistics by Wilczek [23, 24]. Liguori, Mintchev [19] and Goldin, Sharp [16] derived the commutation relations describing an anyon system, i.e., the *anyon commutation relations (ACR).*

In [16], Goldin and Sharp arrived at the commutation relations as a "consequence of the group representations describing anyons, together with the (completely general) intertwining property of the field." Goldin, Sharp [16] constructed a representation of the ACR in the space $\mathfrak{F} = L^2(\Gamma_0(\mathbb{R}^2), m)$, where $\Gamma_0(\mathbb{R}^2)$ denotes the space of finite configurations in $\mathbb{R}^2$, and m is the Lebesgue–Poisson measure on $\Gamma_0(\mathbb{R}^2)$. Note that $L^2(\Gamma_0(\mathbb{R}^2), m)$ can be identified with the symmetric Fock space $\mathcal{F}_s(L^2(\mathbb{R}^2, dx))$.

Liguori, Mintchev [19] and Goldin, Majid [13] constructed an anyonic Fock space $\mathcal{F}_{(q,\bar{q})}(L^2(\mathbb{R}^2, dx))$ and realized the ACR in this space. Note, however, that this representation of the ACR is unitarily equivalent to the representation from [16]. We also refer to [9] for a detailed discussion of the anyonic Fock space.

In [20], a refinement of the ACR was proposed. These new commutation relations are determined by a pair $(q,\bar{q})$ of complex conjugate numbers of modulus 1 and by a real number η. A possible natural choice of η is $\eta = \Re(q)$. In the case of the representation of the ACR in the anyonic Fock space $\mathcal{F}_{(q,\bar{q})}(L^2(\mathbb{R}^2, dx))$, one does not see the difference between the ACR from [16, 19] and the ACR from [20]. Nevertheless, this difference becomes crucial when one tries to construct more complex representations of the ACR.

By using these new commutation relations, the definition of an ACR algebra and that of a gauge-invariant quasi-free state on the ACR algebra were proposed in [20]. Under certain assumptions on the generator K satisfying (5), a class of representations of the ACR corresponding to a gauge-invariant quasi-free state was explicitly constructed. In particular, the operator K was required to be real, i.e., mapping real-valued functions into real-valued functions.

The aim of the present paper is twofold. First, we will present a new discrete model of anyon statistics over a two-dimensional lattice. It should be noted that the ACR in the discrete setting were already discussed in [10, 13]. (See also the references in [20].) However, in both papers [13] and [10], only the case of a one-dimensional lattice was considered. We show in this paper that addition of a second dimension of the lattice allows us to construct a wide class of gauge-invariant quasi-free states on the corresponding ACR algebra. And second, we will show how the class of representations of the ACR in the continuum setting can be extended to include those gauge-invariant quasi-free states for which the generator K satisfying (5) is not necessarily real.

The paper is organized as follows. In Section 2, following [9, 10, 19]), we recall the construction of the Fock space for a generalized statistics and the Fock representation of the corresponding commutation relations.

Section 3 consists of two parts. In the first subsection, we consider the discrete ACR over a one-dimensional lattice, $\mathbb{Z}$. This model was proposed in [10] and was influenced by the discrete model in [13]. We construct a class of non-Fock representations of the ACR indexed by a positive-valued function on $\mathbb{Z}$ (which is bounded by 1 in the case of anyon particles of fermion type). We show that the corresponding vacuum state can be thought of as a gauge-invariant quasi-free in which instead of the determinant/permanent a certain *functional q-determinant* appears.

In the second subsection, we propose a model of discrete ACR over a two-dimensional lattice, $\mathbb{Z}^2$. This model has more resemblance to the continuous case (where the physical space is $\mathbb{R}^2$). As a result we construct a class of gauge-invariant quasi-free representations of the ACR indexed by a positive linear operator in $\ell^2(\mathbb{Z})$ (bounded by 1 in the case of anyon particles of fermion type).

Finally, in Section 4, we discuss the ACR algebra and the representations of the ACR corresponding to the gauge-invariant quasi-free states. We extend here the constructions from [20] to the case where the corresponding operator K satisfying (5) is not necessarily real.

2. Fock representation for a generalized statistics

Let X be a locally compact Polish space and let σ be a Radon measure on X. (The reader who is not familiar with these notions may think of X as $\mathbb{R}^d$ or a discrete space and σ the Lebesgue measure dx or the counting measure, respectively.) Let $X^{(2)}$ be a symmetric subset of X^2 such that

$$\sigma^{\otimes 2}(X^2 \setminus X^{(2)}) = 0.$$

Let $Q : X^{(2)} \to \mathbb{C}$ be a function satisfying $|Q(x,y)| = 1$ and $Q(x,y) = \overline{Q(y,x)}$. Liguori and Mintchev [19] introduced the notion of a *generalized statistics corresponding to the function* Q. Heuristically, this is a family of creation operators ∂_x^+ and annihilation operators ∂_x^- at points $x \in X$ such that ∂_x^+ is the adjoint of ∂_x^- and these operators satisfy the following Q*-commutation relations* (Q-*CR*):

$$\partial_x^+ \partial_y^+ = Q(y,x) \partial_y^+ \partial_x^+, \tag{6}$$
$$\partial_x^- \partial_y^- = Q(y,x) \partial_y^- \partial_x^-, \tag{7}$$
$$\partial_x^- \partial_y^+ = Q(x,y) \partial_y^+ \partial_x^- + \delta(x,y). \tag{8}$$

A rigorous meaning of the operators ∂_x^+ and ∂_x^- and the commutation relations (6)–(8) is given by smearing these relations with functions from the complex L^2-space $\mathcal{H} := L^2(X,\sigma)$. More precisely, one should define linear operators

$$a^+(f) = \int_X f(x)\, \partial_x^+\, \sigma(dx), \qquad a^-(f) = \int_X \overline{f(x)}\, \partial_x^-\, \sigma(dx), \quad f \in \mathcal{H}, \tag{9}$$

on a dense linear subspace $\mathfrak{D}$ of a complex Hilbert space $\mathfrak{F}$ so that $a^+(f)$ depends linearly on f, the adjoint of $a^+(f)$ restricted to $\mathfrak{D}$ is equal to $a^-(f)$, and for all $f, g \in \mathcal{H}$,

$$a^+(f)a^+(g) = \int_{X^2} f(x)g(y)Q(y,x)\, \partial_y^+ \partial_x^+\, \sigma(dx)\, \sigma(dy), \tag{10}$$
$$a^-(f)a^-(g) = \int_{X^2} \overline{f(x)g(y)}\, Q(y,x)\, \partial_y^- \partial_x^-\, \sigma(dx)\, \sigma(dy), \tag{11}$$
$$a^-(f)a^+(g) = \int_{X^2} \overline{f(x)}\, g(y)Q(x,y)\, \partial_y^+ \partial_x^-\, \sigma(dx)\, \sigma(dy) + \int_X g(x)\overline{f(x)}\, \sigma(dx). \tag{12}$$

Note that the operator-valued integrals on the right-hand side of formulas (10)–(12) should also be given rigorous meaning. Note also that the choice $Q \equiv 1$ gives the CCR and the choice $Q \equiv -1$ gives the CAR, see (1)–(3).

To construct the Fock representation of the Q-CR one defines the corresponding Q*-symmetric Fock space*. We denote

$$X^{(n)} := \big\{(x_1, \dots, x_n) \in X^n \mid (x_i, x_j) \in X^{(2)} \text{ for all } 1 \le i < j \le n\big\},$$

and we obviously have $\sigma^{\otimes n}(X^n \setminus X^{(n)}) = 0$. A function $f^{(n)} : X^{(n)} \to \mathbb{C}$ is called Q*-symmetric* if for any $i \in \{1, \dots, n-1\}$ and $(x_1, \dots, x_n) \in X^{(n)}$.

$$f^{(n)}(x_1, \dots, x_n) = Q(x_i, x_{i+1}) f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n).$$

We denote by $\mathcal{H}^{\circledast n}$ the subspace of $\mathcal{H}^{\otimes n}$ that consists of all Q-symmetric functions from $\mathcal{H}^{\otimes n}$. We call $\mathcal{H}^{\circledast n}$ the *nth Q-symmetric tensor power of $\mathcal{H}$.*

Consider the symmetric group S_n of all permutations of $1, \dots, n$. For each $\pi \in S_n$, we define a function $Q_\pi : X^{(n)} \to \mathbb{C}$ by

$$Q_\pi(x_1, \dots, x_n) := \prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} Q(x_i, x_j). \tag{13}$$

Note that, in the case $Q \equiv 1$, we get $Q_\pi \equiv 1$, while in the case $Q \equiv -1$, we get $Q_\pi \equiv (-1)^{|\pi|} = \operatorname{sgn} \pi$. Here $|\pi|$ is the number of inversions in π, i.e., the number of $i < j$ such that $\pi(i) > \pi(j)$. For a function $f^{(n)} : X^n \to \mathbb{C}$, we define

$$(P_n f^{(n)})(x_1, \dots, x_n) := \frac{1}{n!} \sum_{\pi \in S_n} Q_\pi(x_1, \dots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}). \tag{14}$$

The operator P_n determines the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{H}^{\circledast n}$.

For any $f^{(n)} \in \mathcal{H}^{\circledast n}$ and $g^{(m)} \in \mathcal{H}^{\circledast m}$, we define the *$Q$-symmetric tensor product of $f^{(n)}$ and $g^{(m)}$* by

$$f^{(n)} \circledast g^{(m)} := P_{n+m}(f^{(n)} \otimes g^{(m)}).$$

The tensor product $\circledast$ is associative.

We define the *Q-Fock space over $\mathcal{H}$* by

$$\mathcal{F}^Q(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledast n} n!\,.$$

(In particular, for $Q \equiv 1$ $\mathcal{F}^Q(\mathcal{H}) = \mathcal{F}_s(\mathcal{H})$ and for $Q \equiv -1$ $\mathcal{F}^Q(\mathcal{H}) = \mathcal{F}_a(\mathcal{H})$.) The vector $\Omega := (1, 0, 0, \dots) \in \mathcal{F}^Q(\mathcal{H})$ is called the *vacuum*. We also denote by $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})$ the subset of $\mathcal{F}^Q(\mathcal{H})$ consisting of all finite sequences

$$F = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$$

in which $f^{(i)} \in \mathcal{H}^{\circledast i}$ for $i = 0, 1, \dots, n$, $n \in \mathbb{N}$.

For each $f \in \mathcal{H}$, we define a *creation operator* $a^+(f)$ and an *annihilation operator* $a^-(f)$ as linear operators acting on $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})$ that satisfy

$$a^+(f) h^{(n)} := f \circledast h^{(n)}, \quad h^{(n)} \in \mathcal{H}^{\circledast n},$$

and $a^-(f) := (a^+(f))^* \restriction_{\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})}$. Furthermore, for $f \in \mathcal{H}$ and $h^{(n)} \in \mathcal{H}^{\circledast n}$, we have:

$$(a^-(f) h^{(n)})(x_1, \dots, x_{n-1}) = n \int_X \overline{f(u)}\, h^{(n)}(u, x_1, \dots, x_{n-1})\, \sigma(du).$$

Thus, if we introduce formal operators ∂_x^+ and ∂_x^- by formulas (9), we formally get

$$\partial_x^+ h^{(n)} = \delta_x \circledast h^{(n)}, \qquad \partial_x^- h^{(n)} = n h^{(n)}(x, \cdot),$$

where δ_x is the delta function at x. Now, one can easily give a rigorous meaning to the Q-CR (10)–(12) and show that these relations hold.

Indeed, for $h^{(n)} \in \mathcal{H}^{\circledast n}$,

$$\begin{aligned}&\left(a^+(f)a^+(g)h^{(n)}\right)(x_1,\dots,x_{n+2})\\&\quad=\left(\int_{X^2} f(x)g(y)\,\partial_x^+\partial_y^+\,\sigma(dx)\,\sigma(dy)\,h^{(n)}\right)(x_1,\dots,x_{n+2})\\&\quad=P_{n+2}\big(f(x_1)g(x_2)h^{(n)}(x_3,\dots,x_n)\big).\end{aligned}$$

Hence, we naturally define

$$\begin{aligned}&\left(\int_{X^2} f(x)g(y)\,Q(y,x)\partial_y^+\partial_x^+\ \sigma(dx)\,\sigma(dy)\ h^{(n)}\right)(x_1,\dots,x_{n+2})\\&\quad=\left(\int_{X^2} g(x)f(y)\,Q(x,y)\partial_x^+\partial_y^+\ \sigma(dx)\,\sigma(dy)\ h^{(n)}\right)(x_1,\dots,x_{n+2})\\&\quad:=P_{n+2}\big(g(x_1)f(x_2)Q(x_1,x_2)h^{(n)}(x_3,\dots,x_n)\big),\end{aligned}$$

and we indeed have

$$\begin{aligned}&P_{n+2}\big(f(x_1)g(x_2)h^{(n)}(x_3,\dots,x_n)\big)\\&\quad=P_{n+2}\big(g(x_1)f(x_2)Q(x_1,x_2)h^{(n)}(x_3,\dots,x_n)\big).\end{aligned}$$

Thus, (10) holds. By duality, this also implies that (11) holds.

Analogously, we define

$$\begin{aligned}&\left(\int_{X^2} \overline{f(x)}\,g(y)\,Q(x,y)\partial_y^+\partial_x^-\ \sigma(dx)\,\sigma(dy)\ h^{(n)}\right)(x_1,\dots,x_n)\\&\quad:=nP_n\left(\int_X \overline{f(u)}g(x_1)Q(u,x_1)g(x_1)h^{(n)}(u,x_2,\dots,x_n)\,\sigma(du)\right).\end{aligned}\tag{15}$$

On the other hand,

$$\big(a^-(f)a^+(g)h^{(n)}\big)(x_1,\dots,x_n)=(n+1)\int_X\overline{f(u)}\,(g\circledast h^{(n)})(u,x_1,\dots,x_n)\,\sigma(du).\tag{16}$$

By using (14)–(16), one finally shows that formula (12) holds.

We note that, in the obtained representation of the Q-CR, we only used the values of the function Q $\sigma^{\otimes 2}$-almost everywhere.

3. Discrete setting

Following [10, 13], we start with a discussion of the ACR over the one-dimensional lattice.

3.1. One-dimensional lattice

Let $X=\mathbb{Z}$ and let σ be the counting measure on $\mathbb{Z}$. Thus, $L^2(X,\sigma)=\ell^2(X)$. Fix $q\in\mathbb{C}$ of modulus 1 and $\eta\in\{-1,1\}$.

We define $Q : X^2 \to \mathbb{C}$ by

$$Q(x,y) := \begin{cases} q, & \text{if } x < y, \\ \bar{q}, & \text{if } x > y, \\ \eta, & \text{if } x = y. \end{cases}$$

With this choice of Q, the Q-CR (6)–(8) become the *discrete ACR over* $\mathbb{Z}$. In the case $\eta = -1$, one speaks of *anyons of fermion type.*

Let us now construct a class of non-Fock representations of these commutation relations. Let X_1 and X_2 denote two copies of $\mathbb{Z}$ and let $Y := X_1 \sqcup X_2$. Thus, $\ell^2(Y) = \ell^2(X_1) \oplus \ell^2(X_2)$. We define a function $\mathbf{Q} : Y^2 \to \mathbb{C}$ by

$$\mathbf{Q}(x,y) := \begin{cases} Q(x,y), & \text{if either } x, y \in X_1 \text{ or } x, y \in X_2, \\ Q(y,x), & \text{if either } x \in X_1,\ y \in X_2 \text{ or } x \in X_2,\ y \in X_1. \end{cases} \tag{17}$$

(Here and below we use an obvious abuse of notation.) We consider the corresponding creation and annihilation operators, ∂_x^+, ∂_x^- $(x \in Y)$ in $\mathcal{F}^{\mathbf{Q}}(\ell^2(Y))$. Note that these are now rigorously defined operators that satisfy the commutation relations

$$\partial_x^+ \partial_y^+ = \mathbf{Q}(y,x) \partial_y^+ \partial_x^+, \quad \partial_x^- \partial_y^- = \mathbf{Q}(y,x) \partial_y^- \partial_x^-, \tag{18}$$

$$\partial_x^- \partial_y^+ = \mathbf{Q}(x,y) \partial_y^+ \partial_x^- + \delta(x,y). \tag{19}$$

By (19),

$$\partial_x^+ \partial_y^- = \mathbf{Q}(x,y) \partial_y^- \partial_x^+ - \eta\delta(x,y). \tag{20}$$

For $x \in \mathbb{Z}$ considered as an element of X_i $(i \in \{1,2\})$, we denote the corresponding creation and annihilation operators in $\mathcal{F}^{\mathbf{Q}}(\ell^2(Y))$ at point $x \in X_i$ by $\partial_{x,i}^+$ and $\partial_{x,i}^-$, respectively.

Fix functions $K_i : X \to [0,\infty)$ $(i \in \{1,2\})$ and define, for each $x \in X$, linear operators

$$D_x^+ := K_1(x) \partial_{x,1}^- + K_2(x) \partial_{x,2}^+, \tag{21}$$

$$D_x^- := K_1(x) \partial_{x,1}^+ + K_2(x) \partial_{x,2}^-. \tag{22}$$

The following proposition is a direct consequence of (17)–(22).

Proposition 1. *Assume $K_2(x)^2 = 1 + \eta K_1(x)^2$ for all $x \in \mathbb{Z}$. Then the operators D_x^+, D_x^- $(x \in \mathbb{Z})$ given by* (21), (22) *satisfy the discrete ACR over $\mathbb{Z}$.*

Let $\mathbf{A}$ denote the complex algebra generated by the operators D_x^+, D_x^- $(x \in \mathbb{Z})$ and the identity operator. Let τ be the vacuum state on $\mathbf{A}$, i.e.,

$$\tau(A) := (A\Omega, \Omega)_{\mathcal{F}^{\mathbf{Q}}(\ell^2(Y))}, \quad A \in \mathbf{A}. \tag{23}$$

Due to the commutation relation (8), τ is completely determined by the functionals

$$\mathbf{S}^{(k,n)}(x_1, \dots, x_k, y_1, \dots, y_n) := \tau(D_{x_1}^+ \cdots D_{x_k}^+ D_{y_1}^- \dots, D_{y_n}^-), \tag{24}$$

where $k, n \in \mathbb{N}_0$, $k + n \geq 1$.

The following proposition shows that the state τ on $\mathbf{A}$ can be understood as a gauge-invariant quasi-free state on the ACR algebra over $\mathbb{Z}$.

Proposition 2. *We have*

$$\mathbf{S}^{(k,n)} \equiv 0 \quad \textit{if } k \neq n. \tag{25}$$

Furthermore,

$$\mathbf{S}^{(1,1)}(x,y) = K_1(x)^2 \delta_{x,y}, \tag{26}$$

(*where $\delta_{x,y}$ is the Kronecker delta*) *and for $n \geq 2$*

$$\mathbf{S}^{(n,n)}(x_n,\dots,x_1,y_1,\dots,y_n) = \sum_{\pi\in S_n} Q_\pi(x_1,\dots,x_n)\,\mathbf{S}^{(1,1)}(x_l,y_{\pi(l)}). \tag{27}$$

Remark 3. Note that the expression on the right-hand side of formula (27) can be thought of as a *functional Q-determinant of the matrix* $\left[\mathbf{S}^{(1,1)}(x_i,y_j)\right]_{i,j=1,\dots,n}$.

Remark 4. It follows from Proposition 2 that the obtained class of representations of the discrete ACR over $\mathbb{Z}$ is completely determined by the function $K(x) := K_1^2(x)$ defined on $\mathbb{Z}$ and taking values in $[0,\infty)$ if $\eta = 1$ and in $[0,1]$ if $\eta = -1$.

Proof of Proposition 2. Formulas (25), (26) follow immediately from (21)–(24). Using also (13), (14), we get

$$\begin{aligned}
&\mathbf{S}^{(n,n)}(x_n,\dots,x_1,y_1,\dots,y_n)\\
&\quad = K_1(x_1)\cdots K_1(x_n)K(y_1)\cdots K(y_n)\\
&\qquad \times n!\left(\delta_{y_1}\circledast\cdots\circledast\delta_{y_n},\delta_{x_1}\circledast\cdots\circledast\delta_{x_n}\right)_{\ell^2(Y)^{\circledast n}}\\
&\quad = K_1(x_1)\cdots K_1(x_n)K(y_1)\cdots K(y_n)\\
&\qquad \times \left(n!P_n(\delta_{y_1}\otimes\cdots\otimes\delta_{y_n}),\delta_{x_1}\otimes\cdots\otimes\delta_{x_n}\right)_{\ell^2(Y)^{\otimes n}} \qquad (28)\\
&\quad = K_1(x_1)\cdots K_1(x_n)K(y_1)\cdots K(y_n)\sum_{\pi\in S_n} Q_\pi(y_{\pi(1)},\dots,y_{\pi(n)})\prod_{l=1}^n \delta_{y_{\pi(l)},x_l}\\
&\quad = K_1(x_1)^2\cdots K_1(x_n)^2\sum_{\pi\in S_n} Q_\pi(x_1,\dots,x_n)\prod_{l=1}^n \delta_{x_l,y_{\pi(l)}}, \qquad \square
\end{aligned}$$

which implies (12).

3.2. Two-dimensional lattice

We will now present a new discrete model of the ACR over a two-dimensional lattice, $\mathbb{Z}^2$. This construction will have more similarities to the continuous case over the real plane $\mathbb{R}^2$. Furthermore, we will be able to construct a much wider class of representations of the discrete ACR corresponding to a gauge-invariant quasi-free state.

So, let now $X = \mathbb{Z}^2$. We denote $x = (x^1,x^2) \in X$. We again fix a complex number q of modulus 1 and $\eta \in \{-1,1\}$. We define $Q : X^2 \to \mathbb{C}$ by

$$Q(x,y) := \begin{cases} q, & \text{if } x^1 < y^1,\\ \bar{q}, & \text{if } x^1 > y^1,\\ \eta, & \text{if } x^1 = y^1.\end{cases}$$

With this choice of Q, the Q-CR (6)–(8) will be called the *discrete ACR over* $\mathbb{Z}^2$. Analogously to Subsection 3.1, we then define $\ell^2(Y)$ and the operators $\partial^+_{x,i}$ and $\partial^-_{x,i}$ ($x \in X$) in $\mathcal{F}^{\mathbf{Q}}(\ell^2(Y))$.

Each bounded linear operator L in $\ell^2(X)$ can be identified with its matrix $[L(x,y)]_{x,y\in X}$:

$$(Lf)(x) = \sum_{y\in X} L(x,y)f(y), \quad f \in \ell^2(X).$$

Furthermore, for each $x \in X$, we obviously have $L(x,\cdot) \in \ell^2(X)$.

Let K_1 and K_2 be bounded self-adjoint operators in $\ell^2(X)$. We assume that

$$K_i(x,y) = 0 \quad \text{if } x^1 \neq y^1,\ i = 1,2. \tag{29}$$

This assumption is equivalent to the following requirement: for each bounded function $g : \mathbb{Z} \to \mathbb{C}$, we have $LM_{g,1}f = M_{g,1}Lf$ for all $f \in \ell^2(X)$. Here we defined the operator $M_{g,1}$ by

$$(M_{g,1}f)(x) := g(x^1)f(x), \quad f \in \ell^2(X).$$

Now, for each $x \in X$, we define linear operators

$$D^+_x := a^-\big(\overline{K_1(x,\cdot)}, 0\big) + a^+\big(0, K_2(x,\cdot)\big), \tag{30}$$

$$D^-_x := a^+\big(\overline{K_1(x,\cdot)}, 0\big) + a^-\big(0, K_2(x,\cdot)\big). \tag{31}$$

In view of (29), formulas (30), (31) can be written in the form

$$D^+_x = \sum_{y\in X:\, y^1=x^1} \big(K_1(x,y)\partial^-_{y,1} + K_2(x,y)\partial^+_{y,2}\big), \tag{32}$$

$$D^-_x = \sum_{y\in X:\, y^1=x^1} \Big(\overline{K_1(x,y)}\partial^+_{y,1} + \overline{K_2(x,y)}\partial^-_{y,2}\Big). \tag{33}$$

Proposition 5. *Let K_1 and K_2 be bounded positive operators in $\ell^2(\mathbb{Z}^2)$ satisfying* (29). *Assume $K_2^2 = \mathbf{1} + \eta K_1^2$. Then the operators D^+_x, D^-_x ($x \in \mathbb{Z}^2$) given by* (30), (31) *satisfy the discrete ACR over $\mathbb{Z}^2$.*

Proof. We will only prove the less trivial commutation relation between D^-_x and D^+_y. By (18)–(20), (32) and (33), we have

$$\begin{aligned} D^-_x D^+_y &= \sum_{u\in X:\, u^1=x^1} \Big(\overline{K_1(x,u)}\partial^+_{u,1} + \overline{K_2(x,u)}\partial^-_{u,2}\Big) \\ &\qquad \times \sum_{v\in X:\, v^1=y^1} \big(K_1(y,v)\partial^-_{v,1} + K_2(y,v)\partial^+_{v,2}\big) \\ &= \sum_{u\in X:\, u^1=x^1} \sum_{v\in X:\, v^1=y^1} \Big[Q(u,v)\big(K_1(y,v)\partial^-_{v,1} + K_2(y,v)\partial^+_{v,2}\big) \\ &\qquad \times \Big(\overline{K_1(x,u)}\partial^+_{u,1} + \overline{K_2(x,y)}\partial^-_{u,2}\Big) \\ &\qquad + \big(K_2(y,u)K_2(u,x) - \eta K_1(y,u)K_1(u,x)\big)\delta_{u,v}\Big] \end{aligned}$$

$$= Q(x,y)D_y^+D_x^- + \sum_{u\in X}\big(K_2(y,u)K_2(u,x) - \eta K_1(y,u)K_1(u,x)\big)$$
$$= Q(x,y)D_y^+D_x^- + \delta_{x,y}. \qquad \square$$

Similarly to Subsection 3.1, we now define the vacuum state τ on the complex algebra $\mathbf{A}$ generated by the operators D_x^+, D_x^- $(x\in\mathbb{Z}^2)$ and the identity operator. We also define the functionals $\mathbf{S}^{(k,n)}$ by (24). Similarly to Proposition 2, we get

Proposition 6. *Formula* (25) *holds for the just constructed state τ. Furthermore, define the positive bonded linear operator $K := K_1^2$ in $\ell^2(\mathbb{Z}^2)$ with matrix $[K(x,y)]_{x,y\in\mathbb{Z}^2}$. Then*

$$\mathbf{S}^{(1,1)}(x,y) = K(x,y)$$

and for $n\geq 2$, formula (27) *holds.*

Remark 7. It follows from Proposition 6 that the obtained class of representations of the discrete ACR over $\mathbb{Z}^2$ is completely determined by the bounded linear operator K in $\ell^2(\mathbb{Z}^2)$ satisfying $K\geq 0$ if $\eta=1$ and $0\leq K\leq 1$ if $\eta=-1$.

Proof of Proposition 6. Formula (25) follows immediately from (30), (31). Next,

$$\begin{aligned}\mathbf{S}^{(1,1)}(x,y) &= (D_y^-\Omega, D_x^-\Omega)_{\mathcal{F}^{\mathbf{Q}}(\ell^2(Y))}\\ &= (K_1(\cdot,y),K_1(\cdot,x))_{\ell^2(X)}\\ &= \sum_{u\in X}K_1(u,y)K_1(x,u)\\ &= K(x,y),\end{aligned}$$

and for $n\geq 2$, similarly to (28) and using (29), we calculate

$$\begin{aligned}&\mathbf{S}^{(n,n)}(x_n,\dots,x_1,y_1,\dots,y_n)\\ &\quad= (D_{y_1}^-\cdots D_{y_n}^-\Omega, D_{x_1}^-\cdots D_{x_n}^-\Omega)_{\mathcal{F}^{\mathbf{Q}}(\ell^2(Y))}\\ &\quad= n!\big(K_1(\cdot,y_1)\circledast\cdots\circledast K_1(\cdot,y_n), K_1(\cdot,x_1)\circledast\cdots\circledast K_1(\cdot,x_n)\big)_{\ell^2(X)^{\circledast n}}\\ &\quad= \big(n!P_n(K_1(\cdot,y_1)\otimes\cdots\otimes K_1(\cdot,y_n)), K_1(\cdot,x_1)\otimes\cdots\otimes K_1(\cdot,x_n)\big)_{\ell^2(X)^{\otimes n}}\\ &\quad= \sum_{\pi\in S_n}\sum_{(u_1,\dots,u_n)\in X^n} Q_\pi(u_1,\dots,u_n)K_1(u_1,y_{\pi(1)})\cdots K_1(u_n,y_{\pi(n)})\\ &\qquad\quad\times K_1(x_1,u_1)\delta_{u_1^1,x_1^1}\cdots K_1(x_n,u_n)\delta_{u_n^1,x_n^1}\\ &\quad= \sum_{\pi\in S_n} Q_\pi(x_1,\dots,x_n)\sum_{(u_1,\dots,u_n)\in X^n} K_1(u_1,y_{\pi(1)})\cdots K_1(u_n,y_{\pi(n)})\\ &\qquad\quad\times K_1(x_1,u_1)\cdots K_1(x_n,u_n)\\ &\quad= \sum_{\pi\in S_n} Q_\pi(x_1,\dots,x_n)K(x_1,y_{\pi(1)})\cdots K(x_n,y_{\pi(n)}). \qquad \square\end{aligned}$$

4. Continuous setting

Let now $X = \mathbb{R}^2$ and $\sigma(dx) = dx$ is the Lebesgue measure. Thus, $\mathcal{H} = L^2(\mathbb{R}^2, dx)$. We fix $q \in \mathbb{C}$, $|q| = 1$, and set

$$Q(x,y) = \begin{cases} q, & \text{if } x^1 < y^1, \\ \bar{q}, & \text{if } x^1 > y^1. \end{cases} \tag{34}$$

Thus, in the notations of Section 2, we have

$$X^{(2)} = \{(x,y) \in X^2 \mid x^1 \neq y^1\}.$$

4.1. An ACR algebra

We saw in Section 3 that, in order to construct non-Fock representations of the ACR, we had to derive from the commutation relation (8), a commutation relation for $\partial_x^+ \partial_y^-$. By formal multiplication of (8) by $Q(y,x)$ and swapping x and y variables, we get

$$\partial_x^+ \partial_y^- = Q(x,y)\partial_y^- \partial_x^+ - Q(x,x)\delta(x,y).$$

But in the Fock representation of the ACR in the continuous case (Section 2), the value of $Q(x,x)$ is arbitrary, since the set $\{(x,y) \in X^2 \mid x = y\}$ is of zero Lebesgue measure $dx\,dy$. Hence, it can be chosen arbitrary. Thus, we fix any $\eta \in \mathbb{R}$ and set $Q(x,x) := \eta$ for all $x \in X$.

Recall (16). By [9, Proposition 2.7], we have

$$\begin{aligned}(n+1)&(g \circledast h^{(n)})(u, x_1, \dots, x_n)\\ &= g(u)h^{(n)}(x_1,\dots,x_n)\\ &\quad + \sum_{k=1}^{n} Q(u,x_k)Q(x_1,x_k)Q(x_2,x_k)\cdots\\ &\qquad \cdots Q(x_{k-1},x_k)\, g(x_k)h^{(n)}(u,x_1,\dots,\check{x}_k,\dots,x_n),\end{aligned} \tag{35}$$

where $\check{x}_k$ denotes the absence of x_k. Now, let $\varphi^{(2)} : X^2 \to \mathbb{C}$ be a bounded function with compact support. By (16) and (35), we have

$$\begin{aligned}\bigg(\int_{X^2} &\varphi^{(2)}(u,v)\partial_u^- \partial_v^+ \, du\, dv\, h^{(n)}\bigg)(x_1,\dots,x_n)\\ &= \int_X \varphi^{(2)}(u,u)\, du h^{(n)}(x_1,\dots,x_n)\\ &\quad + \sum_{k=1}^{n} \int_X \varphi^{(2)}(u,x_k)Q(u,x_k)Q(x_1,x_k)\cdots\\ &\qquad \cdots Q(x_{k-1},x_k)h^{(n)}(u,x_1,\dots,\check{x}_k,\dots,x_n)\, du.\end{aligned} \tag{36}$$

(Although the calculations in formula (36) look formal, they can, in fact, be given a rigorous meaning, see Section 3 in [20].) Note that in (36), we use the values of

$\varphi^{(2)}(u,v)\,du\,dv$-a.e. and the values of $\varphi(u,u)\,du$-a.e. Formulas (35) and (36) imply

$$\begin{aligned}
&\bigg(\int_{X^2}\varphi^{(2)}(u,v)Q(v,u)\partial_u^-\partial_v^+\,du\,dv\,h^{(n)}\bigg)(x_1,\dots,x_n)\\
&=\eta\int_X\varphi^{(2)}(u,u)\,duh^{(n)}(x_1,\dots,x_n)\\
&\quad+\sum_{k=1}^n\int_X\varphi^{(2)}(u,x_k)Q(x_1,x_k)\cdots Q(x_{k-1},x_k)h^{(n)}(u,x_1,\dots,\check{x}_k,\dots,x_n)\,du\\
&=\eta\bigg(\int_X\varphi^{(2)}(u,u)\,du\bigg)h^{(n)}(x_1,\dots,x_n)\\
&\quad+\bigg(\int_{X^2}\varphi^{(2)}(u,v)\partial_v^+\partial_u^-h^{(n)}\bigg)(x_1,\dots,x_n). \qquad (37)
\end{aligned}$$

Thus,

$$\partial_x^+\partial_y^-=Q(x,y)\partial_y^-\partial_x^+-\eta\delta(x,y), \qquad (38)$$

and this formula holds for any choice of η as the value of $Q(x,x)$.

This suggests us to consider the commutation relations (6)–(8) in the continuous case in the refined form

$$\partial_x^+\partial_y^+=Q(y,x)\partial_y^+\partial_x^+,\quad x\neq y, \qquad (39)$$

$$\partial_x^-\partial_y^-=Q(y,x)\partial_y^-\partial_x^-,\quad x\neq y, \qquad (40)$$

$$\partial_x^-\partial_y^+=Q(x,y)\partial_y^+\partial_x^-+\delta(x,y), \qquad (41)$$

where

$$Q(x,y)=\begin{cases}q, & \text{if } x^1<y^1,\\ \bar{q}, & \text{if } x^1>y^1,\\ \eta, & \text{if } x^1=y^1.\end{cases} \qquad (42)$$

To formalize the commutation relations (39)–(41), a rigorous definition of an ACR algebra was proposed in [20]. Let us briefly recall the main points of this definition.

Consider a sequence $\sharp=(\sharp_1,\dots,\sharp_n)\in\{+,-\}^n$. Let

$$C_\sharp:=\big\{(i,j)\in\{1,\dots,n\}^2\mid i<j,\ \sharp_i\neq\sharp_j\big\}.$$

Let $\mathbb{P}_\sharp$ denote the collection of all subsets $A=\{(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)\}\subset C_\sharp$ such that $i_1,j_1,i_2,j_2,\dots,i_k,j_k$ are all different numbers. We define a measure $m_\sharp$ on $\mathbb{R}^n$ by

$$m_\sharp(ds_1\cdots ds_n)=\bigg(1+\sum_{A=\{(i_1,j_1),\dots,(i_k,j_k)\}\in\mathbb{P}_\sharp}\prod_{l=1}^k\delta(s_{i_l},s_{j_l})\bigg)ds_1\cdots ds_n.$$

For example, if $\sharp=\{+,+\}$, $m_\sharp(ds_1\,ds_2)=ds_1\,ds_2$ and if $\sharp=\{+,-\}$, $m_\sharp(ds_1\,ds_2)=ds_1\,ds_2+\delta(s_1,s_2)\,ds_1\,ds_2$.

We now assume that, for each $\sharp=(\sharp_1,\dots,\sharp_n)\in\{+,-\}^n$, we are given operator-valued integrals

$$\int_{X^n}\varphi^{(n)}(x_1,\dots,x_n)\,\partial_{x_1}^{\sharp_1}\cdots\partial_{x_n}^{\sharp_n}\,dx_1\cdots dx_n \tag{43}$$

that are defined in a complex separable Hilbert space $\mathfrak{F}$ on a dense subspace $\mathfrak{D}$ of $\mathfrak{F}$ and map $\mathfrak{D}$ into itself. The function $\varphi^{(n)}:X^n\to\mathbb{C}$ is supposed to be of the form

$$\varphi^{(n)}(x_1,\dots,x_n)=h_1(x_1)\cdots h_n(x_n)\varkappa^{(n)}(x_1^1,\dots,x_n^1), \tag{44}$$

where $h_1,\dots,h_n\in\mathcal{H}$ and $\varkappa^{(n)}\in L^\infty(\mathbb{R}^n,m_\sharp)$. (Note that the representation of $\varphi^{(n)}$ in the form (44) is not unique and we assume that the operator in (43) does not depend on this representation.) We assume that the product of two operator-valued integrals is given by

$$\begin{aligned}&\int_{X^k}\varphi^{(k)}(x_1,\dots,x_k)\,\partial_{x_1}^{\sharp_1}\cdots\partial_{x_k}^{\sharp_k}\,dx_1\cdots dx_k\\ &\qquad\times\int_{X^m}\psi^{(m)}(x_{k+1},\dots,x_{k+m})\,\partial_{x_{k+1}}^{\sharp_{k+1}}\cdots\partial_{x_{k+m}}^{\sharp_{k+m}}\,dx_{k+1}\cdots dx_{k+m}\\ &\quad=\int_{X^{k+m}}\varphi^{(k)}(x_1,\dots,x_k)\psi^{(m)}(x_{k+1},\dots,x_{k+m})\,\partial_{x_1}^{\sharp_1}\cdots\partial_{x_{k+m}}^{\sharp_{k+m}}\,dx_1\cdots dx_{k+m},\end{aligned}$$

and the adjoint of an operator-valued integral is given by

$$\begin{aligned}&\left(\int_{X^k}\varphi^{(k)}(x_1,\dots,x_k)\,\partial_{x_1}^{\sharp_1}\cdots\partial_{x_k}^{\sharp_k}\,dx_1\cdots dx_k\right)^*\\ &\quad=\left(\int_{X^k}\overline{\varphi^{(k)}(x_1,\dots,x_k)}\,\partial_{x_k}^{\triangle_k}\cdots\partial_{x_1}^{\triangle_1}\,dx_1\cdots dx_k\right),\end{aligned}$$

where

$$\triangle_i:=\begin{cases}+ & \text{if } \sharp_i=-,\\ - & \text{if } \sharp_i=+,\end{cases}\qquad i=1,\dots,k.$$

One makes further assumptions that make rigorous sense of the duality of ∂_x^+ and ∂_x^- and the commutation relations (39)–(41) with the function Q given by (42).

Note that, due to our definition of the measures $m_\sharp$, we never use the value of the function Q on the diagonal $\{(x,y)\in X^2\mid x^1=y^1\}$ when we apply the Q-commutation relations between ∂_x^+, ∂_y^+, or between ∂_x^-, ∂_y^-. On the other hand, the value η appears in calculations when we apply the commutation relations to $\partial_x^-\partial_y^+$ or $\partial_x^+\partial_y^-$. (Note that formula (38) holds now in the smeared form!)

Let $\mathbf{A}$ be the complex algebra generated by the identity operator and the operator-valued integrals of the form (43). We will call $\mathbf{A}$ an *ACR algebra in the continuum.*

Let τ be a state on $\mathbf{A}$. Then due to the commutation relations (39)–(41), τ is completely determined by the values of

$$\tau\biggl(\int_{X^{k+n}}\varphi^{(k+n)}(x_1,\dots,x_{k+n})\partial_{x_1}^+\cdots\partial_{x_k}^+\,\partial_{x_{k+1}}^-\cdots\partial_{x_{k+n}}^-\,dx_1\cdots dx_{k+n}\biggr).$$

Thus, for

$$g_1, \dots, g_k, h_1, \dots, h_n \in \mathcal{H}, \quad \varkappa^{(k+n)} \in L^\infty\big(\mathbb{R}^{k+n}, m_{\{+\}^k \times \{-\}^n}\big),$$

we define

$$\begin{aligned}
&\mathbf{S}^{(k,n)}(g_1, \dots, g_k, h_1, \dots, h_n, \varkappa^{(k+n)}) \\
&\quad := \tau\bigg(\int_{X^{k+n}} g_1(x_1) \cdots g_k(x_k) h_1(x_{k+1}) \cdots h_n(x_{k+n}) \psi^{(k+n)}(x_1^1, \dots, x_{k+n}^1) \\
&\qquad \times \partial^+_{x_1} \cdots \partial^+_{x_k} \partial^-_{x_{k+1}} \cdots \partial^-_{x_{k+n}} \, dx_1 \cdots dx_{k+n} \bigg).
\end{aligned}$$

In particular, if $\varkappa^{(k+n)} \equiv 1$, we get

$$\mathbf{S}^{(k,n)}(g_1, \dots, g_k, h_1, \dots, h_n, 1) = \tau\big(a^+(g_1) \cdots a^+(g_k) a^-(Jh_1) \cdots a^-(Jh_n)\big).$$

Here $J : \mathcal{H} \to \mathcal{H}$ is the antiunitary operator of complex conjugation: $(Jf)(x) := \overline{f(x)}$.

As easily seen, the Fock representation of the commutation relations (6)–(8) with Q given by (34) gives a representation of the commutation relations (39)–(41) with Q given by (42) for any choice of η. In other words, based on the creation and annihilation operators in the Q-symmetric Fock space $\mathcal{F}^Q(L^2(\mathbb{R}^2, dx))$, one constructs an ACR algebra in which the commutation relations (41), (38) hold for any choice of η. For this representation of ACR, we trivially have $\mathbf{S}^{(k,n)} \equiv 0$ for all k and n.

4.2. Non-Fock representations

To construct non-Fock representations of the commutation relations (39)–(41), we proceed by analogy with Section 3.

We denote by X_1, X_2 two copies of $X = \mathbb{R}^2$. We denote $Y := X_1 \sqcup X_2$, and we can obviously extend the Lebesgue measure dx to Y. We define a function $\mathbf{Q} \to \mathbb{C}$ by (17), and we consider the Q-symmetric Fock space $\mathcal{F}^{\mathbf{Q}}(L^2(Y, dx))$. Note that, in this construction, we may think of Q as given by (34) as we do not actually use here the values of Q on the diagonal.

We fix continuous linear operators K_1 and K_2 in $\mathcal{H}$. We assume that these operators commute with any operator of multiplication by a bounded measurable function $\varkappa(x^1)$. The latter condition means that, for $i = 1, 2$ and $f \in \mathcal{H}$,

$$(K_i f)(x^1, x^2) = \big(K_i(x^1) f(x^1, \cdot)\big)(x^2),$$

where for dx^1-a.a. $x^1 \in \mathbb{R}$, $K_i(x^1)$ is a bounded linear operator in $L^2(\mathbb{R}, dx^2)$. For example, K_i may be of the form $\mathbf{1} \otimes \tilde{K}_i$, where $\tilde{K}_i$ is a bounded linear operator in $L^2(\mathbb{R}, dx^2)$.

For $x \in X$, we denote by $\partial^+_{x,i}$ and $\partial^-_{x,i}$ ($i \in \{1, 2\}$) the creation and annihilation operators at the point x being identified with the corresponding point of X_i.

Thus, for $f \in \mathcal{H}$,

$$a^+(f,0) = \int_X f(x)\partial^+_{x,1}\,dx, \qquad a^-(f,0) = \int_X \overline{f(x)}\,\partial^-_{x,1}\,dx,$$
$$a^+(0,f) = \int_X f(x)\partial^+_{x,2}\,dx, \qquad a^-(0,f) = \int_X \overline{f(x)}\,\partial^-_{x,2}\,dx.$$

We now define (formal) operators D^+_x and D^-_x $(x \in X)$ that satisfy, for each $f \in \mathcal{H}$:

$$\int_X f(x)D^+_x\,dx = \int_X (K_1 f)(x)\partial^-_{x,1}\,dx + \int_X (K_2 f)(x)\partial^+_{x,2}\,dx,$$
$$\int_X f(x)D^-_x\,dx := \int_X (JK_1Jf)(x)\partial^+_{x,1}\,dx + \int_X (JK_2Jf)(x)\partial^-_{x,2}\,dx.$$

If we denote

$$A^+(f) := \int_X f(x)D^+_x\,dx,$$
$$A^-(f) := \int_X \overline{f(x)}\,D^-_x\,dx,$$

then

$$A^+(f) := a^+(0, K_2 f) + a^-(JK_1 f, 0),$$
$$A^-(f) := a^-(0, K_2 f) + a^+(JK_1 f, 0).$$

Theorem 8. *Additionally to the above assumption on the operators K_1 and K_2, assume that these operators are self-adjoint and satisfy*

$$K_2^2 = \mathbf{1} + \eta K_1^2.$$

Then the operators D^+_x, D^-_x satisfy the commutation relations (39)–(41) *and generate an ACR algebra* $\mathbf{A}$.

In [20, Theorem 19], Theorem 8 was proved under the assumption that the operators K_i were real, i.e., satisfying $JK_i = K_iJ$, or equivalently $JK_iJ = K_i$ $(i = 1, 2)$. The proof of Theorem 8 is similar to that of [20, Theorem 19], so we omit it. (The proof of Theorem 8 actually extends the proof of Proposition 5 to the continuous setting.)

The definition of a gauge-invariant quasi-free state on an ACR algebra was given in [20, Subsection 2.3]. However, it is technically rather difficult. Below we will present a simplified definition, which will be completely sufficient for our purposes.

Definition 9. We will say that *a state τ on the ACR algebra $\mathbf{A}$ is gauge-invariant quasi-free* if:

- $\mathbf{S}^{(k,n)} \equiv 0$ if $k \neq n$;

- There exists a continuous self-adjoint operator K in $\mathcal{H}$ that commutes with any operator of multiplication by a bounded measurable function $\varkappa(x^1)$ and
$$\mathbf{S}^{(1,1)}(g,h,\varkappa^{(2)}) = \int_X (Kg)(x)h(x)\varkappa^{(2)}(x^1,x^1)\,dx; \tag{45}$$
- For $n \geq 2$,
$$\begin{aligned}&\mathbf{S}^{(n,n)}(g_n,\dots,g_1,h_1,\dots,h_n,\psi^{(2n)})\\ &= \sum_{\pi\in S_n}\int_{X^n}\left(\prod_{i=1}^n (Kg_i)(x_i)h_{\pi(i)}(x_i)\right)\\ &\quad\times \psi^{(2n)}(x_n^1,\dots,x_1^1,x_{\pi^{-1}(1)}^1,\dots,x_{\pi^{-1}(n)}^1)Q_\pi(x_1,\dots,x_n)\,dx_1\cdots dx_n.\end{aligned} \tag{46}$$

Remark 10. Note that Definition 9 implies that
$$\tau\big(a^+(g)a^-(h)\big) = \int_X (Kg)(x)\overline{h(x)}\,dx = (Kg,h)_{\mathcal{H}}$$
and for $n \geq 2$
$$\begin{aligned}&\tau\big(a^+(g_n)\cdots a^+(g_1)a^-(h_1)\cdots a^-(h_n)\big)\\ &\quad= \sum_{\pi\in S_n}\int_{X^n}\left(\prod_{i=1}^n (Kg_i)(x_i)\overline{h_{\pi(i)}(x_i)}\right) Q_\pi(x_1,\dots,x_n)\,dx_1\cdots dx_n.\end{aligned} \tag{47}$$
In particular, in the Fermi case, $Q_\pi \equiv \operatorname{sgn}\pi$ and so
$$\tau\big(a^+(g_n)\cdots a^+(g_1)a^-(h_1)\cdots a^-(h_n)\big) = \sum_{\pi\in S_n}\operatorname{sgn}\pi\prod_{i=i}^n (Kg_i,h_{\pi(i)})_{\mathcal{H}}.$$
In the general case, the right-hand side of (47) can be though of as a *functional Q-determinant in the continuum.*

Theorem 11. *Let* $\mathbf{A}$ *be the ACR algebra from Theorem* 8*. We define a state* τ *on* $\mathbf{A}$ *by*
$$\tau(A) := (A\Omega,\Omega)_{\mathcal{F}^{\mathbf{Q}}(L^2(Y,dx))}, \quad A\in\mathbf{A}.$$
Then τ *is a gauge-invariant quasi-free state on* $\mathbf{A}$ *satisfying* (45), (46) *with* $K=K_1^2$.

Theorem 11 is proved analogously to [20, Theorem 22], see also the proof of Proposition 6.

Corollary 12. *Let* $\eta\geq 0$*. Let* K *be a continuous linear operator in* $\mathcal{H}$*. Assume* K *commutes with any operator of multiplication by a bounded measurable function* $\varkappa(x^1)$*. Let also* $K\geq 0$*. Then there exists a gauge-invariant quasi-free state* τ *on the ACR algebra* $\mathbf{A}$ *that satisfies* (45), (46).

If $\eta<0$*, the latter statement remains true if the operator* K *additionally satisfies* $0\leq K\leq -1/\eta$.

For the proof of Corollary 12, just choose in Theorem 11
$$K_1 := \sqrt{K}, \quad K_2 := \sqrt{\mathbf{1}+\eta K}.$$

References

[1] Araki, H.: *On quasifree states of CAR and Bogoliubov automorphisms*. Publ. Res. Inst. Math. Sci. **6** (1970/71), 385–442.

[2] Araki, H.: *On quasifree states of the canonical commutation relations. II*. Publ. Res. Inst. Math. Sci. **7** (1971/72), 121–152.

[3] Araki, H., Shiraishi, M.: *On quasifree states of the canonical commutation relations. I*. Publ. Res. Inst. Math. Sci. **7** (1971/72), 105–120.

[4] Araki, H., Woods, E.: *Representations of the C.C.R. for a nonrelativistic infinite free Bose gas*. J. Math. Phys. **4** (1963), 637–662.

[5] Araki, H., Wyss, W.: *Representations of canonical anticommutation relations*, Helv. Phys. Acta **37** (1964), 136–159.

[6] Araki, H., Yamagami, S.: *On quasi-equivalence of quasifree states of the canonical commutation relations*. Publ. Res. Inst. Math. Sci. **18** (1982), 703–758 (283–338).

[7] Berezin, F.A.: *The Method of Second Quantization. Academic Press*, New York, London, 1966.

[8] Berezin, F.A.: *Some remarks on the representations of commutation relations*. (In Russian) Uspehi Mat. Nauk **24** (1969), no. 4, 65–88.

[9] Bożejko, M., Lytvynov, E., Wysoczański, J.: *Noncommutative Lévy processes for generalized* (*particularly anyon*) *statistics*. Comm. Math. Phys. **313** (2012), 535–569.

[10] Bożejko, M., Lytvynov, E., Wysoczański, J.: *Fock representations of Q-deformed commutation relations*. J. Math. Phys. **58** (2017), 073501, 19 pp.

[11] Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum-Statistical Mechanics. II. Equilibrium States*. Springer-Verlag, New York, Berlin, 1981.

[12] Dereziński, J., Gérard, C.: *Mathematics of Quantization and Quantum Fields*. Cambridge University Press, Cambridge, 2013.

[13] Goldin, G.A., Majid, S.: *On the Fock space for nonrelativistic anyon fields and braided tensor products*. J. Math. Phys. **45** (2004), 3770–3787.

[14] Goldin, G.A., Menikoff, R., Sharp, D.H.: *Particle statistics from induced representations of a local current group*. J. Math. Phys. **21** (1980), 650–664.

[15] Goldin, G.A., Menikoff, R., Sharp, D.H.: *Representations of a local current algebra in nonsimply connected space and the Aharonov–Bohm effect*. J. Math. Phys. **22** (1981), 1664–1668.

[16] Goldin, G.A., Sharp, D.H.: *Diffeomorphism groups, anyon fields, and q commutators*. Phys. Rev. Lett. **76** (1996), 1183–1187.

[17] Hugenholtz, N.M., Kadison, R.V.: *Automorphisms and quasi-free states of the CAR algebra*. Comm. Math. Phys. **43** (1975), 181–197.

[18] Leinass, J.M., Myrheim, J.: *On the theory of identical particles*. Nuovo Cimento **37 B** (1977), 1–23.

[19] Liguori, A., Mintchev, M.: *Fock representations of quantum fields with generalized statistics*. Comm. Math. Phys. **169** (1995), 635–652.

[20] Lytvynov, Eugene *Gauge-invariant quasi-free states on the algebra of the anyon commutation relations*. Comm. Math. Phys. **351** (2017), 653–687.

[21] Manuceau, J.; Verbeure, A.: *Quasi-free states of the C.C.R.-algebra and Bogoliubov transformations*. Comm. Math. Phys. **9** (1968), 293–302.

[22] Powers, R.T., Størmer, E.: *Free states of the canonical anticommutation relations*. Comm. Math. Phys. **16** (1970), 1–33.

[23] Wilczek, F.: *Quantum mechanics of fractional-spin particles*. Phys. Rev. Lett. **49** (1982), 957–959.

[24] Wilczek, F.: *Magnetic flux, angular momentum, and statistics*. Phys. Rev. Lett. **48** (1982), 1144–1145

Eugene Lytvynov
Department of Mathematics
Swansea University
Singleton Park
Swansea, SA2 8PP, UK
e-mail: e.lytvynov@swansea.ac.uk

Part VIII

Special Topics

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 331–340

Remarks to the Resonance-Decay Problem in Quantum Mechanics from a Mathematical Point of View

Hellmut Baumgärtel

Abstract. The description of bumps in scattering cross-sections by Breit–Wigner amplitudes led in the framework of the mathematical Physics to its formulation as the so-called Resonance-Decay Problem. It consists of a spectral theoretical component and the connection of this component with the construction of decaying states. First the note quotes a solution for scattering systems, where the absolutely continuous parts of the Hamiltonians are semi-bounded and the scattering matrix is holomorphic in the upper half-plane. This result uses the approach developed by Lax and Phillips, where the energy scale is extended to the whole real axis. The relationship of the spectral theoretic part of its solution and corresponding solutions obtained by other approaches is explained in the case of the Friedrichs model. A No-Go theorem shows the impossibility of the total solution within the specific framework of non-relativistic quantum mechanics. This points to the importance of the Lax–Phillips approach. At last, a solution is presented, where the scattering matrix is meromorphic in the upper half-plane.

Mathematics Subject Classification (2010). 81U20 Scattering Theory.

Keywords. Resonance-decay problem, Lax–Phillips approach, Friedrichs model.

1. Introduction

The origin of the resonance-decay problem in non-relativistic quantum mechanics is the observation of bumps in scattering cross-sections and their successful description by the so-called Breit–Wigner formula

$$\mathbb{R} \ni \lambda \to \frac{1}{\pi} \frac{\alpha}{(\lambda - c)^2 + \alpha^2}, \quad \alpha > 0,\ c \in \mathbb{R},$$

where c is the resonance energy E_0. Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha}{(\lambda-c)^2+\alpha^2}d\lambda = 1, \tag{1}$$

the Breit–Wigner formula describes, in a more general scope, a probability distribution of a real numerical quantity varying over the whole real axis $\mathbb{R}$. However in the framework of scattering theory the Breit–Wigner formula is interpreted by the Breit–Wigner amplitude

$$e(\lambda) := (\frac{\alpha}{\pi})^{1/2}\frac{1}{\lambda-(c-i\alpha)}, \quad c-i\alpha =: \zeta \in \mathbb{C}_-,$$

such that Eq. (1) now reads $\int_{-\infty}^{\infty}|e(\lambda)|^2 d\lambda = 1$. This suggests to consider the function e as a state of a quantum-mechanical quantity, whose states are elements of the Hilbert space $L^2(\mathbb{R}, d\lambda)$, where the energy is "diagonalized", i.e., the energy operator is the multiplication operator M in this space and the time-evolution is given by the unitary operator e^{-itM}.

The so-called expectation value for the state e according to the evolution e^{-itM} is given by $(e, e^{-itM}e)$ and the corresponding "Born probability" by $|(e, e^{-itM}e)|^2$. The calculation of the expectation value gives

$$(e, e^{-itM}e) = e^{-\alpha|t|-ict},$$

i.e., with $\zeta := c - i\alpha$ one gets

$$(e, e^{-itM}e) = \begin{cases} e^{-it\zeta}, & t > 0, \\ e^{-it\overline{\zeta}}, & t < 0, \end{cases} \tag{2}$$

and

$$|(e, e^{-itM}e)|^2 = e^{-2\alpha|t|}, \quad t \in \mathbb{R}.$$

That is, the expectation value forms an exponentially decaying semigroup $t \to e^{-it\zeta}$ for $t \geqslant 0$, similarly for $t \leqslant 0$. Within this context the Breit–Wigner amplitude – in particular the decay-semigroup property (2) – suggests the idea that e could be interpreted as an unstable or decaying state, i.e., as an eigenvector of an exponentially decaying semigroup, where ζ is an eigenvalue of its generator. This interpretation of the Breit–Wigner amplitude leads to the problem to derive such a semigroup and the spectrum of its generator from properties of the scattering matrix, in particular from their poles in the lower half-plane. The reason to focus on the poles is explained in Section 2. The formulation of these ideas within the framework of the mathematical scattering theory led to the so-called

Resonance-Decay Problem. Let $\{H, H_0\}$ be an asymptotically complete quantum-mechanical scattering system with scattering operator S. Then one has to construct a non-selfadjoint operator B, generator of a so-called decay-semigroup, depending on H via S, whose eigenvalue spectrum coincides with the set of all poles of the scattering matrix in the lower half-plane, such that the corresponding eigenstates can be interpreted as the hypothetical decaying states, connected with the Breit–Wigner amplitudes.

In other words, the first part of the problem is a *spectral theoretical characterization* of the poles of the scattering matrix in the lower half-plane and the second step is to connect this characterization with the decay problem. Since the seventies this problem induced a vast series of various developments (for a selection of references see [1, 2]). In the non-relativistic quantum mechanics the focus of interest is directed to scattering systems, whose Hamiltonians are semi-bounded. However the mathematical framework of scattering theory also includes scattering systems, where the generator H of the unitary evolution group $\mathbb{R} \ni t \to e^{-itH}$ on a Hilbert space $\mathcal{H}$ together with the multiplication operator $M =: H_0$ on the Hilbert space $L^2(\mathbb{R}, \mathcal{K}, d\lambda)$ form an asymptotic complete scattering system, i.e., the absolutely continuous spectrum of H and H_0 is the whole real axis. The Hilbert space $\mathcal{K}$ represents the multiplicity of the problem.

A scattering theory for such unitary evolutions, which are equipped with so-called outgoing and incoming subspaces was presented by P.D. Lax and R.S. Phillips (see [3] and also [6, Chap. 12]), in particular for mutual orthogonal out- and incoming subspaces. In this case the scattering matrix is holomorphic in $\mathbb{C}_+$, it satisfies the condition $\|S(z)\| \leqslant 1, z \in \mathbb{C}_+$, and Lax and Phillips solved the resonance-decay problem completely. A decisive concept for their solution is an invariant subspace of a special decay-semigroup, defined on the so-called Hardy space $\mathcal{H}^2_+(\mathbb{R}, \mathcal{K}, d\lambda)$ for the upper half-plane, defined by

$$\mathbb{R} \ni t \to Q_+ e^{-itM} \restriction_{\mathcal{H}^2_+}, \tag{3}$$

where Q_+ is the projection from $L^2(\mathbb{R}, \mathcal{K}, d\lambda)$ onto $\mathcal{H}^2_+$. In the following this decay-semigroup is called the characteristic semigroup. At this point it is appropriate to refer to the property (2) of the Breit–Wigner amplitude and the idea mentioned there. Remarkably e is an eigenvector of the decay-semigroup (3) with eigenvalue $e^{-i\zeta t}$. I.e., the Breit–Wigner amplitude appears in the Lax–Phillips approach as an important element of a solution of the resonance-decay problem.

The idea, to use the Lax–Phillips technique also for the resonance-decay problem on the positive energy half-axis, i.e., for semi-bounded Hamiltonians, is based on the property that the linear manifold $P_+\mathcal{H}^2_+$ is dense in the Hilbert space $L^2(\mathbb{R}_+, \mathcal{K}, d\lambda)$ of the reference Hamiltonian H_0, where P_+ is the projection from $L^2(\mathbb{R}, \mathcal{K}, d\lambda)$ on $L^2(\mathbb{R}_+, \mathcal{K}, d\lambda)$. The linear manifold $P_+\mathcal{H}^2_+$ equipped with the norm of $\mathcal{H}^2_+$ *is* $\mathcal{H}^2_+$ itself, since P_+ is injective on $\mathcal{H}^2_+$. The proof of the following result for semi-bounded Hamiltonians H, H_0 applies the Lax–Phillips technique, in particular the properties of the characteristic semigroup (see [2, Theorem 3]):

If the scattering matrix of the scattering system $\{H, H_0\}$ on $\mathbb{R}_+$ is holomorphic continuable into the upper half-plane $\mathbb{C}_+$ and satisfies a certain boundedness condition then an invariant subspace of the characteristic semigroup is constructed such that its restriction to this subspace is a solution of the resonance-decay problem. That is, the spectrum of the generator of this restriction consists exactly of all poles of the scattering matrix and its resolvent set of all points, where the scattering matrix is holomorphic. The eigenvectors for the poles ζ are exactly special Breit–Wigner amplitudes $k/(\lambda - \zeta)$ for certain vectors k, which satisfy the condition $S(\overline{\zeta})^* k = 0$, thus solving the multiplicity problem.

So far, this result solves the first part of the resonance-decay problem, the spectral characterization of the poles, completely. However, the eigenvectors, i.e., the decaying states, are vectors from $\mathcal{H}_+^2 \subset L^2(\mathbb{R}, \mathcal{K}, d\lambda)$. This means that the direct relationship to the positive energy axis and the initial Hilbert space of states is lost. Even it turns out that these states constructed, i.e., the Breit–Wigner amplitudes, cannot be transferred unitarily to the Hilbert space $L^2(\mathbb{R}_+, \mathcal{K}, d\lambda)$ of the reference Hamiltonian $H_0 = M_+$, its multiplication operator, to be decaying states w.r.t. the evolution e^{-itM_+} (see Section 3).

Nevertheless, in sight of the structural mathematical point of view, the result can be interpreted within this Hilbert space: in the dense linear manifold $P_+\mathcal{H}_+^2$ one can introduce without ambiguity the $\mathcal{H}_+^2$-norm. Then the decay-semigroup can be considered as acting on $P_+\mathcal{H}_+^2$ equipped with this stronger norm.

The real weakness of this result is that its ansatz is an abstract one. Basically, it takes into consideration only the (canonical) reference Hamiltonian H_0 for the absolutely continuous spectrum and the scattering operator. The right or justification to use this ansatz goes back to the theorem of Wollenberg (see [5, 6]) together with the fact that the Hamiltonian H, i.e., the interaction, is sometimes unknown. However, the reason for the appearance of poles in the scattering matrix of a scattering system H, H_0 remains to be seen.

2. Friedrichs model

In so-called Friedrichs models the reason for the appearance of poles in the corresponding scattering matrix can be recognized. In these models eigenvalues of the reference operator H_0, which are embedded in its absolutely continuous spectrum are sometimes unstable caused by the interaction of H. They can generate poles of the scattering matrix. Therefore, the spectral theoretical characterization of these poles can be alternatively obtained by the method of “generalized eigenvalues”, for example by so-called Gelfand triples. In this approach the corresponding eigenantilinear forms are usually of the pure Dirac-type (see, e.g., [4, 7, 9] and [8]). Interestingly for the Friedrichs model presented in [7] the solution of the multiplicity problem corresponds exactly to the solution quoted in Section 1 (see [7, Theorem 4.2]). In this paper the Friedrichs model $H := M + \Gamma + \Gamma^*$ on the full energy axis is considered on the Hilbert state space $\mathcal{H} := L^2(\mathbb{R}, \mathcal{K}, d\lambda) \oplus \mathcal{E}$, where $\dim \mathcal{K} < \infty, \dim \mathcal{E} < \infty$ and M the multiplication operator. The operator $\Gamma : \mathcal{E} \to L^2(\mathbb{R}, \mathcal{K}, d\lambda)$ is defined by $(\Gamma e)(\lambda) := M(\lambda)e$, where $M(\lambda) \in \mathcal{L}(\mathcal{E} \to \mathcal{K})$. In this model one obtains for the scattering matrix the expression

$$S_{\mathcal{K}}(\lambda) = \mathbb{1}_{\mathcal{K}} - 2\pi i M(\lambda) L_+(\lambda + i0)^{-1} M(\lambda)^*, \tag{4}$$

where $L_+(\cdot)$ denotes the so-called Livšic matrix on $\mathbb{C}_+$. Now, if one assumes that $M(\cdot)$ is holomorphic on $\mathbb{R}$ and meromorphic continuable, then for a pole ζ of the scattering matrix in $\mathbb{C}_-$ one obtains that the multiplicity of the corresponding Dirac-antilinear form is given by those $k \in \mathcal{K}$, such that $k = M(\zeta)e, e \in \mathcal{E}$, where $L_+(\zeta)e = 0$ and $L_+(\cdot)$ denotes the continuation of the Livšic matrix into the lower half-plane (see [7, Theorem 4.2]). The correspondence to the solution of the multiplicity problem quoted in Section 1 is expressed by

Lemma. *Let $\zeta \in \mathbb{C}_-$ and $k \in \mathcal{K}$. Then $S(\overline{\zeta})^* k = 0$ iff $k = M(\zeta)e$, where $e \in \mathcal{E}$* and *$L_+(\zeta)e = 0$.*

Proof. (i) Let $S(\overline{\zeta})^* k = 0$. Then, according to Eq. (4), one obtains

$$k = -2\pi i M(\zeta)(L_+(\overline{\zeta})^{-1})^* M(\overline{\zeta})^* k$$

Put

$$e := -2\pi i (L_+(\overline{\zeta})^{-1})^* M(\overline{\zeta})^* k = -2\pi i (L_+(\overline{\zeta})^*)^{-1} M(\overline{\zeta})^* k. \tag{5}$$

Since $\zeta \in \mathbb{C}_-$, the Livšic matrix at $\overline{\zeta}$ reads

$$L_+(\overline{\zeta}) = (\overline{\zeta} - H_0)P_{\mathcal{E}} - \int_{-\infty}^{\infty} \frac{M(\lambda)^* M(\lambda)}{\overline{\zeta} - \lambda} d\lambda.$$

Further note

$$L_+(\overline{\zeta})^* = L_-(\zeta),$$

where $L_-(\cdot)$ denotes the Livšic matrix on $\mathbb{C}_-$. According to Eq. (5) one has

$$L_+(\overline{\zeta})^* e = -2\pi i M(\overline{\zeta})^* k = L_-(\zeta)e.$$

For the continuation of $L_+(\cdot)$ into the lower half-plane one obtains

$$L_+(\zeta) = L_-(\zeta) + 2\pi i M(\overline{\zeta})^* M(\zeta).$$

Then one obtains

$$L_+(\zeta)e = -2\pi i M(\overline{\zeta})^*(k - M(\zeta)e).$$

The definition of e in Eq. (5) implies $k = M(\zeta)e$. Therefore $L_+(\zeta)e = 0$ follows.
(ii) Let $k := M(\zeta)e$, where $L_+(\zeta)e = 0$. Then

$$L_+(\overline{\zeta})^* e = L_-(\zeta)e = -2\pi i M(\overline{\zeta})^* M(\zeta)e,$$

hence

$$e = -2\pi i (L_+(\overline{\zeta})^*)^{-1} M(\overline{\zeta})^* M(\zeta)e$$

and

$$k = M(\zeta)e = -2\pi i M(\zeta)(L_+(\overline{\zeta})^*)^{-1} M(\overline{\zeta})^* k$$

follows, i.e., $S(\overline{\zeta})^* k = 0$. □

Similar results one obtains for Friedrichs models on the positive half-axis.

The scattering matrices of Friedrichs models may have poles in the upper half-plane. Insofar the extension of the result mentioned in Section 1 to these cases is obvious. For example, if in the Friedrichs model considered one puts $\dim \mathcal{K} = 1$ and $\Gamma e(\lambda) := \pi^{-1/2}(\lambda + i)^{-1}$, then $\zeta := i$ is a pole of the scattering matrix.

3. A no-go-theorem

In Section 1 a solution of the resonance-decay problem for a scattering system on the positive half-line, proved in [2], is quoted and critically considered. It was mentioned that the decaying states constructed there cannot be transferred to the Hilbert space of the reference Hamiltonian. This section contains a proof for this assertion.

Theorem 1. *There is no state $\phi \in L^2(\mathbb{R}_+, \mathcal{K}, d\lambda)$, $\|\phi\| = 1$, such that the Born probability w.r.t. the unitary time evolution generated by M_+ is exponentially decaying, i.e., such that*

$$|(\phi, e^{-itM_+}\phi)|^2 = e^{-2\alpha t}, \quad t > 0,$$

for some constant $\alpha > 0$.

Proof. Born probabilities are symmetric w.r.t. future and past, i.e., they depend only on $|t|$. Assume that there is a state ϕ and a constant $\alpha > 0$ such that

$$|(\phi, e^{-itM_+}\phi)|^2 = e^{-2\alpha|t|}, \quad t \in \mathbb{R}.$$

Then $|(\phi, e^{-itM_+}\phi)| = e^{-\alpha|t|}$ and

$$(\phi, e^{-itM_+}\phi) = \int_0^\infty e^{-it\lambda}|\phi(\lambda)|^2_{\mathcal{K}} d\lambda = e^{-\alpha|t|+i\beta(t)},$$

where $\beta(\cdot)$ is real-valued, continuous and one has $\beta(-t) = -\beta(t)$. Define

$$g(\lambda) = \begin{cases} |\phi(\lambda)|^2_{\mathcal{K}}, & \lambda > 0 \\ 0, & \lambda < 0 \end{cases}$$

Then $g \in L^1(\mathbb{R}, d\lambda)$ and

$$\int_{-\infty}^\infty e^{-it\lambda} g(\lambda) d\lambda = e^{-\alpha|t|+i\beta(t)}.$$

The function on the right-hand side is a L^2-function, where

$$\int_{-\infty}^\infty \alpha|e^{-\alpha|t|+i\beta(t)}|^2 dt = 1,$$

i.e., there is a function $f \in L^2(\mathbb{R}, d\lambda)$ such that

$$F(f)(t) = \hat{f}(t) = \alpha^{1/2} e^{-\alpha|t|+i\beta(t)},$$

where $\|f\|_{L^2} = 1$ and F denotes the Fourier transform. That is, one obtains

$$\int_{-\infty}^\infty e^{-it\lambda}(2\pi\alpha)^{-1/2} f(\lambda) d\lambda = e^{-\alpha|t|+i\beta(t)} = \int_{-\infty}^\infty e^{-it\lambda} g(\lambda) d\lambda.$$

Since the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ w.r.t. the L^1-norm and dense in $L^2(\mathbb{R})$ w.r.t. the L^2-norm, according to a standard argument in the theory of the Fourier transformation it follows that $(2\pi\alpha)^{-1/2} f = g$, i.e., one obtains $g \in L^2(\mathbb{R})$.

Now g has the property $g = P_+ g$. This means $\hat{f}$ is an element of $\mathcal{H}^2_-(\mathbb{R})$, the Hardy space of the lower half-plane, i.e., one gets $\hat{f} = Q_- \hat{f}$, where Q_- denotes the projection onto $\mathcal{H}^2_-$. Because of

$$P_+ = F^{-1} Q_- F$$

it follows that the inverse Fourier transform $F^{-1}\hat{f}$ is necessarily from $P_+ L^2(\mathbb{R}, dx)$, i.e., the function

$$x \to \int_{-\infty}^{\infty} e^{ixt} e^{-\alpha|t| + i\beta(t)} dt$$

vanishes for $x < 0$. However, the function

$$h(z) := \int_{-\infty}^{\infty} e^{izt} e^{-\alpha|t| + i\beta(t)} dt$$

is well defined within the stripe $|\operatorname{Im} z| < \alpha$ and a holomorphic function there. Therefore, since this function vanishes for $z = x < 0$ it vanishes identically, hence also for $x > 0$, i.e., one obtains $g = f = 0$, a contradiction. □

4. A result for scattering systems with poles of the scattering matrix in the upper half-plane

An extension of the result mentioned in Section 1 in this direction is suggested in Section 2. An incomplete version of the following result can be found already in [2, Theorem 2], incomplete because of a flaw in the proof. Surprisingly it turns out that not only the poles in the lower half-plane cause decaying states, but also holomorphic points ζ there may generate such states, but only in the case that $\overline{\zeta}$ is a pole (in the upper half-plane) with a special property of its main part.

Theorem 2. *Assume that the scattering matrix of the scattering system $\{H, H_0\}$ on $\mathbb{R}$ satisfies the following conditions:*

(I) *It is meromorphic in $\mathbb{C}_+$ with at most finitely many poles,*
(II) $\|S(z)\| < K,\ K > 0,\ z \in \mathbb{C}_+\ |z| > R$*, where R is sufficiently large,*
(III) *there are no complex-conjugated poles,*
(IV) *there is at least one pole in $\mathbb{C}_-$.*

Then the spectrum $\operatorname{spec} B_+ \subset \mathbb{C}_-$ *of the generator B_+ of the restriction of the characteristic semigroup to the subspace $\mathbb{T}_+ \subset \mathcal{H}^2_+$ is described as follows:*

(i) *$\zeta \in \mathbb{C}_-$ is an eigenvalue of B_+ iff (a) ζ is a pole of $S(\cdot)$* or *(b) $\overline{\zeta}$ is a pole of $S(\cdot)$ and the operator coefficient A of the leading term of the main part of the pole $\overline{\zeta}$ is not invertible.*
(ii) *$\zeta \in \mathbb{C}_-$ is a point of the resolvent set* $\operatorname{res} B_+$ *of B_+ iff* (a) *$S(\zeta)$ and $S(\overline{\zeta})$ exist, i.e., $\zeta, \overline{\zeta}$ are holomorphic points of $S(\cdot)$ or* (b) *$S(\zeta)$ exists and $\overline{\zeta}$ is a pole of $S(\cdot)$ and A is invertible, i.e., A^{-1} exists.*

Proof. Let $\eta_1, \eta_2, \dots, \eta_r$ be the poles in $\mathbb{C}_+$ with the multiplicities $g_1, g_2, \dots, g_r$. Put $g := \sum_{j=1}^r g_j$. Let $p(\cdot)$ be the polynomial of degree g, defined by $p(\lambda) := \prod_{j=1}^r (\lambda - \eta_j)^{g_j}$. Put $\mathcal{M}_+ := S\mathcal{N}_+$, where $\mathcal{N}_+$ is the linear manifold of $\mathcal{H}_+^2$ of all functions u of the form $u(\lambda) := \frac{p(\lambda)}{(\lambda+i)^g} w(\lambda)$, where $w \in \mathcal{H}_+^2$. Further put $\mathcal{T}_+ := \mathcal{H}_+^2 \ominus \mathcal{M}_+$.

Proof of (i): Let ζ be an eigenvalue of B_+. Then a corresponding eigenvector has necessarily the form $f(\lambda) = \frac{k_0}{\lambda - \zeta}$ for some $k_0 \in \mathcal{K}$ and one has

$$\int_{-\infty}^{\infty} (\frac{k_0}{\lambda - \zeta}, S(\lambda)u(\lambda))_{\mathcal{K}} d\lambda = \int_{-\infty}^{\infty} \frac{1}{\lambda - \overline{\zeta}} (k_0, S(\lambda)u(\lambda))_{\mathcal{K}} d\lambda = 0, \quad u \in \mathcal{N}_+.$$

First let $\overline{\zeta}$ be a holomorphic point of $S(\cdot)$. Then one obtains

$$\int_{-\infty}^{\infty} \frac{1}{\lambda - \overline{\zeta}} (S(\lambda)^* k_0, u(\lambda))_{\mathcal{K}} d\lambda = 2\pi i (S(\overline{\zeta})^* k_0, u(\overline{\zeta}))_{\mathcal{K}} = 0,$$

hence $S(\overline{\zeta})^* k_0 = 0$ follows, since every $k \in \mathcal{K}$ is possible for $u(\overline{\zeta})$. This means that $S(\overline{\zeta})^*$ is not invertible, i.e., ζ is a pole and (a) is true. Therefore, one can assume that $\overline{\zeta}$ is a pole. Then one gets

$$\begin{aligned}\int_{-\infty}^{\infty} (\frac{k_0}{\lambda - \zeta}, S(\lambda)\frac{p(\lambda)}{(\lambda+i)^g} w(\lambda))_{\mathcal{K}} d\lambda &= \int_{-\infty}^{\infty} \frac{1}{\lambda - \overline{\zeta}} (k_0, S(\lambda)p(\lambda)\frac{w(\lambda)}{(\lambda+i)^g})_{\mathcal{K}} d\lambda \\ &= 0 = 2\pi i (k_0, (S(\cdot)p(\cdot))(\overline{\zeta}) \frac{w(\overline{\zeta})}{(\overline{\zeta}+i)^g})_{\mathcal{K}} \\ &= 2\pi i (k_0, c(\overline{\zeta}) A \frac{w(\overline{\zeta})}{(\overline{\zeta}+i)^g})_{\mathcal{K}},\end{aligned}$$

where $(S(\cdot)p(\cdot))(\overline{\zeta}) = c(\overline{\zeta})A$, $c(\overline{\zeta}) \neq 0$ and $A \neq 0$, where A is the leading term of the main part of the pole $\overline{\zeta}$, i.e., one obtains $(A^* k_0, k) = 0$ for all $k \in \mathcal{K}$, hence $A^* k_0 = 0$, i.e., A is not invertible and (b) is true.

For the reversal let ζ be a pole of $S(\cdot)$ and $S(\overline{\zeta})^* k_0 = 0$ or let $\overline{\zeta}$ be a pole and $A^* k_0 = 0$ for some $k_0 \in \mathcal{K}, k_0 \neq 0$. Then all calculations are reversible.

Proof of (ii): Let $\zeta \in \operatorname{res} B_+$. If ζ is a pole then ζ is an eigenvalue, hence it cannot be a member of $\operatorname{res} B_+$. If $S(\zeta)$ exists and $\overline{\zeta}$ is a pole, but A is not invertible then ζ is again an eigenvalue. Reversal:

(a): In this case $S(\zeta)$ and $S(\overline{\zeta})$ exist. Then ζ is not an eigenvalue, hence

$$(B_+ - \zeta \mathbb{1})^{-1}$$

exists. According to the "closed graph theorem" it is sufficient to show that

$$\operatorname{ima}(B_+ - \zeta \mathbb{1}) = \mathcal{T}_+$$

is true, i.e., if $g \in \mathcal{T}_+$ then one has to construct a function $f \in \operatorname{dom} B_+$ such that

$$(B_+ - \zeta \mathbb{1}) f = g.$$

In any case f is an element from the domain of the generator of the full characteristic semigroup. Therefore it is sufficient to construct f as an element of $\mathcal{T}_+ = \mathcal{H}^2_+ \ominus \mathcal{M}_+$ such that

$$f(\lambda) = \frac{g(\lambda) - k_0}{\lambda - \zeta},$$

where $k_0 \in \mathcal{K}$ is a suitable vector. Since $g \perp \mathcal{M}_+$, i.e., $g \perp S\mathcal{N}_+$ or $S^* g \perp \mathcal{N}_+$ one has

$$S^* \frac{\overline{p(\cdot)} g(\cdot)}{(\cdot - i)^g} \perp \mathcal{H}^2_+,$$

i.e., the function

$$\lambda \to h(\lambda) := S(\lambda)^* \frac{\overline{p(\lambda)}}{(\lambda - i)^g} g(\lambda)$$

is an element of $\mathcal{H}^2_-$. The corresponding expression for f reads

$$\frac{S(\lambda)^* \overline{p(\lambda)}}{(\lambda - i)^g} f(\lambda) = (\lambda - \zeta)^{-1} \Big(h(\lambda) - \frac{S(\lambda)^* \overline{p(\lambda)}}{(\lambda - i)^g} k_0\Big). \tag{6}$$

Since $\zeta \in \mathbb{C}_-$, on has $p(\zeta) \neq 0$. In order that the right-hand side of Eq. (6) is from $\mathcal{H}^2_-$, one has to put

$$h(\zeta) = \frac{S(\zeta)^* \overline{p(\zeta)}}{(\zeta - i)^g} k_0 = \frac{S(\overline{\zeta})^{-1} \overline{p(\zeta)}}{(\zeta - i)^g} k_0.$$

Hence k_0 is uniquely determined by

$$k_0 = \frac{(\zeta - i)^g}{\overline{p(\zeta)}} S(\zeta) h(\zeta).$$

(b): In this case $S(\zeta)$ exists but $\overline{\zeta}$ is a pole of $S(\cdot)$ and A is invertible. The function h is defined as before. Now the equation (6) is written in the form

$$\frac{(S(\cdot)p(\cdot))^*(\lambda)}{(\lambda - i)^g} f(\lambda) = (\lambda - \zeta)^{-1} \Big(h(\lambda) - \frac{(S(\cdot)p(\cdot))^*(\lambda)}{(\lambda - i)^g} k_0\Big).$$

In this case one has $(S(\cdot)p(\cdot))(\overline{\zeta}) = cA$, where $c \neq 0$ because $\overline{\zeta}$ is one of the poles η_j and A is again the leading term of the main part of the pole $\overline{\zeta}$, i.e.,

$$(S(\cdot)p(\cdot))^*(\zeta) = \overline{c} A^*$$

and one obtains k_0 uniquely from the equation

$$h(\zeta) = \frac{\overline{c}}{(\zeta - i)^g} A^* k_0,$$

i.e.,

$$k_0 = (\overline{c})^{-1} (\zeta - i)^g (A^*)^{-1} h(\zeta). \qquad \square$$

References

[1] Baumgärtel, H., *The Resonance-Decay Problem in Quantum Mechanics*, in: Geometric Methods in Physics, XXX Workshop 2011, Trends in Mathematics, 165–174, Springer Basel 2013.

[2] Baumgärtel, H., *Resonances of quantum mechanical scattering systems and Lax–Phillips scattering theory*, J. Math. Phys. **51**, 113508 (2010).

[3] Lax, P.D. and Phillips, R.S., *Scattering Theory* Academic Press, New York 1967.

[4] Bohm, A. and Gadella, M.: *Dirac Kets, Gamov Vectors and Gelfand Triplets*, Lecture Notes in Physics, Vol. **348**, Springer Berlin 1989.

[5] Wollenberg, M., *On the inverse problem in the abstract theory of scattering*, ZIMM-Preprint Akad. Wiss. DDR, Berlin 1977.

[6] Baumgärtel, H. and Wollenberg, M., *Mathematical Scattering Theory*. Birkhäuser Basel Boston Stuttgart 1983.

[7] Baumgärtel, H., *Generalized Eigenvectors for Resonances in the Friedrichs Model and Their Associated Gamov Vectors*, Rev. Math. Phys. **18**, 61–78 (2006).

[8] Baumgärtel, H, Kaldass, H. and Komy, S., *On spectral properties of the resonances for selected potential scattering systems*, J. Math. Phys. **50**, 023511 (2009).

[9] Baumgärtel, H., *Spectral and Scattering Theory of Friedrichs Models on the Positive Half Line with Hilbert–Schmidt Perturbations*, Annales Henri Poincaré (AHP) 10, Nr. 1, 123–143 (2009).

Hellmut Baumgärtel
Mathematical Institute
University of Potsdam, Germany
e-mail: baumg@uni-potsdam.de

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 341–346

Dynamical Generation of Graphene

M. Myronova and E. Bourret

Abstract. In recent years, the astonishing physical properties of carbon nanostructures have been discovered and are nowadays being intensively studied. We introduce how to obtain a graphene sheet using group theoretical methods and how to construct a graphene layer using the method of dynamical generation of quasicrystals. Both approaches can be formulated in such a way that the points of an infinite graphene sheet are generated. The main objective is to describe how to generate graphene step by step from a single point of $\mathbb{R}^2$.

Mathematics Subject Classification (2010). Group theory, Mathematical crystallography.

Keywords. Dynamical generation, graphene, nanotubes, congruence classes.

1. Introduction

Graphene is a two-dimensional Euclidean plane tiled by regular hexagons, these hexagons being all of the same size. The vertices of the hexagons are usually taken as carbon atoms, but other graphene-like structures were also observed [4, 5].

Although the most promising nanomaterials are graphene and carbon nanotubes, their geometrical structures remain so far unexplored. Owing to their exceptional physical, chemical and mechanical properties, they found an increasing variety of applications [6].

The mathematical way to obtain a graphene sheet and the related nanotubes is to use the finite reflection groups. In general, we should use the Lie algebras A_2 and G_2 (or respectively their groups $SU(3)$ and $G(2)$) to construct a layer of graphene, because both of them yield triangular lattices. In this paper, we consider two mathematical methods to construct graphene. Both will provide identical graphene layers.

The first method used to obtain a mathematical model for the graphene is the construction using the simple Lie group $SU(3)$. It gives us an opportunity to define the congruence classes for the points of its weight lattice and, as a result, to obtain an hexagonal tilling of $\mathbb{R}^2$ by removing the points of one of the congruence classes.

The second method we used is inspired by the process of dynamical generation of quasicrystals [1]. It was shown that, by using the Coxeter groups H_2, H_3 and H_4, quasicrystals can be constructed from a single point of $\mathbb{C}^n$ $(n = 2, 3, 4)$. In our case, we consider the Lie group $SU(3)$ and we define a step-by-step construction of the graphene from a single point of the plane $\mathbb{R}^2$. Furthermore, the method of dynamical generation could be used to obtain any crystallographic structure.

2. Root and weigh lattices of A_n

Let $\Phi = \{\alpha_1, \dots, \alpha_n\}$ be a root system of rank n of the Lie algebra A_n in real Euclidean space $\mathbb{R}^n$ [2]. It is determined by the Coxeter–Dynkin diagram shown in Figure 1.

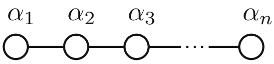


FIGURE 1. Coxeter–Dynkin diagram of the Lie group $SU(n+1)$.

The set of simple roots α_i, $i = 1, \dots, n$ of the root system Φ of A_n is called an α-basis. Thus, for the Coxeter–Dynkin diagram, there is the corresponding Cartan matrix, which gives the geometry of the α-basis:

$$\mathcal{C}_{jk} = \frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle}, \quad \text{where} \quad j, k = 1, 2, \dots, n. \tag{1}$$

The set of ω_k, $k = 1, \dots, n$ is called the set of fundamental weights and forms the ω-basis (or the basis of fundamental weights). It is convenient to work mostly in the ω-basis. Therefore, we need to convert the α-basis using the duality relation of the bases:

$$\frac{2\langle \alpha_j, \omega_k \rangle}{\langle \alpha_j, \alpha_j \rangle} \equiv \langle \alpha_j^\vee, \omega_k \rangle = \delta_{jk}, \quad \text{where} \quad j, k \in \{1, 2, \dots, n\}. \tag{2}$$

The link between the α- and ω-bases is also given by the Cartan matrix $\mathcal{C}$ (1) and its inverse $\mathcal{C}^{-1}$:

$$\alpha_j = \sum_{k=1}^{n} \mathcal{C}_{jk}\omega_k \quad \text{and} \quad \omega_j = \sum_{k=1}^{n} (\mathcal{C}_{jk}^{-1})\alpha_k. \tag{3}$$

It is important to introduce the root lattice $\mathcal{Q}$ given by the set of all linear combinations of the simple roots α_i:

$$\mathcal{Q} = \Big\{ \sum_{i=1}^{n} a_i\alpha_i \mid a_i \in \mathbb{Z} \Big\} \equiv \bigoplus_i \mathbb{Z}\alpha_i \equiv \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n. \tag{4}$$

The positive root lattice $\mathcal{Q}_+$ of $\mathcal{Q}$ is defined as:

$$\mathcal{Q}_+ = \Big\{ \sum_{i=1}^{n} a_i \alpha_i \mid a_i = \mathbb{Z}^{\geq 0} \Big\} \equiv a_1\alpha_1 + \cdots + a_n\alpha_n. \tag{5}$$

Likewise, we introduce the weight lattice $\mathcal{P}$ and the cone of dominant weight $\mathcal{P}_+$:

$$\mathcal{P} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n \quad \text{and} \quad \mathcal{P}_+ = \mathbb{Z}^{\geq 0}\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\omega_n. \tag{6}$$

In general, the points of the weight lattice $\mathcal{P}$ of the Lie group $SU(n+1)$ can be split into $(n+1)$ congruence classes denoted as $\mathcal{K}_k$, $k = \mathbb{Z}^{\geq 0}$ [3]. Each point of $\mathcal{P}$ belongs precisely to one congruence class and the splitting is defined as

$$\begin{aligned} &x = a_1\omega_1 + a_2\omega_2 + \ldots + a_n\omega_n \in K_k, \\ &na_1 + (n-1)a_2 + \cdots + 2a_{n-1} + a_n = k \mod n+1. \end{aligned} \tag{7}$$

3. Construction of the graphene from the Lie algebra A_2

The most appropriate way to construct a mathematical model for the graphene layer is to use the simple Lie algebra A_2. The root system of A_2 and its Coxeter–Dynkin diagram are shown in Figure 2.

FIGURE 2. The root system of Lie algebra A_2 and it's Coxeter–Dynkin diagram are shown from left to right, respectively.

The simple roots α_1 and α_2 span a real Euclidean space $\mathbb{R}^2$. The geometric relations between them are:

$$\langle\alpha_1, \alpha_1\rangle = \langle\alpha_2, \alpha_2\rangle = 2, \quad \langle\alpha_1, \alpha_2\rangle = -1 \quad \text{and} \quad \angle(\alpha_1, \alpha_2) = \frac{2\pi}{3}.$$

The Cartan matrix of A_2 and its inverse are defined from (1) as follows:

$$\mathcal{C}_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \mathcal{C}_{A_2}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{8}$$

As we mentioned before, the link between α- and ω-bases of A_2 is given by the Cartan matrix (8). Hence, we can write explicitly:

$$\alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = -\omega_1 + 2\omega_2, \quad \omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2.$$

For the A_2 case, the expressions for the root lattice $\mathcal{Q}$ (4) as well as for the weight lattice $\mathcal{P}$ (6) can be simplified:

$$\mathcal{Q}_{A_2} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \quad \mathcal{P}_{A_2} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2. \tag{9}$$

From now on, there are two ways to construct a graphene sheet using the Lie algebra A_2. The first one is to consider the root lattice $\mathcal{Q}_{A_2}$ and the second one is to use the weight lattice $\mathcal{P}_{A_2}$. Both of these lattices are triangular in $\mathbb{R}^2$. Root and weight lattices coincide, but not all the points of $\mathcal{Q}_{A_2}$ belong to $\mathcal{P}_{A_2}$ (Fig. 3).

The construction of the graphene using the root lattice $\mathcal{Q}_{A_2}$ starts from finding the proximity cells, called Voronoi domains or Brillouin zones, for each of the lattice points. In this case, the proximity cells are regular hexagons which tile $\mathbb{R}^2$ (Fig. 3). We obtain the graphene sheet by removing all the points of the lattice $\mathcal{Q}_{A_2}$ while retaining the hexagons of proximity cells.

However, an interesting case appears when we use $\mathcal{P}_{A_2}$. As was defined in the previous section, the points of $\mathcal{P}_{A_2}$ can be split into the three mutually congruent classes $\mathcal{K}_k$, $k = 0, 1, 2$, by applying the rule from Equation (7):

$$x = a_1\omega_1 + a_2\omega_2 \in \mathcal{K}_k, \quad \text{where} \quad 2a_1 + a_2 = k \bmod 3 \quad \text{and} \quad k = 0, 1, 2.$$

The result of this splitting is the following: the points of the congruence class $\mathcal{K}_0$ represent the points of $\mathcal{Q}_{A_2}$, the points of $\mathcal{K}_1$ represent the points of $\mathcal{Q}_{A_2} + \omega_1$, and the points of $\mathcal{K}_2$ represent the points of $\mathcal{Q}_{A_2} + \omega_2$ (Fig. 3).

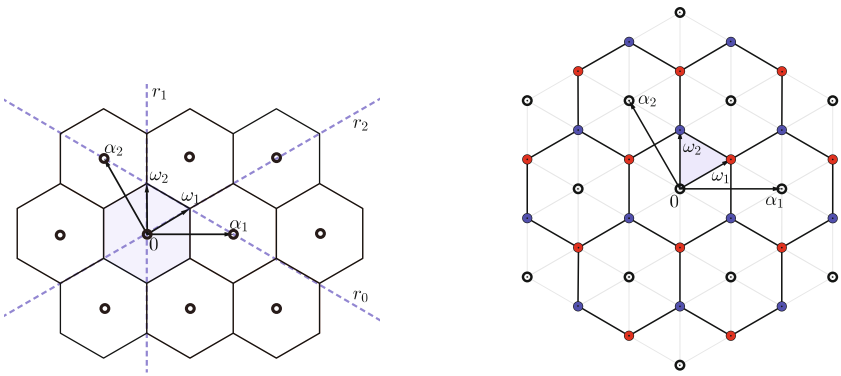


FIGURE 3. On the left, a fragment of the Q_{A_2} is shown. The shaded region stands for the Voronoi domain. On the right, a fragment of the $\mathcal{P}_{A_2}$ is shown. Points marked by white nodes belong to $\mathcal{Q}_{A_2}$ and the congruence class $\mathcal{K}_0$. Points marked by red and blue nodes belong to $\mathcal{P}_{A_2}$ and $\mathcal{K}_1$, $\mathcal{K}_2$, respectively. The shaded region stands for the fundamental domain of $\mathcal{P}_{A_2}$.

Consequently, removing the points of $\mathcal{K}_0$, $\mathcal{K}_1$ or $\mathcal{K}_2$ yields the hexagonal structure which represents a graphene layer. For example, in Figure 3, we disregarded the points of $\mathcal{K}_0$.

Note that even though both constructions start with the triangular lattices, the graphene structure does not form a lattice. A graphene sheet can be refined and the construction will be based on the refinement of the lattices $\mathcal{P}_{A_2}$ and $\mathcal{Q}_{A_2}$.

4. Method of dynamical generation

In this section we describe a step-by-step process to build a graphene sheet starting from a single point of $\mathbb{R}^2$. We consider the weight lattice $\mathcal{P}_{A_2}$. We will be using two different steps, namely $\mathcal{S}_1$ and $\mathcal{S}_2$ (Fig. 4):

$$\mathcal{S}_1 = (\omega_1, -\omega_1 + \omega_2, -\omega_2), \quad \mathcal{S}_2 = (\omega_2, \omega_1 - \omega_2, -\omega_1).$$

The points of $\mathcal{S}_1$ belong to the $\mathcal{K}_1$ and the points of $\mathcal{S}_2$ belong to the $\mathcal{K}_2$.

As the seed point of our construction we can choose any point of $\mathcal{P}_{A_2}$. If the chosen point belongs to the congruence class $\mathcal{K}_0$ then the first step can be either $\mathcal{S}_1$ or $\mathcal{S}_2$. If it belongs to $\mathcal{K}_1$ then the next step should be $\mathcal{S}_1$. Finally, if it belongs to $\mathcal{K}_2$ then the next step should be $\mathcal{S}_2$.

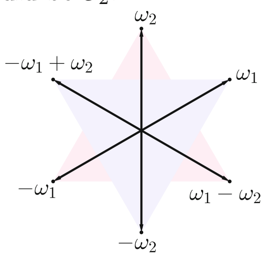


FIGURE 4. The steps of the dynamical generation of the graphene are shown. The blue and red regions correspond to the steps $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.

By applying this rule to a single point of $\mathbb{R}^2$, we dynamically generate the points of our structure in the following way: three vectors of the starting step define three points of the plane. If any of these points coincides with an already existing point of the graphene structure, we disregard it. If it is a new point, it should be kept. The more steps we do, the bigger the graphene structure gets and after an infinite number of steps the graphene layer is complete.

For example, from Figure 3 we see that adding a point of $\mathcal{K}_1$ to another point of $\mathcal{K}_1$ yields a point of $\mathcal{K}_2$ and adding a point of $\mathcal{K}_2$ to another point of $\mathcal{K}_2$ yields a point of $\mathcal{K}_1$. One should also note that adding a point of $\mathcal{K}_1$ to a point of $\mathcal{K}_2$ yield a point of $\mathcal{K}_0$ which does not belong to the graphene structure. Therefore, such an addition is not allowed.

However, an interesting situation arises when removing one, two, or even three vectors from each of the steps $\mathcal{S}_1$ or $\mathcal{S}_2$. For example, consider the following combinations of the steps:

(a) $\mathcal{S}_1 = (\omega_1, -\omega_1 + \omega_2, -\omega_2)$, $\quad \mathcal{S}_2 = (\omega_2, 0, 0)$;

(b) $\mathcal{S}_1 = (\omega_1, 0, -\omega_2)$, $\quad \mathcal{S}_2 = (\omega_2, 0, -\omega_1)$;

(c) $\mathcal{S}_1 = (\omega_1, -\omega_1 + \omega_2, -\omega_2)$, $\quad \mathcal{S}_2 = (0, 0, 0)$.

This way we can obtain sheets of graphene that will not cover the entire plane $\mathbb{R}^2$. The resulting graphene structures are shown in Figure 5.

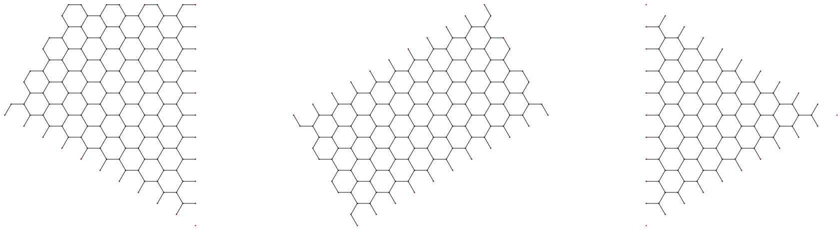


Figure 5. The resulting graphene layers are shown for the combinations of steps $\mathcal{S}_1$ and $\mathcal{S}_2$ from (a), (b) and (c) from left to right, respectively.

Acknowledgment

We would like to thank Dr. J. Patera for useful comments and helpful discussions. We are grateful to the Natural Sciences and Engineering Research Council of Canada and the MIND Research Institute of Irvine, California, for partial support of the work.

References

[1] R.V. Moody, J. Patera, *Dynamical Generation of Quasicrystals*, Lett. Math. Phys. **36**, 291, 1996.

[2] A. Klimyk, J. Patera, *Orbit functions,* SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) **2**, 006, 60 pages, math-ph/0601037, 2006.

[3] F.W. Lemire, J. Patera, *Congruence classes of finite representations of simple Lie superalgebras,* J. Math. Phys., **23**, 1409–1414, 1982.

[4] M.S. Dresselhaus, G. Dresselhaus, P.C. Eklund, *Science of Fullerenes and Carbon nanotubes,* Academic Press, 1996.

[5] P.J.F. Harris, *Carbon Nanotubes and Related Structures,* Cambridge University Press, Cambridge, 1999.

[6] Shabi Abbas Zaidi, *Graphene: A Comprehensive Review on Its Utilization in Carbon Paste Electrodes for Improved Sensor Performances,* Int. J. Electrochem. Sci., **8**, 11337–11355, 2013.

M. Myronova and E. Bourret
Département de physique, Centre de recherches mathématique,
Université de Montréal
C.P. 6128, Centre-ville, Montréal
H3C 3J7, Québec, Canada
e-mail: maria.myronova@umontreal.ca
emmanuel.bourret@umontreal.ca

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 347–354

Eight Kinds of Orthogonal Polynomials of the Weyl Group C_2 and the Tau Method

Tomasz Czyżycki and Jiří Hrivnák

Abstract. The four kinds of the classical Chebyshev polynomials are generalized to eight kinds of two-variable polynomials of the Weyl group C_2. The admissible shift of the weight lattice and the four sign homomorphisms of C_2 generate eight types of the underlying hybrid character functions. The construction method of the resulting shifted four kinds of polynomials is detailed. The tau method for the approximation of solutions of differential equations using these two-variable polynomials is discussed.

Mathematics Subject Classification (2010). 33C52; 42A10; 65N30.

Keywords. Weyl groups, orthogonal polynomials, tau method.

1. Introduction

The set $\Delta = \{\alpha_1, \alpha_2\}$ of simple roots of the root system C_2 consists of the simple short root α_1 and the simple long root α_2, given by coordinates in the standard orthonormal basis of $\mathbb{R}^2$ with scalar product $\langle\, ,\, \rangle$ as

$$\alpha_1 = [1, 0], \quad \alpha_2 = [-1, 1].$$

Four types of lattices are related to the root system Δ, the root lattice Q, the weight lattice P, the coweight lattice $P^\vee$ and the dual root lattice $Q^\vee$ determined by

$$\begin{aligned}
Q &= \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \\
Q^\vee &= \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee, \quad \text{with } \alpha_1^\vee = [2, 0], \qquad \alpha_2^\vee = [-1, 1], \\
P &= \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \quad \text{with } \omega_1 = \left[\frac{1}{2}, \frac{1}{2}\right], \quad \omega_2 = [0, 1], \\
P^\vee &= \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee, \quad \text{with } \omega_1^\vee = [1, 1], \qquad \omega_2^\vee = [0, 1].
\end{aligned}$$

The Weyl group W of the root system C_2 is generated by two reflections r_α, $\alpha \in \Delta$ which are orthogonal to the simple roots and intersect at the origin. The affine

Weyl group is the semidirect product $W^{\mathrm{aff}} = Q^\vee \rtimes W$ and the dual affine Weyl group is the semidirect product $\widehat{W}^{\mathrm{aff}} = Q \rtimes W$.

Admissible shifts $\rho^\vee, \rho \in \mathbb{R}^2$ are required to preserve the invariance of the shifted weight and the dual weight lattices,

$$W(\rho^\vee + P^\vee) = \rho^\vee + P^\vee, \quad W(\rho + P) = \rho + P.$$

For any irreducible crystallographic root system, all admissible shifts are classified in [3]. For the case C_2, there exist precisely two shifts of the form

$$\rho = \frac{1}{2}\omega_2 = \left[0, \frac{1}{2}\right], \ \rho^\vee = \frac{1}{2}\omega_1^\vee = \left[\frac{1}{2}, \frac{1}{2}\right]. \tag{1}$$

Any vectors in the ω-basis are denoted in round brackets, i.e., $\rho = \frac{1}{2}\omega_2 = \left(0, \frac{1}{2}\right)$. The fundamental domain F of the affine Weyl group W^{aff}, which consists of exactly one point from each W^{aff}-orbit, is the triangle given by its vertices $\left\{0, \frac{1}{2}\omega_1^\vee, \omega_2^\vee\right\}$. The dual fundamental domain $F^\vee$ of the dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ is the triangle given by its vertices $\left\{0, \omega_1, \frac{1}{2}\omega_2\right\}$. The calculation of the functions $\varepsilon : \mathbb{R}^2 \to \mathbb{N}$ and $h_M^\vee : \mathbb{R}^2 \to \mathbb{N}$, $M \in \mathbb{N}$ defined by the relation

$$\varepsilon(x) = \frac{|W|}{|\mathrm{Stab}_{W^{\mathrm{aff}}}(x)|}, \quad h_M^\vee(x) = \left|\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}\left(\frac{x}{M}\right)\right|, \tag{2}$$

is demonstrated for the C_2 case in [3]. The four sign homomorphisms $\sigma : W \to \{\pm 1\}$, introduced in [1], are for C_2 denoted by $\mathbf{1}$, σ^e, σ^s and σ^l. The explicit form of the eight sets $F^\sigma(\rho) \subset F$ and eight sets $F^{\sigma\vee}(\rho^\vee) \subset F^\vee$, with ρ and $\rho^\vee$ either zero or given by (1), follows from equations (57) and (61) in [3], i.e., the sets $F^\sigma(\rho) \subset F$ are of the explicit form

$$F^\sigma(\rho) = \{y_1^{\sigma,\rho}\omega_1^\vee + y_2^{\sigma,\rho}\omega_2^\vee \mid y_0^{\sigma,\rho} + 2y_1^{\sigma,\rho} + y_2^{\sigma,\rho} = 1\},$$

where

$$\begin{aligned}
y_0^{\mathbf{1},0}, y_1^{\mathbf{1},0}, y_2^{\mathbf{1},0} \geq 0, &\quad y_0^{\mathbf{1},\rho} > 0, y_1^{\mathbf{1},\rho}, y_2^{\mathbf{1},\rho} \geq 0,\\
y_0^{\sigma^e,0}, y_1^{\sigma^e,0}, y_2^{\sigma^e,0} > 0, &\quad y_0^{\sigma^e,\rho} \geq 0, y_1^{\sigma^e,\rho}, y_2^{\sigma^e,\rho} > 0,\\
y_0^{\sigma^s,0} \geq 0, y_1^{\sigma^s,0} > 0, y_2^{\sigma^s,0} \geq 0, &\quad y_0^{\sigma^s,\rho}, y_1^{\sigma^s,\rho} > 0, y_2^{\sigma^s,\rho} \geq 0,\\
y_0^{\sigma^l,0} > 0, y_1^{\sigma^l,0} \geq 0, y_2^{\sigma^l,0} > 0, &\quad y_0^{\sigma^l,\rho}, y_1^{\sigma^l,\rho} \geq 0, y_2^{\sigma^l,\rho} > 0,
\end{aligned}$$

and the sets $F^{\sigma\vee}(\rho^\vee) \subset F^\vee$ are of the explicit form

$$F^{\sigma\vee}(\rho^\vee) = \left\{z_1^{\sigma,\rho^\vee}\omega_1 + z_2^{\sigma,\rho^\vee}\omega_2 \mid z_0^{\sigma,\rho^\vee} + z_1^{\sigma,\rho^\vee} + 2z_2^{\sigma,\rho^\vee} = 1\right\},$$

where

$$\begin{aligned} z_0^{\mathbb{1},0}, z_1^{\mathbb{1},0}, z_2^{\mathbb{1},0} &\geq 0, & z_0^{\mathbb{1},\rho^\vee} &> 0, z_1^{\mathbb{1},\rho^\vee}, z_2^{\mathbb{1},\rho^\vee} \geq 0,\\ z_0^{\sigma^e,0}, z_1^{\sigma^e,0}, z_2^{\sigma^e,0} &> 0, & z_0^{\sigma^e,\rho^\vee} &\geq 0, z_1^{\sigma^e,\rho^\vee}, z_2^{\sigma^e,\rho^\vee} > 0,\\ z_0^{\sigma^s,0}, z_1^{\sigma^s,0} > 0, z_2^{\sigma^s,0} &\geq 0, & z_0^{\sigma^s,\rho^\vee} &\geq 0, z_1^{\sigma^s,\rho^\vee} > 0, z_2^{\sigma^s,\rho^\vee} \geq 0,\\ z_0^{\sigma^l,0}, z_1^{\sigma^l,0} \geq 0, z_2^{\sigma^l,0} &> 0, & z_0^{\sigma^l,\rho^\vee} &> 0, z_1^{\sigma^l,\rho^\vee} \geq 0, z_2^{\sigma^l,\rho^\vee} > 0. \end{aligned}$$

According to [7], defining four vectors ρ^σ by $\rho^{\mathbb{1}} = (0,0), \rho^l = (0,1), \rho^s = (1,0), \rho^e = (1,1)$, four types of Weyl-orbit functions $\phi^\sigma_{\lambda+\rho^\sigma} : \mathbb{R}^2 \to \mathbb{C}$ of C_2 are given by

$$\phi^\sigma_{\lambda+\rho^\sigma}(x) = \sum_{\mu\in O(\lambda+\rho^\sigma)} \sigma(w)\, e^{2\pi i\langle\mu,x\rangle}, \quad x,\lambda \in \mathbb{R}^2,$$

where $O(\mu)$ denotes the W-orbit of $\mu \in \mathbb{R}^2$.

2. Eight kinds of polynomials of C_2

Four kinds of the classical Chebyshev polynomials of one variable are generalized to the case of the Weyl group C_2. To this aim, eight shifted generalized characters $\chi^\sigma_{\rho,\lambda}$ of C_2 are introduced as ratios

$$\chi^\sigma_{\rho,\lambda}(x) = \frac{\phi^\sigma_{\lambda+\nu^\sigma_\rho}(x)}{\phi^\sigma_{\nu^\sigma_\rho}(x)},$$

with the symbols ν^σ_0 determined as $\nu^\sigma_0 = \rho^\sigma$ and for the non-trivial admissible shift ρ as $\nu^{\mathbb{1}}_\rho = \nu^{\sigma^l}_\rho = \left(0,\frac{1}{2}\right)$ and $\nu^{\sigma^s}_\rho = \nu^{\sigma^e}_\rho = \left(1,\frac{1}{2}\right)$. The eight characters $\chi^\sigma_{\rho,\lambda}$ are, as Weyl group invariant exponential sums, polynomials $\mathbb{T}^\sigma_{\rho,\lambda}(X_1,X_2)$ in basic two C-functions

$$X_1 = \phi^{\mathbb{1}}_{\omega_1}, \quad X_2 = \phi^{\mathbb{1}}_{\omega_2}, \tag{3}$$

i.e., it holds that

$$\mathbb{T}^\sigma_{\rho,\lambda}(X_1(x),X_2(x)) = \chi^\sigma_{\rho,\lambda}(x).$$

Expressing the basic two characters $\chi^{\mathbb{1}}_{0,\lambda}$ and $\chi^{\sigma^e}_{0,\lambda}$ as polynomials in the basic two C-functions yields the polynomials $C_{(a,b)}(X_1,X_2)$ and $S_{(a,b)}(X_1,X_2)$,

$$C_{(a,b)}(X_1(x),X_2(x)) = \phi^{\mathbb{1}}_{(a,b)}(x), \quad S_{(a,b)}(X_1(x),X_2(x)) = \frac{\phi^{\sigma^e}_{(a+1,b+1)}(x)}{\phi^{\sigma^e}_{(1,1)}(x)}$$

of the first and second kind from [4]. The hybrid characters $\chi^{\sigma^s}_{0,\lambda}$ and $\chi^{\sigma^l}_{0,\lambda}$ polynomials $S^s_{(a,b)}(X_1,X_2)$ and $S^l{}_{(a,b)}(X_1,X_2)$,

$$S^s_{(a,b)}(X_1(x),X_2(x)) = \frac{\phi^{\sigma^s}_{(a+1,b)}(x)}{\phi^{\sigma^s}_{(1,0)}(x)}, \quad S^l_{(a,b)}(X_1(x),X_2(x)) = \frac{\phi^{\sigma^l}_{(a,b+1)}(x)}{\phi^{\sigma^l}_{(0,1)}(x)},$$

studied in connection with cubature rules in [7], form the polynomials of the third and fourth kinds.

The novel polynomials $C^\rho_\lambda(X_1,X_2)$ of the fifth kind, which are obtained from the shifted character function $\chi^{\mathbf{1}}_{\rho,\lambda}$,

$$C^\rho_{(a,b)}(X_1(x),X_2(x)) = \frac{\phi^{\mathbf{1}}_{(a,b+\frac{1}{2})}(x)}{\phi^{\mathbf{1}}_{(0,\frac{1}{2})}(x)},$$

satisfy the following two general recursion relations

$$\begin{aligned}
X_1C^\rho_{(a,b)} &= C^\rho_{(a+1,b)} + C^\rho_{(a-1,b+1)} + C^\rho_{(a+1,b-1)} + C^\rho_{(a-1,b)},\ a,b\geq 2\\
X_2C^\rho_{(a,b)} &= C^\rho_{(a,b+1)} + C^\rho_{(a+2,b-1)} + C^\rho_{(a-2,b+1)} + C^\rho_{(a,b-1)},\ a\geq 3, b\geq 2
\end{aligned}$$

and additional special recursions,

$$\begin{aligned}
X_1C^\rho_{(a,1)} &= C^\rho_{(a+1,1)} + C^\rho_{(a-1,2)} + C^\rho_{(a+1,0)} + C^\rho_{(a-1,1)},\ a\geq 2\\
X_1C^\rho_{(a,0)} &= C^\rho_{(a+1,0)} + C^\rho_{(a-1,1)} + C^\rho_{(a,0)} + C^\rho_{(a-1,0)},\ a\geq 2\\
X_1C^\rho_{(1,b)} &= C^\rho_{(2,b)} + 2C^\rho_{(0,b+1)} + C^\rho_{(2,b-1)} + 2C^\rho_{(0,b)},\ b\geq 1\\
X_1C^\rho_{(0,b)} &= C^\rho_{(1,b)} + C^\rho_{(1,b-1)},\ b\geq 1\\
X_2C^\rho_{(a,1)} &= C^\rho_{(a,2)} + C^\rho_{(a+2,0)} + C^\rho_{(a-2,2)} + C^\rho_{(a,0)},\ a\geq 3\\
X_2C^\rho_{(a,0)} &= C^\rho_{(a,1)} + C^\rho_{(a-1,0)} + C^\rho_{(a+1,0)} + C^\rho_{(a-2,1)},\ a\geq 3\\
X_2C^\rho_{(2,b)} &= C^\rho_{(2,b+1)} + C^\rho_{(4,b-1)} + 2C^\rho_{(0,b+1)} + C^\rho_{(2,b-1)},\ b\geq 2\\
X_2C^\rho_{(1,b)} &= C^\rho_{(1,b+1)} + C^\rho_{(3,b-1)} + C^\rho_{(1,b-1)} + C^\rho_{(1,b)},\ b\geq 1\\
X_2C^\rho_{(0,b)} &= C^\rho_{(0,b+1)} + C^\rho_{(2,b-1)} + C^\rho_{(0,b-1)},\ b\geq 1.
\end{aligned}$$

Several initial polynomials, which allow recursive calculation of any polynomial of the fifth kind, are the following,

$$\begin{aligned}
&C^\rho_{(0,0)} = 1,\ C^\rho_{(1,0)} = X_1-2,\ C^\rho_{(0,1)} = X_2-X_1+1,\ C^\rho_{(1,1)} = -X_1^2+X_1X_2+2,\\
&C^\rho_{(2,0)} = X_1^2-X_1-2X_2-2,\ C^\rho_{(0,2)} = X_2^2-X_1^2-X_1X_2+X_1+3X_2+1.
\end{aligned}$$

The polynomials $S^\rho_\lambda(X_1,X_2)$ of the sixth kind, which are obtained from the shifted character function $\chi^{\sigma^e}_{\rho,\lambda}$,

$$S^\rho_{(a,b)}(X_1(x),X_2(x)) = \frac{\phi^{\sigma^e}_{(a+1,b+\frac{1}{2})}(x)}{\phi^{\sigma^e}_{(1,\frac{1}{2})}(x)}$$

satisfy two general recursion relations

$$\begin{aligned}
X_1S^\rho_{(a,b)} &= S^\rho_{(a+1,b)} + S^\rho_{(a-1,b+1)} + S^\rho_{(a+1,b-1)} + S^\rho_{(a-1,b)},\ a,b\geq 2,\\
X_2S^\rho_{(a,b)} &= S^\rho_{(a,b+1)} + S^\rho_{(a+2,b-1)} + S^\rho_{(a-2,b+1)} + S^\rho_{(a,b-1)},\ a\geq 3, b\geq 2.
\end{aligned}$$

Special recursion relations are similar to relations for C^ρ-polynomials and several initial polynomials of the sixth kind are

$$\begin{aligned}
&S^\rho_{(0,0)} = 1, \quad S^\rho_{(1,0)} = X_1 + 1, \quad S^\rho_{(0,1)} = X_1 + X_2 + 2,\\
&S^\rho_{(1,1)} = X_1^2 + X_1X_2 + X_1 - 1,\ S^\rho_{(2,0)} = X_1^2 + X_1 - X_2 - 2,\\
&S^\rho_{(0,2)} = X_2^2 - X_1^2 + X_1X_2 + 4X_2 + 3.
\end{aligned}$$

The polynomials $S^{s,\rho}_\lambda(X_1, X_2)$ of the seventh kind, which are obtained from the shifted character function $\chi^{\sigma^s}_{\rho,\lambda}$,

$$S^{s,\rho}_{(a,b)}(X_1(x), X_2(x)) = \frac{\phi^{\sigma^s}_{(a+1,b+\frac{1}{2})}(x)}{\phi^{\sigma^s}_{(1,\frac{1}{2})}(x)},$$

satisfy two general recursion relations

$$\begin{aligned}
X_1S^{s,\rho}_{(a,b)} &= S^{s,\rho}_{(a+1,b)} + S^{s,\rho}_{(a-1,b+1)} + S^{s,\rho}_{(a+1,b-1)} + S^{s,\rho}_{(a-1,b)},\ a, b \geq 2,\\
X_2S^{s,\rho}_{(a,b)} &= S^{s,\rho}_{(a,b+1)} + S^{s,\rho}_{(a+2,b-1)} + S^{s,\rho}_{(a-2,b+1)} + S^{s,\rho}_{(a,b-1)},\ a \geq 3, b \geq 2.
\end{aligned}$$

Other recursion relations are similar to relations for C^ρ-polynomials and several initial polynomials of the seventh kind are

$$\begin{aligned}
&S^{s,\rho}_{(0,0)} = 1,\ S^{s,\rho}_{(1,0)} = X_1 - 1,\ S^{s,\rho}_{(0,1)} = -X_1 + X_2 + 2,\\
&S^{s,\rho}_{(1,1)} = 1 - X_1^2 + X_1X_2 + X_1,\ S^{s,\rho}_{(2,0)} = X_1^2 - X_1 - X_2 - 2,\\
&S^{s,\rho}_{(0,2)} = X_2^2 - X_1^2 - X_1X_2 + 4X_2 + 3.
\end{aligned}$$

The polynomials $S^{l,\rho}_\lambda(X_1, X_2)$ of the eighth kind, which are obtained from the shifted character function $\chi^{\sigma^l}_{\rho,\lambda}$,

$$S^{l,\rho}_{(a,b)}(X_1, X_2) = \frac{\phi^{\sigma^l}_{(a,b+\frac{1}{2})}(x)}{\phi^{\sigma^l}_{(0,\frac{1}{2})}(x)},$$

satisfy two general recursion relations

$$\begin{aligned}
X_1S^{l,\rho}_{(a,b)} &= S^{l,\rho}_{(a+1,b)} + S^{l,\rho}_{(a-1,b+1)} + S^{l,\rho}_{(a+1,b-1)} + S^{l,\rho}_{(a-1,b)},\ a, b \geq 2,\\
X_2S^{l,\rho}_{(a,b)} &= S^{l,\rho}_{(a,b+1)} + S^{l,\rho}_{(a+2,b-1)} + S^{l,\rho}_{(a-2,b+1)} + S^{l,\rho}_{(a,b-1)},\ a \geq 3, b \geq 2.
\end{aligned}$$

Other recursion relations are similar to relations for C^ρ-polynomials and several initial polynomials of the eighth kind are

$$\begin{aligned}
&S^{l,\rho}_{(0,0)} = 1,\ S^{l,\rho}_{(1,0)} = X_1 + 2,\ S^{l,\rho}_{(0,1)} = X_1 + X_2 + 1,\\
&S^{l,\rho}_{(1,1)} = X_1^2 + X_1X_2 - 2,\ S^{l,\rho}_{(2,0)} = X_1^2 + X_1 - 2X_2 - 2,\\
&S^{l,\rho}_{(0,2)} = X_2^2 - X_1^2 + X_1X_2 - X_1 + 3X_2 + 1.
\end{aligned}$$

3. Weight functions and discrete orthogonality

Taking any natural number $M \in \mathbb{N}$, the sets $F^\sigma(\rho)$ and $F^{\sigma\vee}(\rho^\vee)$ induce sixteen discrete point sets $F^\sigma_M(\rho,\rho^\vee) \subset F$,

$$F^\sigma_M(\rho,\rho^\vee) = \left[\frac{1}{M}(\rho^\vee + P^\vee)\right] \cap F^\sigma(\rho) \tag{4}$$

and the corresponding sixteen shifted weight sets

$$\Lambda^\sigma_M(\rho,\rho^\vee) = (\rho + P) \cap M F^{\sigma\vee}(\rho^\vee). \tag{5}$$

The X-transform of C_2 is a mapping $X : \mathbb{R}^2 \to \mathbb{R}^2$ given for any $x \in \mathbb{R}^2$ as

$$X(x) = (X_1(x), X_2(x)),$$

and induces sixteen discrete point sets

$$\Omega^\sigma_M(\rho,\rho^\vee) = X\left(F^\sigma_M(\rho,\rho^\vee)\right).$$

The sets $\Omega^1_{10}(0,0)$ and $\Omega^1_{10}(0,\rho^\vee)$ and are depicted in Figure 1.

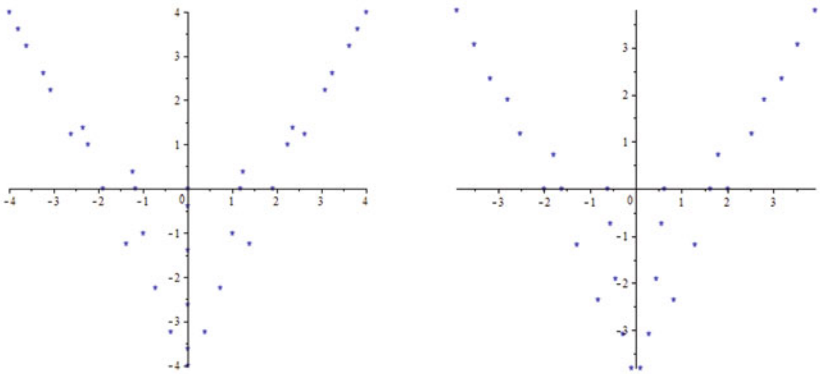


Figure 1. The left panel depicts the points of the non-shifted set $\Omega^1_{10}(0,0)$. The right panel depicts the points of the shifted set $\Omega^1_{10}(0,\rho^\vee)$.

The restrictions of the mapping X to the point sets $F^\sigma_M(\rho,\rho^\vee)$ are denoted by $X^\sigma_M(\rho,\rho^\vee)$. Since the mappings $X^\sigma_M(\rho,\rho^\vee)$ are one-to-one, it holds for the numbers of points that $|\Omega^\sigma_M(\rho,\rho^\vee)| = |\Lambda^\sigma_M(\rho,\rho^\vee)|$, and the discrete function $\widetilde{\varepsilon} : \Omega^\sigma_M(\rho,\rho^\vee) \to \mathbb{N}$, given for any $y \in \Omega^\sigma_M(\rho,\rho^\vee)$ by

$$\widetilde{\varepsilon}(y) = \varepsilon\left(\left(X^\sigma_M(\rho,\rho^\vee)\right)^{-1} y\right),$$

is well defined.

The polynomial weight functions $w^\sigma_\rho(X_1,X_2)$, defined by

$$w^\sigma_\rho(X_1(x),X_2(x)) = [\phi^\sigma_{\nu^\sigma_\rho}(x)]^2,$$

are for the novel four classes of the shifted polynomials of the form

$$
\begin{aligned}
w_\rho^{1}(X_1,X_2) &= 2X_1 + X_2 + 4,\\
w_\rho^{\sigma^e}(X_1,X_2) &= 2X_1^3 - X_1^2X_2 - 4X_1^2 - 8X_1X_2 + 4X_2^2 + 16X_2,\\
w_\rho^{\sigma^s}(X_1,X_2) &= 2X_1^3 + X_1^2X_2 + 4X_1^2 - 8X_1X_2 - 4X_2^2 - 16X_2,\\
w_\rho^{\sigma^l}(X_1,X_2) &= -2X_1 + X_2 + 4.
\end{aligned}
$$

Thus, the discrete orthogonality of Weyl orbit functions in [3] induces for all $\lambda, \lambda' \in -\nu_\rho^\sigma + \Lambda_M^\sigma(\rho,\rho^\vee)$ the discrete orthogonality of all eight classes of shifted polynomials $\mathbb{T}^\sigma_{\rho,\lambda}$ of the form

$$\sum_{y\in\Omega_M^\sigma(\rho,\rho^\vee)} \widetilde{\varepsilon}(y) w_\rho^\sigma(y)\, \mathbb{T}^\sigma_{\rho,\lambda}(y)\mathbb{T}^\sigma_{\rho,\lambda'}(y) = 16M^2 \cdot h_M^\vee(\lambda+\nu_\rho^\sigma)\cdot \delta_{\lambda,\lambda'}.$$

4. Tau method

The tau method, called also Lanczos approximation method, is the method of searching for approximate solutions of differential equations in the form of finite sum of functions from a given family. The main idea is to approximate the solution of a given differential equation by solving exactly an approximate problem. The crucial point is including in the solution additional terms with arbitrary coefficient τ. Consider the differential equation

$$u_{xy} = u,$$

with initial conditions $u(x,0) = x^2, u_y(x,0) = 2$. The approximate solution is assumed in the form of sum of C^ρ-polynomials up to degree 3,

$$F(x,y) = a_{00}C^\rho_{(0,0)}(x,y) + \cdots + a_{03}C^\rho_{(0,3)}(x,y).$$

Lanczos approximation method requires rewriting all partial derivatives of F to the form of the underlying C^ρ-polynomials, hence the equation

$$
\begin{aligned}
F_{xy}(x,y) = (a_{11} - a_{02} + 6a_{21} - 3a_{30} - 11a_{03})C^\rho_{(0,0)}&\\
+ (2a_{12} - 2a_{03})C^\rho_{(0,1)} + (2a_{21} - 6a_{03})C^\rho_{(1,0)} + \tau_1 C^\rho_{(1,1)} + \cdots&\\
= F(x,y) = a_{00}C^\rho_{(0,0)} + a_{01}C^\rho_{(0,1)} + a_{10}C^\rho_{(1,0)} + \cdots + a_{03}C^\rho_{(0,3)}&
\end{aligned}
$$

is obtained. Solving this system of linear equations and requiring the initial conditions, the approximate solution of the form

$$F(x,y) = 2y + \frac{1}{3}xy^2 + x^2 - 2y^2 - \frac{10}{27}y^3$$

is determined.

Acknowledgment

The authors are grateful for partial support by Project No. 7AMB16PL043.

References

[1] J. Hrivnák, L. Motlochová, J. Patera, *On discretization of tori of compact simple Lie groups II*, J. Phys. A: Math. Theor. **45** (2012) 255201.

[2] R.V. Moody, J. Patera, *Orthogonality within the families of C-, S-, and E- functions of any compact semisimple Lie group*, SIGMA 2 (2006) 076.

[3] T. Czyżycki and J. Hrivnák, *Generalized discrete orbit function transforms of affine Weyl groups*, J. Math. Phys. **55** (2014) 113508.

[4] M. Nesterenko, J. Patera and A. Tereszkiewicz, *Orthogonal Polynomials of Compact Simple Lie Groups*, Hindawi Publ. Corp. Int. J. of Math. and Math. Sc., 2011, ID 969424.

[5] E.L. Ortiz, K.S. Pun, *Numerical solution of nonlinear partial differential equations with the Tau method*, Journ. of Comp. and Appl. Math. 12,13 (1985) 511–516.

[6] V. Druskin, L. Knizhnerman, *On application of the Lanczos method to solution of some partial differential equations*, Journ. of Comp. and Appl. Math. 50 (1994) 255–262.

[7] R.V. Moody, L. Motlochová, J. Patera, *Gaussian cubature arising from hybrid characters of simple Lie groups*, J. Fourier Anal. Appl. (2014), 20, Issue 6, 1257–1290.

Tomasz Czyżycki
Institute of Mathematics
University of Białystok
K. Ciołkowskiego 1M
PL-15–245 Białystok, Poland
e-mail: tomczyz@math.uwb.edu.pl

Jiří Hrivnák
Czech Technical University in Prague
Břehová 7
CZ-115 19 Prague 1, Czech Republic
e-mail: jiri.hrivnak@fjfi.cvut.cz

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 355–373

Links Between Quantum Chaos and Counting Problems

Aleksander Yu. Orlov

Abstract. I show that Hurwitz numbers may be generated by certain correlation functions which appear in quantum chaos.

Mathematics Subject Classification (2010). 05A15, 14N10, 17B80, 35Q51, 35Q53, 35Q55, 37K20, 37K30.

Keywords. Hurwitz number, Klein surface, products of random matrices.

First, in short we present two different topics: Hurwitz numbers which appear in counting of branched covers of Riemann and Klein surfaces, and the study of spectral correlation functions of products of random matrices which belong to independent (complex) Ginibre ensembles.

There are a lot of studies on extracting information about Hurwitz numbers, on one side, from integrable systems, as it was done in [21, 51, 52] and further developed in [6, 7, 15, 18, 23, 27, 43, 44, 48, 49, 66] (see also reviews [29] and [33]) and from matrix integrals [22, 34, 39] on the other. (Actually the point that there is a special family of tau functions which were introduced in [35] and in [55] and studied in [24–26, 53, 56, 58, 59] where links with matrix models were written down which describe perturbation series in coupling constants of a number of matrix models, and these very tau functions, called hypergeometric ones, count also special types of Hurwitz numbers. This article is based on [48, 59] and [54]. In the last paper we put known results in quantum chaos [1–3]. The results of our work should be compared to ones obtained in [5, 31] and [12].

1. Counting of branched covers

Let us consider a connected compact surface without boundary Ω and a branched covering $f : \Sigma \to \Omega$ by a connected or non-connected surface Σ. We will consider a covering f of the degree d. It means that the preimage $f^{-1}(z)$ consists of d points $z \in \Omega$ except some finite number of points. This points are called *critical values of f*.

Consider the preimage $f^{-1}(z) = \{p_1, \dots, p_\ell\}$ of $z \in \Omega$. Denote by δ_i the degree of f at p_i. It means that in the neighborhood of p_i the function f is homeomorphic to $x \mapsto x^{\delta_i}$. The set $\Delta = (\delta_1, \dots, \delta_\ell)$ is the partition of d, that is called *topological type of* z.

For a partition Δ of a number $d = |\Delta|$ denote by $\ell(\Delta)$ the number of the non-vanishing parts ($|\Delta|$ and $\ell(\Delta)$ are called the weight and the length of Δ, respectively). We denote a partition and its Young diagram by the same letter. Denote by $(\delta_1, \dots, \delta_\ell)$ the Young diagram with rows of length $\delta_1, \dots, \delta_\ell$ and corresponding partition of $d = \sum \delta_i$.

Fix now points $z_1, \dots, z_{\rm F}$ and partitions $\Delta^{(1)}, \dots, \Delta^{({\rm F})}$ of d. Denote by

$$\widetilde{C}_{\Omega(z_1,\dots,z_{\rm F})}(d; \Delta^{(1)}, \dots, \Delta^{({\rm F})})$$

the set of all branched covering $f : \Sigma \to \Omega$ with critical points $z_1, \dots, z_{\rm F}$ of topological types $\Delta^{(1)}, \dots, \Delta^{({\rm F})}$.

Coverings $f_1 : \Sigma_1 \to \Omega$ and $f_2 : \Sigma_2 \to \Omega$ are called isomorphic if there exists a homeomorphism $\varphi : \Sigma_1 \to \Sigma_2$ such that $f_1 = f_2\varphi$. Denote by $\mathtt{Aut}(f)$ the group of automorphisms of the covering f. Isomorphic coverings have isomorphic groups of automorphisms of degree $|\mathtt{Aut}(f)|$.

Consider now the set $C_{\Omega(z_1,\dots,z_{\rm F})}(d; \Delta^{(1)}, \dots, \Delta^{({\rm F})})$ of isomorphic classes in $\widetilde{C}_{\Omega(z_1,\dots,z_{\rm F})}(d; \Delta^{(1)}, \dots, \Delta^{({\rm F})})$. This is a finite set. The sum

$$H^{{\rm E},{\rm F}}(d; \Delta^{(1)}, \dots, \Delta^{({\rm F})}) = \sum_{f \in C_{\Omega(z_1,\dots,z_{\rm F})}(d; \Delta^{(1)}, \dots, \Delta^{({\rm F})})} \frac{1}{|\mathtt{Aut}(f)|} \quad , \tag{1}$$

does not depend on the location of the points $z_1, \dots, z_{\rm F}$ and is called *Hurwitz number*. Here F denotes the number of the branch points, and E is the Euler characteristic of the base surface.

In case it will not produce a confusion we admit 'trivial' profiles (1^d) among $\Delta^1, \dots, \Delta^{\rm F}$ in (1) keeping the notation $H^{{\rm E},{\rm F}}$ though the number of critical points now is less than F.

In case we count only connected covers Σ we get the *connected* Hurwitz numbers $H^{{\rm E},{\rm F}}_{\rm con}(d; \Delta^{(1)}, \dots, \Delta^{({\rm F})})$.

The Hurwitz numbers arise in different fields of mathematics: from algebraic geometry to integrable systems. They are well studied for orientable Ω. In this case the Hurwitz number coincides with the weighted number of holomorphic branched coverings of a Riemann surface Ω by other Riemann surfaces, having critical points $z_1, \dots, z_{\rm F} \in \Omega$ of the topological types $\Delta^{(1)}, \dots, \Delta^{({\rm F})}$, respectively. The well known isomorphism between Riemann surfaces and complex algebraic curves gives the interpretation of the Hurwitz numbers as the numbers of morphisms of complex algebraic curves.

Similarly, the Hurwitz number for a non-orientable surface Ω coincides with the weighted number of the dianalytic branched coverings of the Klein surface without boundary by another Klein surface and coincides with the weighted number of morphisms of real algebraic curves without real points [11, 45, 46]. An extension of the theory to all Klein surfaces and all real algebraic curves leads to Hurwitz numbers for surfaces with boundaries may be found in [9, 47].

Riemann–Hurwitz formula related the Euler characteristic of the base surface $\textsc{e}$ and the Euler characteristic of the d-fold cover $\textsc{e}'$ as follows:

$$\textsc{e}' = d\textsc{e} + \sum_{i=1}^{\textsc{f}} \left(\ell(\Delta^{(i)}) - d\right) = 0 \tag{2}$$

where the sum ranges over all branch points $z_i\,, i = 1, 2, \ldots$ with ramification profiles given by partitions $\Delta^i\,, i = 1, 2, \ldots$, respectively, and $\ell(\Delta^{(i)})$ denotes the length of the partition $\Delta^{(i)}$ which is equal to the number of the preimages $f^{-1}(z_i)$ of the point z_i.

Example 1. Let $f : \Sigma \to \mathbb{CP}^1$ be a covering without critical points. Then, each d-fold cover is the disjoint union of d Riemann spheres: $\mathbb{CP}^1 \coprod \cdots \coprod \mathbb{CP}^1$, then $|\mathtt{Aut} f| = d!$ and $H^{2,0}(d) = \frac{1}{d!}$

Example 2. Let $f : \Sigma \to \mathbb{CP}^1$ be a d-fold covering with two critical points with the profiles $\Delta^{(1)} = \Delta^{(2)} = (d)$. (One may think of $f = x^d$.) Then $H^{2,2}(d; (d), (d)) = \frac{1}{d}$. Let us note that Σ is connected in this case (therefore $H^{2,2}(d; (d), (d)) = H^{2,2}_{\mathrm{con}}(d; (d), (d))$) and its Euler characteristic $\textsc{e}' = 2$.

Example 3. The generating function for the Hurwitz numbers $H^{2,2}(d; (d), (d))$ from the previous Example may be written as

$$F(h^{-1}\mathbf{p}^{(1)}, h^{-1}\mathbf{p}^{(2)}) := h^{-2} \sum_{d>0} H^{2,2}_{\mathrm{con}}(d; (d), (d)) p_d^{(1)} p_d^{(2)} = h^{-2} \sum_{d>0} \frac{1}{d} p_d^{(1)} p_d^{(2)}$$

Here $\mathbf{p}^{(i)} = (p_1^{(i)}, p_2^{(i)}, \ldots)$, $i = 1, 2$ are two sets of formal parameters. The powers of the auxiliary parameter $\frac{1}{h}$ count the Euler characteristic of the cover $\textsc{e}'$ which is 2 in our example. Then thanks to the known general statement about the link between generating functions of "connected" and "disconnected" Hurwitz numbers (see for instance [36]) one can write down the generating function for the Hurwitz numbers for covers with two critical points, $H^{2,2}(d; \Delta^{(1)}, \Delta^{(2)})$, as follows:

$$\begin{aligned}
\tau(h^{-1}\mathbf{p}^{(1)}, h^{-1}\mathbf{p}^{(2)}) &= \mathrm{e}^{F(h^{-1}\mathbf{p}^{(1)}, h^{-1}\mathbf{p}^{(2)})} \\
&= \mathrm{e}^{h^{-2}\sum_{d>0} \frac{1}{d} p_d^{(1)} p_d^{(2)}} = \sum_{d\geq 0} \sum_{\Delta^{(1)}, \Delta^{(2)}} H^{2,2}(d; \Delta^{(1)}, \Delta^{(2)})\, h^{-\textsc{e}'} \mathbf{p}^{(1)}_{\Delta^{(1)}} \mathbf{p}^{(2)}_{\Delta^{(2)}}
\end{aligned} \tag{3}$$

where $\mathbf{p}^{(i)}_{\Delta^{(i)}} := p^{(i)}_{\delta_1^{(i)}} p^{(i)}_{\delta_2^{(i)}} p^{(i)}_{\delta_3^{(i)}} \cdots$, $i = 1, 2$ and where $\textsc{e}' = \ell(\Delta^{(1)}) + \ell(\Delta^{(2)})$ in agreement with (2) where we put $\textsc{f} = 2$. From (3) it follows that the profiles of both critical points coincide, otherwise the Hurwitz number vanishes. Let us denote this profile by Δ, $|\Delta| = d$ and from the last equality we get

$$H^{2,2}(d; \Delta, \Delta) = \frac{1}{z_\Delta}$$

Here

$$z_\Delta = \prod_{i=1}^{\infty} i^{m_i}\, m_i! \tag{4}$$

where m_i denotes the number of parts equal to i of the partition Δ (then the partition Δ is often denoted by $(1^{m_1}2^{m_2}\cdots)$).

Example 4. Let $f : \Sigma \to \mathbb{RP}^2$ be a covering without critical points. Then, if Σ is connected, then $\Sigma = \mathbb{RP}^2$, $\deg f = 1$ or $\Sigma = S^2$, $\deg f = 2$. Next, if $d = 3$, then $\Sigma = \mathbb{RP}^2 \coprod \mathbb{RP}^2 \coprod \mathbb{RP}^2$ or $\Sigma = \mathbb{RP}^2 \coprod S^2$. Thus $H^{1,0}(3) = \frac{1}{3!} + \frac{1}{2!} = \frac{2}{3}$.

Example 5. Let $f : \Sigma \to \mathbb{RP}^2$ be a covering with a single critical point with profile Δ, and Σ is connected. Note that due to (2) the Euler characteristic of Σ is $\mathrm{E}' = \ell(\Delta)$. (One may think of $f = z^d$ defined in the unit disc where we identify z and $-z$ if $|z| = 1$.) In case we cover the Riemann sphere by the Riemann sphere $z \to z^m$ we get two critical points with the same profiles. However we cover $\mathbb{RP}^2$ by the Riemann sphere, then we have the composition of the mapping $z \to z^m$ on the Riemann sphere and the factorization by antipodal involution $z \to -\frac{1}{\bar z}$. Thus we have the ramification profile (m, m) at the single critical point 0 of $\mathbb{RP}^2$. The automorphism group is the dihedral group of the order $2m$ which consists of rotations by $\frac{2\pi}{m}$ and antipodal involution $z \to -\frac{1}{\bar z}$. Thus we get that

$$H^{1,1}_{\mathrm{con}}\left(2m; (m, m)\right) = \frac{1}{2m}.$$

From (2) we see that $1 = \ell(\Delta)$ in this case. Now let us cover $\mathbb{RP}^2$ by $\mathbb{RP}^2$ via $z \to z^d$. From (2) we see that $\ell(\Delta) = 1$. For even d we have the critical point 0, in addition each point of the unit circle $|z| = 1$ is critical (a folding), while from the beginning we restrict our consideration only on isolated critical points. For odd $d = 2m - 1$ there is the single critical point 0, the automorphism group consists of rotations on the angle $\frac{2\pi}{2m-1}$. Thus in this case

$$H^{1,1}_{\mathrm{con}}\left(2m-1; (2m-1)\right) = \frac{1}{2m-1}.$$

Example 6. The generating series of the connected Hurwitz numbers with a single critical point from the previous Example is

$$F(h^{-1}\mathbf{p}) = \frac{1}{h^2}\sum_{m>0} p_m^2 H^{1,1}_{\mathrm{con}}\left(2m; (m, m)\right) + \frac{1}{h}\sum_{m>0} p_{2m-1} H^{1,1}_{\mathrm{con}}\left(2m-1; (2m-1)\right)$$

where $H^{1,1}_{\mathrm{con}}$ describes d-fold covering either by the Riemann sphere ($d = 2m$) or by the projective plane ($d = 2m - 1$). We get the generating function for Hurwitz numbers with a single critical point

$$\begin{aligned} \tau(h^{-1}\mathbf{p}) &= \mathrm{e}^{F(h^{-1}\mathbf{p})} = \mathrm{e}^{\frac{1}{h^2}\sum_{m>0}\frac{1}{2m}p_m^2 + \frac{1}{h}\sum_{m \mathrm{odd}}\frac{1}{m}p_m} \\ &= \sum_{d>0}\sum_{\substack{\Delta \\ |\Delta|=d}} h^{-\ell(\Delta)}\mathbf{p}_\Delta H^{1,a}(d; \Delta) \end{aligned} \tag{5}$$

where $a = 0$ and if $\Delta = (1^d)$, and where $a = 1$ and otherwise. Then $H^{1,1}(d;\Delta)$ is the Hurwitz number describing d-fold covering of $\mathbb{RP}^2$ with a single branch point of type $\Delta = (d_1,\ldots,d_l)$, $|\Delta| = d$ by a (not necessarily connected) Klein surface of Euler characteristic $\mathrm{E}' = \ell(\Delta)$. For instance, for $d = 3$, $\mathrm{E}' = 1$ we get $H^{1,1}(3;\Delta) = \frac{1}{3}\delta_{\Delta,(3)}$. For unbranched coverings (that is for $a = 0$, $\mathrm{E}' = d$) we get the corresponding formula.

Tau functions. Let us note that the expression presented in (3), namely,

$$\tau_1^{\mathrm{2KP}}(h^{-1}\mathbf{p}^{(1)}, h^{-1}\mathbf{p}^{(2)}) = \mathrm{e}^{h^{-2}\sum_{d>0}\frac{1}{d}p_d^{(1)}p_d^{(2)}} \tag{6}$$

coincides with the simplest two-component KP tau function with two sets of higher times $h^{-1}\mathbf{p}^{(i)}$, $i = 1,2$, while (5) may be recognized as the simplest non-trivial tau function of the BKP hierarchy of Kac and van de Leur [30]

$$\tau_1^{\mathrm{BKP}}(h^{-1}\mathbf{p}) = \mathrm{e}^{\frac{1}{h^2}\sum_{m>0}\frac{1}{2m}p_m^2 + \frac{1}{h}\sum_{m\,\mathrm{odd}}\frac{1}{m}p_m} \tag{7}$$

written down in [57]. In (3) and in (5) the higher times are rescaled as $p_m \to h^{-1}p_m$, $m > 0$ as it is common in the study of the integrable dispersionless equations where only the top power of the 'Plank constant' h is taken into account. For instance, see [50] where the counting of coverings of the Riemann sphere by Riemann spheres was related to the so-called Laplacian growth problem [42, 64]. About the quasiclassical limit of the DKP hierarchy see [4]. The rescaling is also common for tau functions used in two-dimensional gravity where the powers of $h^{-\mathrm{E}}$ group contributions of surfaces of Euler characteristic E to the 2D gravity partition function [14]. In the context of the links between Hurwitz numbers and integrable hierarchies the rescaling $\mathbf{p} \to h^{-1}\mathbf{p}$ was considered in [29] and in [49]. In our case the role similar to h plays N^{-1}, where N is the size of matrices in matrix integrals.

With the help of these tau functions we shall construct integral over matrices. To do this we present the variables $\mathbf{p}^{(i)}$, $i = 1,2$ and $\mathbf{p}$ as traces of a matrix we are interested in. We write $\mathbf{p}(X) = (p_1(X), p_2(X), \ldots)$, where

$$p_m(X) = \mathrm{tr}\,X^m = \sum_{i=1}^{N} x_i^m \tag{8}$$

and where $x_1,\ldots,x_N$ are eigenvalues of X.

In this case we use non-bold capital letters for the matrix argument and our tau functions are tau functions of the matrix argument:

$$\tau_1^{\mathrm{2KP}}(X,\mathbf{p}) := \tau_1^{\mathrm{2KP}}(\mathbf{p}(X),\mathbf{p}) = \sum_\lambda s_\lambda(X)s_\lambda(\mathbf{p}) = \mathrm{e}^{\mathrm{tr}\,V(X,\mathbf{p})} = \prod_{i=1}^{N}\mathrm{e}^{\sum_{m=1}^{\infty}\frac{1}{m}x_i^m p_m} \tag{9}$$

where x_i are eigenvalues of X, and $\mathbf{p} = (p_1, p_2, \dots)$ is a semi-infinite set of parameters, and

$$\tau_1^{\mathrm{BKP}}(X) := \tau_1^{\mathrm{BKP}}(\mathbf{p}(X)) = \sum_\lambda s_\lambda(X) = \prod_{i=1}^{N}(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}. \tag{10}$$

Here s_λ denotes the Schur function, see Appendix A. We recall the fact [38], which we shall need: if X is $N \times N$ matrix, then

$$s_\lambda(X) = 0, \qquad \text{if} \quad \ell(\lambda) > N \tag{11}$$

where $\ell(\lambda)$ is the length of a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_\ell > 0$.

For further purposes we need the following spectral invariants of a matrix X:

$$\mathbf{P}_\Delta(X) := \prod_{i=1}^{\ell} p_{\delta_i}(X) \tag{12}$$

where $\Delta = (\delta_1, \dots, \delta_\ell)$ is a partition and each p_{δ_i} is defined by (8)

In our notation one can write

$$\tau_1^{\mathrm{2KP}}(X,Y) = \tau_1^{\mathrm{2KP}}(\mathbf{p}(X), \mathbf{p}(Y)) = \sum_\Delta \frac{1}{z_\Delta}\mathbf{P}_\Delta(X)\mathbf{P}_\Delta(Y). \tag{13}$$

Combinatorial approach. The study of the homomorphisms between the fundamental group of the base Riemann surface of genus g (the Euler characteristic is respectively $\mathrm{E} = 2 - 2g$) with F marked points and the symmetric group in the context of the counting of the non-equivalent d-fold covering with given profiles Δ^i, $i = 1, \dots, \mathrm{F}$ results in the following equation (for instance, for the details, see Appendix A written by Zagier for the Russian edition of [36] or works [20, 40])

$$\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} X_1 \cdots X_{\mathrm{F}} = 1, \tag{14}$$

where $a_j, b_j, X_i \in S_d$ and where each X_i belongs to the cycle class C_{Δ^i}. Then the Hurwitz number $H^{2-2g,\mathrm{F}}(d; \Delta^1, \dots, \Delta^{\mathrm{F}})$ is equal to the number of solutions of equation (14) divided by the order of symmetric group S_d (to exclude the equivalent solutions obtained by the conjugation of all factors in (14) by elements of the group. In the geometrical approach each conjugation means the re-enumeration of d sheets of the cover).

For instance Example 3 considered above counts non-equivalent solutions of the equation $X_1X_2 = 1$ with given cycle classes C_{Δ^1} and C_{Δ^2}. Solutions of this equation consist of all elements of class C_{Δ^1} and inverse elements, so $\Delta^2 = \Delta^1 =: \Delta$. The number of elements of any class C_Δ (the cardinality of $|C_\Delta|$) divided by $|\Delta|!$ is $\frac{1}{z_\Delta}$ as we got in Example 3.

For Klein surfaces (see [20, 41]) instead of (14) we get

$$\prod_{j=1}^{g} R_j^2 X_1 \cdots X_{\mathrm{F}} = 1, \tag{15}$$

where $R_j, X_i \in S_d$ and where each X_i belongs to the cycle class C_{Δ^i}. In (15), g is the so-called genus of non-orientable surface which is related to its Euler characteristic E as $\mathrm{E} = 1 - g$. For the projective plane ($\mathrm{E} = 1$) we have $g = 0$, for the Klein bottle ($\mathrm{E} = 1$) $g = 1$.

Consider unbranched covers of the torus (equation (14)), projective plane and the Klein bottle (15)). In this we put each $X_i = 1$ in (14)) and (15)). Here we present three pictures, for the torus ($\mathrm{E} = 0$), the real projective plane ($\mathrm{E} = 1$) and Klein bottle ($\mathrm{E} = 0$) which may be obtained by the identification of square's edges. We get $aba^{-1}b^{-1} = 1$ for torus, $abab = 1$ for the projective plane and $abab^{-1} = 1$ for the Klein bottle.

Consider unbranched coverings ($\mathrm{F} = 0$). For the real projective plane we have $g = 1$ in (15) only one $R_1 = ab$. If we treat the projective plane as the unit disk with identified opposite points of the boarder $|z| = 1$, then R is related to the path from z to $-z$. For the Klein bottle ($g = 2$ in (15)) there are $R_1 = ab$ and $R_2 = b^{-1}$.

To avoid confusion in what follows we will use the notion of genus and the notations g only for Riemann surfaces, while the notion of the Euler characteristic E we shall use both for orientable and non-orientable surfaces.

2. Random matrices. Complex Ginibre ensemble

On this subject there is an extensive literature, for instance see [1–3, 61, 62].

We will consider integrals over complex matrices $Z_1, \ldots, Z_n$ where the measure is defined as

$$d\Omega(Z_1, \ldots, Z_n) = \prod_{\alpha=1}^{n} d\mu(Z_\alpha) = c \prod_{\alpha=1}^{n} \prod_{i,j=1}^{N} d\Re(Z_\alpha)_{ij} d\Im(Z_\alpha)_{ij} \mathrm{e}^{-|(Z_\alpha)_{ij}|^2} \tag{16}$$

where the integration range is $\mathbb{C}^{N^2} \times \cdots \times \mathbb{C}^{N^2}$ and where c is the normalization constant defined via $\int d\Omega(Z_1, \ldots, Z_n) = 1$.

We treat this measure as the probability measure. The related ensemble is called the ensemble of n independent complex Ginibre ensembles. The expectation of a quantity f which depends on entries of the matrices $Z_1, \ldots, Z_n$ is defined by

$$\mathbb{E}(f) = \int f(Z_1, \ldots, Z_n) d\Omega(Z_1, \ldots, Z_n).$$

Let us introduce the following products

$$X := (Z_1C_1) \cdots (Z_nC_n) \tag{17}$$

$$Y_t := (\tilde{C}_n Z_n^\dagger)(\tilde{C}_{n-1} Z_{n-1}^\dagger) \cdots (\tilde{C}_{t+1} Z_{t+1}^\dagger)(\tilde{C}_1 Z_1^\dagger)(\tilde{C}_2 Z_2^\dagger) \cdots (\tilde{C}_t Z_t^\dagger), \quad 0 < t < n, \tag{18}$$

where Z_α, C_α, $\tilde{C}_\alpha$ are complex $N \times N$ matrices and where $Z_\alpha^\dagger$ is the Hermitian conjugate of Z_α. We consider each matrix Z_α, $\alpha = 1, \ldots, n$ as the random matrix which belongs to the complex Ginibre ensemble numbered by α while the given matrices C_α and $\tilde{C}_\alpha$ are treated as sources. We are interested in correlation functions of spectral invariants of matrices X and Y_t. Actually all answers we shall obtain depend on pairwise products $C_\alpha \tilde{C}_\alpha$, therefore in what follows we put each $\tilde{C}_\alpha$ equal to the unity matrix I_N.

We denote by $x_1, \dots, x_N$ and by $y_1, \dots, y_N$ the eigenvalues of the matrices X and Y_t, respectively. Given partitions $\lambda = (\lambda_1, \dots, \lambda_l)$, $\mu = (\mu_1, \dots, \mu_k)$, $l, k \le N$, let us introduce the following spectral invariants

$$\mathbf{P}_\lambda(X) = p_{\lambda_1}(X) \cdots p_{\lambda_l}(X), \qquad \mathbf{P}_\mu(Y_t) = p_{\mu_1}(Y_t) \cdots p_{\mu_k}(Y_t) \tag{19}$$

where each $p_m(X)$ is defined via (8).

For a given partition λ, such that $d := |\lambda| \le N$, let us consider the spectral invariant $\mathbf{P}_\lambda$ of the matrix XY_t (see (12)). We have

Theorem 1. *X and Y_t are defined by* (17)–(18). *Denote* $\mathrm{E} = 2 - 2g$.

(A) *Let $n > t = 2g \ge 0$. Then*

$$\begin{aligned}\mathbb{E}\left(\mathbf{P}_\lambda(XY_{2g})\right) = z_\lambda &\sum_{\substack{\Delta^1, \dots, \Delta^{n-2g+1} \\ |\lambda| = |\Delta^j| = d,\, j \le n-2g+1}} H^{2-2g, n+2-2g}(d; \lambda, \Delta^1, \dots, \Delta^{n-2g+1}) \\ &\times P_{\Delta^{n-2g+1}}(C'C'') \prod_{i=1}^{n-2g} P_{\Delta^i}(C_{2g+i})\end{aligned} \tag{20}$$

where

$$C' = C_1 \cdots C_{2g-1}, \qquad C'' = C_2 C_4 \cdots C_{2g} \tag{21}$$

(B) *Let $n > t = 2g + 1 \ge 1$. Then*

$$\begin{aligned}\mathbb{E}\left(\mathbf{P}_\lambda(XY_{2g+1})\right) = z_\lambda &\sum_{\substack{\Delta^1, \dots, \Delta^{n-2g+1} \\ |\lambda| = |\Delta^j| = d,\, j \le n-2g+1}} H^{2-2g, n+2-2g}(d; \lambda, \Delta^1, \dots, \Delta^{n-2g+1}) \\ &\times P_{\Delta^{n-2g}}(C') P_{\Delta^{n-2g+1}}(C'') \prod_{i=1}^{n-2g-1} P_{\Delta^i}(C_{2g+1+i}),\end{aligned} \tag{22}$$

where

$$C' = C_1 C_3 \cdots C_{2g+1}, \qquad C'' = C_2 C_4 \cdots C_{2g} \tag{23}$$

Corollary 1. *Let $|\lambda| = d \le N$ as before, and let each $C_i = I_N$ ($N \times N$ unity matrix). Then*

$$\frac{1}{z_\lambda}\mathbb{E}\left(\mathbf{P}_\lambda(XY_{2g})\right) = \frac{1}{z_\lambda}\mathbb{E}\left(\mathbf{P}_\lambda(XY_{2g+1})\right) = N^{nd - \ell(\lambda)} \sum_{\mathrm{E}'} N^{\mathrm{E}'} S_{\mathrm{E}}^{\mathrm{E}'}(\lambda) \tag{24}$$

where $\mathrm{E} = 2 - 2g$ *and where*

$$S_{\mathrm{E}}^{\mathrm{E}'}(\lambda) := \sum_{\substack{\Delta^1, \dots, \Delta^{n+\mathrm{E}-1} \\ \sum_{i=1}^{n+\mathrm{E}-1} \ell(\Delta^i) = L}} H^{\mathrm{E}, n+\mathrm{E}}(d; \lambda, \Delta^1, \dots, \Delta^{n+\mathrm{E}-1}), \quad L = -\ell(\lambda) + nd + \mathrm{E}' \tag{25}$$

is the sum of Hurwitz numbers counting all d-fold coverings with the following properties:

(i) *the Euler characteristic of the base surface is* E
(ii) *the Euler characteristic of the cover is* E'
(iii) *there are at most* $\mathrm{F} = n + \mathrm{E}$ *critical points*

The item (ii) in the corollary follows from the equality $\mathbf{P}_\Delta(I_N) = N^{\ell(\Delta)}$ (see (12) and (8)) and from the Riemann–Hurwitz relation which relates Euler characteristics of a base and a cover via branch points profile's lengths (see (2)):

$$\sum_{i=1}^{n+\mathrm{E}-1} \ell(\Delta^i) = -\ell(\lambda) + (\mathrm{F} - \mathrm{E})d + \mathrm{E}'$$

In our case $\mathrm{F} - \mathrm{E} = n$.

Theorem 2. *X and Y_t are defined by* (17)–(18).

(A) *If* $|\lambda| \neq |\mu|$ *then* $\mathbb{E}\left(\mathbf{P}_\lambda(X)\mathbf{P}_\mu(Y_t)\right) = 0$.
(B) *Let* $|\lambda| = |\mu| = d$ *and* $n - 1 > t = 2g + 1 \geq 1$. *Then*

$$\mathbb{E}\left(\mathbf{P}_\lambda(X)\mathbf{P}_\mu(Y_{2g+1})\right) = z_\lambda z_\mu \sum_{\substack{\Delta^1,\ldots,\Delta^{n-2g} \\ |\lambda|=|\Delta^j|=d,\ j\leq n-2g}} H^{2-2g,n+2-2g}(d;\lambda,\mu,\Delta^1,\ldots,\Delta^{n-2g})$$
$$\times P_{\Delta^{n-2g-1}}(C')P_{\Delta^{n-2g}}(C_n C'') \prod_{i=1}^{n-2g-2} P_{\Delta^i}(C_{2g+1+i}), \tag{26}$$

where C' and C'' are given by (21).
(C) *Let* $|\lambda| = |\mu|$ $n > t = 2g \geq 0$. *Then*

$$\mathbb{E}\left(\mathbf{P}_\lambda(X)\mathbf{P}_\mu(Y_{2g})\right) = z_\lambda z_\mu \sum_{\substack{\Delta^1,\ldots,\Delta^{n-2g} \\ |\lambda|=|\Delta^j|=d,\ j\leq n-2g}} H^{2-2g,n+2-2g}(d;\lambda,\mu,\Delta^1,\ldots,\Delta^{n-2g})$$
$$\times P_{\Delta^{n-2g+1}}(C'C_nC'') \prod_{i=1}^{n-2g} P_{\Delta^i}(C_{2g+i}), \tag{27}$$

where C' and C'' are given by (23).

Corollary 2. *Let* $|\lambda| = d \leq N$ *as before, and let each* $C_i = I_N$. *Then*

$$\begin{aligned}\frac{1}{z_\lambda z_\mu}\mathbb{E}\left(\mathbf{P}_\lambda(X)\mathbf{P}_\lambda(Y_{2g})\right) &= \frac{1}{z_\lambda z_\mu}\mathbb{E}\left(\mathbf{P}_\lambda(X)\mathbf{P}_\lambda(Y_{2g+1})\right) \\ &= \frac{1}{z_\lambda}\mathbb{E}\left(\mathbf{P}_\lambda(XY_{2g})\right) = \frac{1}{z_\lambda}\mathbb{E}\left(\mathbf{P}_\lambda(XY_{2g+1})\right).\end{aligned} \tag{28}$$

Theorem 3. *X and Y_t are defined by* (17)–(18).

(A) *Let $n-1>t=2g+1\ge 0$. Then*

$$\mathbb{E}\left(\mathbf{P}_\lambda(X)\tau_1^{\mathrm{BKP}}(Y_{2g+1})\right)=z_\lambda \sum_{\substack{\Delta^1,\dots,\Delta^{n-2g}\\ |\lambda|=|\Delta^j|=d,\, j\le n-2g}} H^{1-2g,n+1-2g}(d;\lambda,\Delta^1,\dots,\Delta^{n-2g})$$
$$\times P_{\Delta^{n-2g-1}}(C')P_{\Delta^{n-2g}}(C_nC'')\prod_{i=1}^{n-2g-2}P_{\Delta^i}(C_{2g+1+i}) \tag{29}$$

where C' and C'' are given by (21).

(B) *Let $n>t=2g\ge 0$. Then*

$$\mathbb{E}\left(\mathbf{P}_\lambda(X)\tau_1^{\mathrm{BKP}}(Y_{2g})\right)=z_\lambda z_\mu \sum_{\substack{\Delta^1,\dots,\Delta^{n-2g}\\ |\lambda|=|\Delta^j|=d,\, j\le n-2g}} H^{1-2g,n+1-2g}(d;\lambda,\Delta^1,\dots,\Delta^{n-2g})$$
$$\times P_{\Delta^{n+1-2g}}(C'C_nC'')\prod_{i=1}^{n-2g}P_{\Delta^i}(C_{2g+i}) \tag{30}$$

where C' and C'' are given by (23).

The sketch of proof. The characteristic Frobenius-type formula by Mednykh–Pozdnyakova–Jones [20, 41]

$$H^{\mathrm{E},k}(d;\Delta^1,\dots,\Delta^k)=\sum_{\substack{\lambda\\|\lambda|=d}}\left(\frac{\dim\lambda}{d!}\right)^{\mathrm{E}}\varphi_\lambda(\Delta^1)\cdots\varphi_\lambda(\Delta^k) \tag{31}$$

where $\dim\lambda$ is the dimension of the irreducible representation of S_d, and

$$\varphi_\lambda(\Delta^{(i)}):=|C_{\Delta^{(i)}}|\,\frac{\chi_\lambda(\Delta^{(i)})}{\dim\lambda},\quad \dim\lambda:=\chi_\lambda\left((1^d)\right) \tag{32}$$

$\chi_\lambda(\Delta)$ is the character of the symmetric group S_d evaluated at a cycle type Δ, and χ_λ ranges over the irreducible complex characters of S_d (they are labeled by partitions $\lambda=(\lambda_1,\dots,\lambda_\ell)$ of a given weight $d=|\lambda|$). It is supposed that $d=|\lambda|=|\Delta^1|=\dots=|\Delta^k|$. $|C_\Delta|$ is the cardinality of the cycle class C_Δ in S_d.

Then we use the characteristic map relation [38]:

$$s_\lambda(\mathbf{p})=\frac{\dim\lambda}{d!}\left(p_1^d+\sum_{\substack{\Delta\\|\Delta|=d}}\varphi_\lambda(\Delta)\mathbf{p}_\Delta\right) \tag{33}$$

where $\mathbf{p}_\Delta=p_{\Delta_1}\cdots p_{\Delta_\ell}$ and where $\Delta=(\Delta_1,\dots,\Delta_\ell)$ is a partition whose weight coincides with the weight of λ: $|\lambda|=|\Delta|$. Here

$$\dim\lambda=d!s_\lambda(\mathbf{p}_\infty),\qquad \mathbf{p}_\infty=(1,0,0,\dots) \tag{34}$$

is the dimension of the irreducible representation of the symmetric group S_d. We imply that $\varphi_\lambda(\Delta)=0$ if $|\Delta|\ne|\lambda|$.

Then we know how to evaluate the integral with the Schur function via Lemma used in [59] and [48, 49] (for instance see [38] for the derivation).

Lemma 1. *Let A and B be normal matrices* (*i.e., matrices diagonalizable by unitary transformations*)*. Then below* $p_\infty = (1, 0, 0, \dots)$.

$$\int_{\mathbb{C}^{n^2}} s_\lambda(AZBZ^+) e^{-\operatorname{tr} ZZ^+} \prod_{i,j=1}^{n} d^2 Z = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(p_\infty)} \tag{35}$$

and

$$\int_{\mathbb{C}^{n^2}} s_\mu(AZ)s_\lambda(Z^+B) e^{-\operatorname{tr} ZZ^+} \prod_{i,j=1}^{n} d^2 Z = \frac{s_\lambda(AB)}{s_\lambda(p_\infty)}\delta_{\mu,\lambda}\,. \tag{36}$$

To prove Theorem 1 we use the property that we can equate the integral over $E(\tau^{2\mathrm{KP}}(XY_y))$ using this lemma and (6) and then compare it to the same integral where now we use (9). To prove Theorem 2 in the similar way we equate $E(\tau^{2\mathrm{KP}}(X)\tau^{2\mathrm{KP}}(Y_y))$. To prove Theorem 3 we similarly $E(\tau^{2\mathrm{KP}}(X)\tau^{2\mathrm{KP}}(Y_y))$ in the same way taking into account also (10).

Appendix A. Partitions and Schur functions

Let us recall that the characters of the unitary group $\mathbb{U}(N)$ are labeled by partitions and coincide with the so-called Schur functions [38]. A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is a set of nonnegative integers λ_i which are called parts of λ and which are ordered as $\lambda_i \geq \lambda_{i+1}$. The number of non-vanishing parts of λ is called the length of the partition λ, and will be denoted by $\ell(\lambda)$. The number $|\lambda| = \sum_i \lambda_i$ is called the weight of λ. The set of all partitions will be denoted by $\mathbb{P}$.

The Schur function labeled by λ may be defined as the following function in variables $x = (x_1, \dots, x_N)$:

$$s_\lambda(x) = \frac{\det\left[x_j^{\lambda_i - i + N}\right]_{i,j}}{\det\left[x_j^{-i+N}\right]_{i,j}} \tag{37}$$

in case $\ell(\lambda) \leq N$ and vanishes otherwise. One can see that $s_\lambda(x)$ is a symmetric homogeneous polynomial of degree $|\lambda|$ in the variables $x_1, \dots, x_N$, and $\deg x_i = 1$, $i = 1, \dots, N$.

Remark. In case the set x is the set of eigenvalues of a matrix X, we also write $s_\lambda(X)$ instead of $s_\lambda(x)$.

There is a different definition of the Schur function as quasi-homogeneous non-symmetric polynomial of degree $|\lambda|$ in other variables, the so-called power sums, $\mathbf{p} = (p_1, p_2, \dots)$, where $\deg p_m = m$.

For this purpose let us introduce

$$s_{\{h\}}(\mathbf{p}) = \det[s_{(h_i + j - N)}(\mathbf{p})]_{i,j},$$

where $\{h\}$ is any set of N integers, and where the Schur functions $s_{(i)}$ are defined by $e^{\sum_{m>0}\frac{1}{m}p_m z^m} = \sum_{m\geq 0} s_{(i)}(\mathbf{p})z^i$. If we put $h_i = \lambda_i - i + N$, where N is not less than the length of the partition λ, then

$$s_\lambda(\mathbf{p}) = s_{\{h\}}(\mathbf{p}). \tag{38}$$

The Schur functions defined by (37) and by (38) are equal, $s_\lambda(\mathbf{p}) = s_\lambda(x)$, provided the variables $\mathbf{p}$ and x are related by the power sums relation

$$p_m = \sum_i x_i^m. \tag{39}$$

In case the argument of s_λ is written as a non-capital bold letter one uses definition (38), and we imply definition (37) in case the argument is not bold and non-capital letter, and in case the argument is capital letter which denotes a matrix, then it implies the definition (37) with $x = (x_1, \dots, x_N)$ being the eigenvalues.

It may be easily checked that

$$s_\lambda(\mathbf{p}) = (-1)^{|\lambda|} s_{\lambda^{\mathrm{tr}}}(-\mathbf{p}) \tag{40}$$

where λ^{tr} is the partition conjugated to λ (in [38] it is denoted by λ^*). The Young diagram of the conjugated partition is obtained by the transposition of the Young diagram of λ with respect to its main diagonal. One gets $\lambda_1 = \ell(\lambda^{\mathrm{tr}})$.

Appendix B. Matrix integrals as generating functions of Hurwitz numbers from [48, 49]

In case the base surface is $\mathbb{CP}^1$ the set of examples of matrix integrals generating Hurwitz numbers were studied in works [7, 12, 13, 34, 36, 39, 66]. One can show that the perturbation series in coupling constants of these integrals (Feynman graphs) may be related to TL (KP and two-component KP) hypergeometric tau functions. It actually means that these series generate Hurwitz numbers with at most two arbitrary profiles. (An arbitrary profile corresponds to a certain term in the perturbation series in the coupling constants which are higher times. The TL and 2-KP hierarchies there are two independent sets of higher times which yields two critical points for Hurwitz numbers.)

Here, very briefly, we will write down few generating series for the $\mathbb{RP}^2$ Hurwitz numbers. These series may be not tau functions themselves but may be presented as integrals of tau functions of matrix argument. (The matrix argument, which we denote by a capital letter, say X, means that the power sum variables $\mathbf{p}$ are specified as $p_i = \mathrm{tr}\, X^i$, $i > 0$. Then instead of $s_\lambda(\mathbf{p})$, $\tau(\mathbf{p})$ we write $s_\lambda(X)$ and $\tau(X)$). If a matrix integral in examples below is a BKP tau function then it generates Hurwitz numbers with a single arbitrary profile and all other are subjects of restrictions identical to those in $\mathbb{CP}^1$ case mentioned above. In all examples $V(x,\mathbf{p}) := \sum_{m>0}\frac{1}{m}x^m p_m$. We also recall the notation $\mathbf{p}_\infty = (1,0,0,\dots)$ and that numbers $H^{\mathrm{E,F}}(d;\dots)$ are Hurwitz numbers only in case $d \le N$, N is the size of matrices.

For more details of the $\mathbb{RP}^2$ case see [48]. New development in [48] with respect to the consideration in [59] is the usage of products of matrices. Here we shall consider a few examples. All examples include the simplest BKP tau function, of matrix argument X, [57] defined by

$$\begin{aligned}\tau_1^{\mathrm{B}}(X) &:= \sum_\lambda s_\lambda(X) = \mathrm{e}^{\frac12\sum_{m>0}\frac1m(\operatorname{tr}X^m)^2+\sum_{m>0,\mathrm{odd}}\frac1m\operatorname{tr}X^m}\\ &= \frac{\det^{\frac12}\frac{1+X}{1-X}}{\det^{\frac12}\left(\mathbb{I}_N\otimes\mathbb{I}_N - X\otimes X\right)}\end{aligned} \tag{41}$$

as the part of the integration measure. Other integrands are the simplest KP tau functions $\tau_1^{2\mathrm{KP}}(X,\mathbf{p}) := \mathrm{e}^{\operatorname{tr}V(X,\mathbf{p})}$ where the parameters $\mathbf{p}$ may be called coupling constants. The perturbation series in coupling constants are expressed as sums of products of the Schur functions over partitions and are similar to the series we considered in the previous sections.

Example B1. The projective analog of Okounkov's generating series for double Hurwitz series as a model of normal matrices. From the equality

$$\left(2\pi\zeta_1^{-1}\right)^{\frac12}\mathrm{e}^{\frac{(n\zeta_0)^2}{2\zeta_1}}\mathrm{e}^{\zeta_0 nc+\frac12\zeta_1c^2} = \int_{\mathbb{R}}\mathrm{e}^{x_in\zeta_0+(cx_i-\frac12x_i^2)\zeta_1}dx_i,$$

in a similar way as was done in [58] using $\varphi_\lambda(\Gamma)=\sum_{(i.j)\in\lambda}(j-i)$, one can derive

$$\mathrm{e}^{n|\lambda|\zeta_0}\mathrm{e}^{\zeta_1\varphi_\lambda(\Gamma)}\delta_{\lambda,\mu} = \mathrm{K}\int s_\lambda(M)s_\mu(M^\dagger)\det\left(MM^\dagger\right)^{n\zeta_0}\mathrm{e}^{-\frac12\zeta_1\operatorname{tr}\left(\log\left(MM^\dagger\right)\right)^2}dM,$$

where K is unimportant multiplier, M is a normal matrix with eigenvalues $z_1,\dots,z_N$, $\log|z_i|=x_i$ and

$$dM = d_*U\prod_{i<j}|z_i-z_j|^2\prod_{i=1}^N d^2z_i.$$

Then the $\mathbb{RP}^2$ analogue of Okounkov's generating series may be presented as the following integral ([51]) may be written

$$\begin{aligned}&\sum_{\substack{\lambda\\ \ell(\lambda)\le N}}\mathrm{e}^{n|\lambda|\zeta_0+\zeta_1\varphi_\lambda(\Gamma)}s_\lambda(\mathbf{p})\\ &\quad=\mathrm{K}\int\mathrm{e}^{V(M,\mathbf{p})}\mathrm{e}^{\zeta_0n\operatorname{tr}\log\left(MM^\dagger\right)-\frac12\zeta_1\left(\operatorname{tr}\log\left(MM^\dagger\right)\right)^2}\tau_1^{\mathrm{B}}(M^\dagger)dM\ .\end{aligned} \tag{42}$$

Recall that in the work [51] there were studied Hurwitz numbers with an arbitrary number of simple branch points and two arbitrary profiles. In our analog, describing the coverings of the projective plane, an arbitrary profile only one, because, unlike the Toda lattice, the hierarchy of BKP has only one set of (continuous) higher times.

A similar representation of the Okounkov $\mathbb{CP}^1$ was earlier presented in [8].

Below we use the following notations

- d_*U is the normalized Haar measure on $\mathbb{U}(N)$: $\int_{\mathbb{U}(N)} d_*U = 1$.
- Z is a complex matrix

$$d\Omega(Z, Z^\dagger) = \pi^{-n^2} \mathrm{e}^{-\operatorname{tr}\left(ZZ^\dagger\right)} \prod_{i,j=1}^{N} d\Re Z_{ij} d\Im Z_{ij}.$$

- Let M be a Hermitian matrix the measure is defined

$$dM = \prod_{i\le j} d\Re M_{ij} \prod_{i<j} d\Im M.$$

It is known [38]

$$\int s_\lambda(Z) s_\mu(Z^\dagger)\, d\Omega(Z, Z^\dagger) = (N)_\lambda \delta_{\lambda,\mu}, \tag{43}$$

where $(N)_\lambda := \prod_{(i.j)\in\lambda}(N + j - i)$ is the Pochhammer symbol related to λ. A similar relation was used in [7, 26, 53, 58, 59], for models of Hermitian, complex and normal matrices.

By $\mathbb{I}_N$ we shall denote the $N \times N$ unit matrix. We recall that

$$s_\lambda(\mathbb{I}_N) = (N)_\lambda s_\lambda(\mathbf{p}_\infty)\,, \qquad s_\lambda(\mathbf{p}_\infty) = \frac{\dim\lambda}{d!}, \quad d = |\lambda|.$$

Example B2. Three branch points. The generating function for $\mathbb{RP}^2$ Hurwitz numbers with three ramification points, having three arbitrary profiles:

$$\sum_{\lambda,\, \ell(\lambda)\le N} \frac{s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\Lambda) s_\lambda(\mathbf{p}^{(2)})}{\left(s_\lambda(\mathbf{p}_\infty)\right)^2} = \int \tau_1^{\mathrm{B}}\left(Z_1 \Lambda Z_2\right) \prod_{i=1,2} \mathrm{e}^{V\left(\operatorname{tr} Z_i^\dagger,\, \mathbf{p}^{(i)}\right)}\, d\Omega(Z_i, Z_i^\dagger). \tag{44}$$

If $\mathbf{p}^{(2)} = \mathbf{p}(\mathtt{q},\mathtt{t})$ with any given parameters $\mathtt{q},\mathtt{t}$, and $\Lambda = \mathbb{I}_N$ then (44) is the hypergeometric BKP tau function.

Example B3. 'Projective' Hermitian two-matrix model. The following integral

$$\int \tau_1^{\mathrm{B}}(cM_2) \mathrm{e}^{\operatorname{tr} V(M_1,\mathbf{p}) + \operatorname{tr}(M_1 M_2)} dM_1 dM_2 = \sum_\lambda c^{|\lambda|} (N)_\lambda s_\lambda(\mathbf{p}),$$

where M_1, M_2 are Hermitian matrices is an example of the hypergeometric BKP tau function.

Example B4. Unitary matrices. Generating series for projective Hurwitz numbers with arbitrary profiles in n branch points and restricted profiles in other points:

$$\begin{aligned} &\int \mathrm{e}^{\operatorname{tr}(cU_1^\dagger \cdots U_{n+m}^\dagger)} \left(\prod_{i=n+1}^{n+m} \tau_1^{\mathrm{B}}(U_i) d_*U_i\right) \left(\prod_{i=1}^{n} \tau_1^{\mathrm{KP}}(U_i, \mathbf{p}^{(i)}) d_*U_i\right) \\ &\quad = \sum_{d\ge 0} c^d \,(d!)^{1-m} \sum_{\substack{\lambda,\, |\lambda|=d \\ \ell(\lambda)\le N}} \left(\frac{\dim\lambda}{d!}\right)^{2-m} \left(\frac{s_\lambda(\mathbb{I}_N)}{\dim\lambda}\right)^{1-m-n} \prod_{i=1}^{n} \frac{s_\lambda(\mathbf{p}^{(i)})}{\dim\lambda}. \end{aligned} \tag{45}$$

Here $\mathbf{p}^{(i)}$ are parameters. This series generate certain linear combination of Hurwitz numbers for base surfaces with Euler characteristic $2-m$, $m \geq 0$. In case $n=1$ this BKP tau function may be viewed as an analogue of the generating function of the so-called non-connected Bousquet–Melou–Schaeffer numbers (see Example 2.16 in [33]). In case $n=m=1$ we obtain the following BKP tau function

$$\int \tau_1^{\mathrm{B}}(U_2) e^{\operatorname{tr} V(U_1,\mathbf{p})+\operatorname{tr}(cU_1^\dagger U_2^\dagger)} d_*U_1 d_*U_2 = \sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} c^{|\lambda|} \frac{s_\lambda(\mathbf{p})}{(N)_\lambda}.$$

Example B5. Integrals over complex matrices. A pair of examples. An analogue of Belyi curves generating function [13, 66] is as follows:

$$\sum_{l=1}^{N} N^l \sum_{\substack{\Delta^{(1)},\dots,\Delta^{(n+1)} \\ \ell(\Delta^{n+1})=l}} c^d H^{\mathrm{E},n+1}(d;\Delta^{(1)},\dots,\Delta^{(n+1)}) \prod_{i=1}^{n} \mathbf{p}^{(i)}_{\Delta^{(i)}}$$
$$= \sum_\lambda c^{|\lambda|} \frac{(d!)^{m-2}(N)_\lambda}{(\dim\lambda)^{m-2}} \prod_{i=1}^{n} \frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} \tag{46}$$
$$= \int e^{\operatorname{tr}(cZ_1^\dagger\cdots Z_{n+m}^\dagger)} \left(\prod_{i=n+1}^{n+m} \tau_1^{\mathrm{B}}(Z_i) d\Omega(Z_i,Z_i^\dagger)\right)\left(\prod_{i=1}^{n} \tau_1^{\mathrm{KP}}(Z_i,\mathbf{p}^{(i)}) d\Omega(Z_i,Z_i^\dagger)\right)$$

where $\mathrm{E}=2-m$ is the Euler characteristic of the base surface.

The series in the following example generates the projective Hurwitz numbers themselves where to get rid of the factor $(N)_\lambda$ in the sum over partitions we use mixed integration over $\mathbb{U}(N)$ and over complex matrices:

$$\sum_{\Delta^{(1)},\dots,\Delta^{(n)}} c^d H^{1,n}(d;\Delta^{(1)},\dots,\Delta^{(n)}) \prod_{i=1}^{n} \mathbf{p}^{(i)}_{\Delta^{(i)}} = \sum_{\lambda,\,\ell(\lambda)\leq N} c^{|\lambda|} \frac{\dim\lambda}{d!} \prod_{i=1}^{n} \frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)}$$
$$= \int \tau_1^{\mathrm{KP}}(cU^\dagger Z_1^\dagger\cdots Z_k^\dagger,\mathbf{p}^{(n)}) \tau_1^{\mathrm{B}}(U) d_*U \prod_{i=1}^{n-1} \tau_1^{\mathrm{KP}}(Z_i,\mathbf{p}^{(i)}) d\Omega(Z_i,Z_i^\dagger). \tag{47}$$

Here Z, Z_i, $i=1,\dots,n-1$ are complex $N\times N$ matrices and $U\in\mathbb{U}(N)$. As in the previous examples one can specify all sets $\mathbf{p}^{(i)}=\mathbf{p}(\mathtt{q}_i,\mathtt{t}_i)$, $i=1,\dots,n$ except a single one which in this case has the meaning of the BKP higher times.

Acknowledgment

The work has been funded by RFBR grant 18-01-00273a, the RAS Program "Fundamental problems of nonlinear mechanics", and by the Russian Academic Excellence Project '5-100'. I thank A. Odzijewicz and University of Białystok for warm hospitality which allowed to write this work. I am grateful to S. Natanzon, A. Odzijewicz, J. Harnad, A. Mironov (ITEP) and to van de Leur for various remarks concerning the questions connected with this work. Special gratitude to E. Strakhov for the fact that he drew my attention to the works on quantum chaos devoted to the products of random matrices and for fruitful discussions.

References

[1] G. Akemann, J.R. Ipsen, M. Kieburg, *Products of Rectangular Random Matrices: Singular Values and Progressive Scattering*, arXiv:1307.7560

[2] G. Akemann, T. Checinski, M. Kieburg, *Spectral correlation functions of the sum of two independent complex Wishart matrices with unequal covariances*, arXiv: 1502.01667

[3] G. Akemann, E. Strahov, *Hard edge limit of the product of two strongly coupled random matrices*, arXiv:1511.09410

[4] V. Akhmedova, T. Takebe, A. Zabrodin, Multi-variable reductions of the dispersionless DKP hierarchy, arXiv:1707.01528, accepted J. Phys. A. https://doi.org/10.1088/1751-8121/aa9457

[5] A. Alexandrov, Matrix models for random partitions. Nucl. Phys. B **851**, 620–650 (2011)

[6] A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, *Integrability of Hurwitz Partition Functions. I. Summary*, J. Phys. A: Math. Theor.45 (2012) 045209, arXiv: 1103.4100

[7] A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, *On KP-integrable Hurwitz functions*, JHEP 11(2014) 080, arXiv: 1405.1395

[8] A. Alexandrov and A.V. Zabrodin *Free fermions and tau-functions*, J. Geom. Phys. 67 (2013) pp. 37–80; arXiv:1212.6049

[9] A.A. Alexeevski and S.M. Natanzon, *Noncommutative two-dimansional field theories and Hurwitz numbers for real algebraic curves*, Selecta Math. N.S. v. 12 (2006), n. 3, pp. 307–377, arXiv:math/0202164

[10] A.V. Alekseevskii and S.M. Natanzon, *The algebra of bipartite graphs and Hurwitz numbers of seamed surfaces*, Izvestiya Mathematics 72:4 (2008) pp. 627–646

[11] N.L. Alling and N. Greenleaf, *Foundation of the theory of Klein surfaces*, Springer-Verlag, 1971, Leture Notes in Math. v. 219

[12] J. Ambjorn and L. Chekhov, *The matrix model for hypergeometric Hurwitz number*, Theoret. and Math. Phys., 1 81:3 (2014), 1486–1498; arXiv:1409.3553

[13] J. Ambjorn and L.O. Chekhov, *The matrix model for dessins d'enfants*, Ann. Inst. Henri Poincaré D, 1:3 (2014), 337–361; arXiv:1404.4240

[14] E. Brezin and V. Kazakov, *Exactly solvable field theories of closed strings*, Phys Lett **B236** pp. 144–150 (1990); D. Gross and A. Migdal, *Nonperturbative two-dimensional quantum gravity*, Phys. Rev. Lett. **64** p. 127 (1990)

[15] S.R. Carrel, *The non-orientable map asymptotic constant* p_g, arxiv:1406.1760

[16] L.O. Chekhov, *The Harer–Zagier recursion for an irregular spectral curve*, J. Geom. Phys., 110 (2016), 30–43, arXiv: 1512.09278

[17] R. Dijkgraaf, *Mirror symmetry and elliptic curves, The Moduli Space of Curves*, R. Dijkgraaf, C. Faber, G. van der Geer (editors), Progress in Mathematics, 129, Birkhäuser, 1995

[18] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin and L. Spitz, *Polynomiality of Hurwitz numbers, Bouchard–Marino conjecture, and a new proof of the ELSV formula*, arXiv:1307.4729

[19] T. Ekedahl, S.K. Lando, V. Shapiro and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, C.R. Acad. Sci. Paris Ser. I. Math. Vol. 146, N2, pp. 1175–1180 (1999)

[20] Gareth A. Jones, *Enumeration of Homomorphisms and Surface-Coverings*, Quart. J. Math. Oxford (2), 46 (1995), pp. 485–507

[21] I.P. Goulden and D.M. Jackson, *The KP hierarchy, branched covers, and triangulations*, Advances in Mathematics, **219** pp. 932–951, 2008

[22] I.P. Goulden, M. Guay-Paquet and J. Novak, *Monotone Hurwitz numbers and HCIZ integral*, Ann. Math. Blaise Pascal **21** pp. 71–99 (2014)

[23] M. Guay-Paquet and J. Harnad, *2D Toda tau-functions as combinatorial generating functions*, Letters in Mathematical Physics 105, pp. 827–852 (2015)

[24] J. Harnad and A.Yu. Orlov, *Scalar product of symmetric functions and matrix integrals*, Theoretical and mathematical physics 137 (3), pp. 1676–1690 (2003)

[25] J. Harnad and A.Yu. Orlov, *Matrix integrals as Borel sums of Schur function expansions*, Symmetry and Perturbation Theory 2002, Cala Gonoone (Sardinia), May 1–26, pp. (2002). Proceedings, pp. 116–123 (World Scientific, Singapore, eds. S. Abenda, G. Gaeta); arXiv:nlin/0209035

[26] J. Harnad and A.Yu. Orlov, *Fermionic construction of partition functions for two matrix models and perturbative Schur functions expansions*, J. Phys. A 39, pp. 8783–8809 (2006)

[27] J. Harnad and A.Yu. Orlov, *Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths*, Commun. Math. Phys. **338** (2015) pp. 267–284 arxiv: math.ph/1407.7800

[28] J. Harnad, *Multispecies quantum Hurwitz numbers*, SIGMA **11**, 097 (2015); arXiv: 1410.8817

[29] J. Harnad, *Weighted Hurwitz numbers and hypergeometric τ-functions: an overview*, AMS Proc. Symp. Pure Math. 93 (2016) pp. 289–333; arXiv:1504.03408

[30] V. Kac and J. van de Leur, *The Geometry of Spinors and the Multicomponent BKP and DKP Hierarchies*, CRM Proceedings and Lecture Notes **14** (1998) pp. 159–202

[31] V.A. Kazakov, M. Staudacher, T. Wynter, *Character Expansion Methods for Matrix Models of Dually Weighted Graphs*, Commun. Math. Phys. 177 (1996) 451–468; arXiv:hep-th/9502132

[32] M. Kazarian and S. Lando, *Combinatorial solutions to integrable hierarchies*, Uspekhi Mat. Nauk 70 (2015), no. 3(423), pp. 77–106. English translation: 2015 Russ. Math. Surv. 70, pp. 453–482; arXiv:1512.07172

[33] M.E. Kazarian and S.K. Lando, *An algebro-geometric proof of Witten's conjecture*, J. Amer. Math. Soc. 20:4 (2007), pp. 1079–1089

[34] M. Kazarian and P. Zograph, *Virasoro constraints and topological recursion for Grothendieck's dessin counting*, arxiv1406.5976

[35] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Generalized Kazakov-Migdal-Kontsevich Model: group theory aspects*, International Journal of Mod. Phys. A10 (1995) p. 2015

[36] S.K. Lando, A.K. Zvonkin, Graphs on Surfaces and their Applications, Encyclopaedia of Mathematical Sciences, Volume 141, with appendix by D. Zagier, Springer, N.Y. (2004)

[37] J.W. van de Leur, *Matrix Integrals and Geometry of Spinors*, J. of Nonlinear Math. Phys. **8**, pp. 288–311 (2001)

[38] I.G. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford, (1995)

[39] R. de Mello Koch and S. Ramgoolam, *From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group*, arXiv: 1002.1634

[40] A.D. Mednykh, *Determination of the number of nonequivalent covering over a compact Riemann surface*, Soviet Math. Dokl., 19(1978), pp. 318–320

[41] A.D. Mednykh and G.G. Pozdnyakova,*The number of nonequivalent covering over a compact nonorientable surface*, Sibirs. Mat. Zh, 27(1986), +- 1, pp. 123–131, 199

[42] M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin, *Integrable Structure of Interface Dynamics*, Phys. Rev. Lett. 84 (2000) 5106–5109

[43] A.D. Mironov, A.Yu. Morozov and S.M. Natanzon, *Complect set of cut-and-join operators in the Hurwitz–Kontsevich theory*, Theor. and Math. Phys. 166:1, (2011), pp. 1–22; arXiv:0904.4227

[44] A.D. Mironov, A.Yu. Morozov and S.M. Natanzon, *Algebra of differential operators associated with Young diagramms*, J. Geom. and Phys. n. 62 (2012), pp. 148–155

[45] S.M. Natanzon, *Klein surfaces*, Russian Math. Surv., 45:6(1990), pp. 53–108

[46] S.M. Natanzon, *Moduli of Riemann surfaces, real algebraic curves and their superanalogs*, Translations of Math. Monograph, AMS, Vol. 225 (2004), 160 pp.

[47] S.M. Natanzon, *Simple Hurwitz numbers of a disk*, Funk. Analysis ant its applications, v. 44 (2010), n. 1, pp. 44–58

[48] S.M. Natanzon and A.Yu. Orlov, *Hurwitz numbers and BKP hierarchy*, arXiv: 1407.832

[49] S.M. Natanzon and A.Yu. Orlov, *BKP and projective Hurwitz numbers*, Letters in Mathematical Physics, 107(6), 1065–1109 (2017); arXiv:1501.01283

[50] S.M. Natanzon and A. Zabrodin, *Toda hierarchy, Hurwitz numbers and conformal dynamics*, Int. Math. Res. Notices 2015 (2015) 2082–2110

[51] A. Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Lett. **7**, pp. 447–453 (2000). See also arxivmath-004128.

[52] A. Okounkov and R. Pandharipande, *Gromov–Witten theory, Hurwitz theory and completed cycles*, Annals of Math. **163** p. 517 (2006); arxiv.math.AG/0204305

[53] A.Yu. Orlov, *Soliton theory, symmetric functions and matrix integrals*, Acta Applicandae Mathematica 86 (1–2), pp. 131–158 (2005)

[54] A.Yu. Orlov *Hurwitz numbers and products of random matrices*, Theoretical and Mathematical Physics 193(3) pp. 1282–1323 (2017); arxiv:1701.02296

[55] A.Yu. Orlov and D. Scherbin, *Fermionic representation for basic hypergeometric functions related to Schur polynomials*, arXiv preprint nlin/0001001

[56] A.Yu. Orlov and D. Scherbin, *Hypergeometric solutions of soliton equations*, Theoretical and Mathematical Physics 128 (1), pp. 906–926 (2001)

[57] A.Yu. Orlov, T. Shiota and K. Takasaki, *Pfaffian structures and certain solutions to BKP hierarchies I. Sums over partitions*, accepted by JMP; arXiv: math-ph/12014518

[58] A.Yu. Orlov and T. Shiota, *Schur function expansion for normal matrix model and associated discrete matrix models*, Physics Letters A 343 (5), pp. 384–396

[59] A.Yu. Orlov, *New solvable matrix integrals*, Intern. J. Mod. Phys. **A 19** (suppl 02), pp. 276–93 (2004)

[60] A.Yu. Orlov and E. Strahov, *Products of random matrices and τ-functions*, May 2015, preprint

[61] E. Strahov, *Dynamical correlation functions for products of random matrices*, arXiv:1505.02511

[62] E. Strahov, *Differential equations for singular values of products of Ginibre random matrices*, arXiv:1403.6368

[63] K. Ueno and K. Takasaki, *Toda lattice hierarchy*, *Adv. Stud. Pure Math.* **4**, 1–95 (1984)

[64] A. Zabrodin, New applications of non-hermitian random matrices, Ann. Henri Poincaré 4 Suppl. 2 (2003) S851–S861, cond-mat/0210331; R. Teodorescu, E. Bettelheim, O. Agam, A. Zabrodin and P. Wiegmann, Normal random matrix ensemble as a growth problem, Nucl. Phys. B704 (2005) 407–444

[65] V.E. Zakharov and A.B. Shabat, J. Funct. Anal. Appl. 8, 226 (1974), 13, p. 166 (1979)

[66] P. Zograf, *Enumeration of Gronthendieck's dessons and KP hierarchy*, arXiv: 1312.2538 (2013)

Aleksander Yu. Orlov
Institute of Oceanology
Nahimovskii Prospekt 36
Moscow 117997, Russia

and

National Research University
Higher School of Economics
International Laboratory of Representation Theory
and Mathematical Physics
20 Myasnitskaya Ulitsa
Moscow 101000, Russia
e-mail: orlovs@ocean.ru

Part IX

Extended Abstracts of the Lectures at "School on Geometry and Physics"

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 377–382

Integral Invariants (Poincaré–Cartan) and Hydrodynamics

Marián Fecko

Mathematics Subject Classification (2010). Primary 53Z05; Secondary 76B47.

Keywords. Poincaré and Cartan integral invariant, vortex line, Helmholtz theorem.

1. Introduction

There are several ways how hydrodynamics of ideal fluid may be treated geometrically. In particular, it may be viewed as an application of the theory of *integral invariants* due to Poincaré and Cartan (see Refs. [1, 2], or, in modern presentation, Refs. [3, 4]). Then, the original Poincaré version of the theory refers to the *stationary* (time-independent) flow, described by the stationary *Euler equation*, whereas Cartan's extension embodies the full, possibly time-dependent, situation.

Although the approach via integral invariants is far from being the best known, it has some nice features which, hopefully, make it worth spending some time. Namely, the form in which the Euler equation is expressed in this approach, turns out to be ideally suited for extracting important (and useful) classical consequences of the equations remarkably easily (see more details in Ref. [4]). This refers, in particular, to the behavior of *vortex lines*, discovered long ago by Helmholtz.

2. Poincaré integral invariants

Consider a manifold M endowed with dynamics given by a *vector field* v

$$\dot{\gamma} = v \qquad\qquad \dot{x}^i = v^i(x). \tag{1}$$

The field v generates the dynamics (time evolution) via its flow $\Phi_t \leftrightarrow v$. We will call the structure *phase space*

$$(M, \Phi_t \leftrightarrow v) \qquad\qquad \textit{phase space}. \tag{2}$$

In this situation, let us have a k-form α and consider its integrals over various k-chains (k-dimensional surfaces) c on M. Due to the flow Φ_t corresponding to v, the k-chains flow away, $c \mapsto \Phi_t(c)$. Compare the value of the integral of α over the original c and integral over $\Phi_t(c)$. If, *for any chain* c, the two integrals are equal, it reflects a remarkable property of the form α with respect to the field v. We call it (absolute) integral invariant:

$$\int_{\Phi_t(c)} \alpha = \int_c \alpha \qquad \Leftrightarrow \qquad \int_c \alpha \quad \text{is } \textit{integral invariant}. \tag{3}$$

For *infinitesimal* $t \equiv \epsilon$ we have

$$\int_{\Phi_\epsilon(c)} \alpha = \int_c \alpha + \epsilon \int_c \mathcal{L}_v \alpha \tag{4}$$

(here $\mathcal{L}_v$ is the *Lie derivative* along v). If (3) is to be true *for each* c, we get from (4)

$$\mathcal{L}_v \alpha = 0. \tag{5}$$

Sometimes, however, it may be enough that the integral only behaves invariantly when restricted to *k-cycles* (i.e., chains whose boundary vanish, $\partial c = 0$). We speak of *relative* integral invariants. Then the condition (5) can be weakened to

$$\mathcal{L}_v \alpha = d\tilde{\beta} \tag{6}$$

for some $\tilde{\beta}$. (So, α is to be *Lie-invariant modulo exact* form.) Using Cartan's formula $i_v d + d i_v = \mathcal{L}_v$, the condition (6) may also be rewritten as

$$i_v d\alpha = d\beta. \tag{7}$$

Therefore, the *main statement* on relative (Poincaré) invariants reads:

$$\boxed{i_v d\alpha = d\beta} \qquad \Leftrightarrow \qquad \oint_c \alpha = \quad \textit{relative} \text{ invariant w.r.t. } \Phi_t \leftrightarrow v. \tag{8}$$

2.1. Stationary Euler equation

The *Stationary Euler equation* for the ideal (inviscid) fluid reads (see, e.g., Ref. [5])

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p/\rho - \boldsymbol{\nabla}\Phi. \tag{9}$$

Here the mass density ρ, the velocity field $\mathbf{v}$, the pressure p and the potential Φ of the volume force field (gz for the usual gravitational field) are functions of $\mathbf{r}$.

It turns out (see Ref. [4]) that for *barotropic* fluid (when $\boldsymbol{\nabla} p/\rho = \boldsymbol{\nabla} P$, where P is the *enthalpy* (heat function) *per unit mass*) it may be rewritten in the form of Eq. (7) with a particular choice of α and β:

$$\boxed{i_v d\tilde{v} = -d\mathcal{B}} \qquad \textit{Euler equation} \tag{10}$$

where

$$\tilde{v} := \mathbf{v} \cdot d\mathbf{r} \qquad (\equiv g(v, \,\cdot\,) \;\equiv \flat_g v) \tag{11}$$

is the *velocity* 1-*form* standardly associated with the velocity vector field $v = v^i \partial_i$ in terms of "lowering of index" ($\equiv \flat_g$ procedure) and

$$\mathcal{B} := v^2/2 + P + \Phi \qquad \textit{Bernoulli function.} \tag{12}$$

2.2. Vortex lines equation

Vortex lines, $\gamma(\lambda) \leftrightarrow \mathbf{r}(\lambda)$, are field lines of the *vorticity vector field* $\boldsymbol{\omega}$, which is the curl of the *velocity* field $\mathbf{v}$. So, they satisfy $\boldsymbol{\omega} \times \mathbf{r}' = 0$ (the prime symbolizes *tangent* vector).

Now we have (see the machinery explained in § 8.5 of Ref. [6])

$$\tilde{v} = \mathbf{v} \cdot d\mathbf{r} \tag{13}$$

$$d\tilde{v} = (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S} \;\equiv\; \boldsymbol{\omega} \cdot d\mathbf{S} \tag{14}$$

$$i_{\gamma'} d\tilde{v} = (\boldsymbol{\omega} \times \mathbf{r}') \cdot d\mathbf{r} \tag{15}$$

The *vorticity* 2-*form* $d\tilde{v}$, present in Eq. (10), is of crucial importance. It encodes complete information about the vorticity vector field $\boldsymbol{\omega}$ and, as we see from (15),

$$\boxed{i_{\gamma'} d\tilde{v} = 0} \qquad \textit{vortex line equation} \tag{16}$$

expresses the fact that $\gamma(\lambda)$ is a vortex line.

2.3. Why the form of Eq. (10) is so convenient

For several reasons:

1. Application of i_v on both sides gives

$$v\mathcal{B} = 0 \qquad \textit{Bernoulli equation} \tag{17}$$

   (saying that $\mathcal{B}$ is constant along *stream*lines).
2. Application of $i_{\gamma'}$ on both sides (where γ' is from (16)) gives

$$\gamma'\mathcal{B} = 0 \tag{18}$$

   (saying that $\mathcal{B}$ is constant along *vortex*-lines).
3. Setting $d\tilde{v} = 0$ (when the flow is *irrotational*) leads to

$$\mathcal{B} = \text{const.} \tag{19}$$

   (a version of Bernoulli equation; $\mathcal{B}$ is, then, constant *in bulk* of the fluid).
4. Just looking at (8), (10) and (11) we get

$$\oint_c \mathbf{v} \cdot d\mathbf{r} = \text{const.} \qquad \textit{Kelvin's theorem} \tag{20}$$

   (velocity circulation is conserved *w.r.t. the flow*).
5. Just looking at (8), (16) and using the Stokes theorem gives

$$\int_S \boldsymbol{\omega} \cdot d\mathbf{S} = \text{const.} \qquad \textit{Helmholtz theorem} \tag{21}$$

   (the *strength* of the vortex tube is constant *along the tube*).
6. Application of d on both sides gives very quickly... see Section 2.4.

2.4. Helmholtz theorem on frozen vortex lines – stationary case

Application of d on both sides of (10) results in

$$\mathcal{L}_v(d\tilde{v}) = 0, \qquad \text{i.e.,} \qquad \Phi_t^*(d\tilde{v}) = d\tilde{v} \qquad \Phi_t \leftrightarrow v. \tag{22}$$

So, the vorticity 2-form $d\tilde{v}$ is *invariant* w.r.t. the flow of the fluid.

Let us define a *distribution* $\mathcal{D}$ in terms of $d\tilde{v}$:

$$\mathcal{D} := \{\text{vectors } w \text{ such that } \ i_w d\tilde{v} = 0 \ \text{ holds}\}. \tag{23}$$

Due to the Frobenius criterion the distribution is integrable (see Refs. [4], [6]). From (15) and (16) we see that the distribution is one-dimensional (at those points where $\boldsymbol{\omega} \neq 0$) and that its integral surfaces coincide with vortex lines. Since the distribution $\mathcal{D}$ is invariant w.r.t. $\Phi_t \leftrightarrow v$, its integral surfaces (i.e., vortex lines) are invariant w.r.t. $\Phi_t \leftrightarrow v$, too. But this means that (another) *Helmholtz* theorem is true: vortex lines *move with the fluid* (are *frozen into* the fluid; see Refs.[7–9]).

3. Cartan integral invariants

Cartan proposed, as a first step, to study the dynamics given in (1) and (2) on $M \times \mathbb{R}$ (the *extended* phase space; the *time* coordinate is added) rather than on M. Using the natural projection

$$\pi : M \times \mathbb{R} \to M \quad (m, t) \mapsto m \quad (x^i, t) \mapsto x^i \tag{24}$$

the forms α and β (from the Poincaré theory) may be pulled-back from M onto $M \times \mathbb{R}$ and then combined into a single k-form

$$\sigma = \hat{\alpha} + dt \wedge \hat{\beta}. \tag{25}$$

(Here, we denote $\hat{\alpha} = \pi^*\alpha$ and $\hat{\beta} = \pi^*\beta$.) In a similar way, define a vector field

$$\xi = \partial_t + v. \tag{26}$$

Its flow clearly consists of the flow $\Phi_t \leftrightarrow v$ on the M factor combined with the trivial lapsing of time in the $\mathbb{R}$ factor (so, it is "the same flow"). A simple check (see Ref. [4]) reveals that the equation

$$i_\xi d\sigma = 0 \tag{27}$$

is equivalent to (7). And the main statement (8) takes the form

$$\boxed{i_\xi d\sigma = 0} \qquad \Leftrightarrow \qquad \oint_c \sigma = \quad \textit{relative} \text{ invariant.} \tag{28}$$

The first *new* result by Cartan (w.r.t. Poincaré) is the following observation: Take *any* two cycles in $M \times \mathbb{R}$ which encircle the common *tube of solutions* (here "solutions" mean integral curves of ξ, i.e., solutions of the dynamics as seen from $M \times \mathbb{R}$). Then, *still*, integrals of σ over c_1 and c_2 give the same number (a simple proof see in Ref. [4]).

The further Cartan generalization is stronger and much more interesting for us. Namely, (25) might also be regarded as a decomposition of the *most general*

k-form σ on $M\times\mathbb{R}$. In this case, $\hat{\alpha}$ and $\hat{\beta}$ need not be obtained by the pull-back from M. Rather, they are the most general *spatial* forms on $M\times\mathbb{R}$. In comparison with just pull-backs, they may be *time-dependent*, i.e., it *may happen* that $\mathcal{L}_{\partial_t}\hat{\alpha}\neq 0$ and/or $\mathcal{L}_{\partial_t}\hat{\beta}\neq 0$. (In coordinate presentation, their *components* may depend on time.)

It turns out that the proof of (28) does not use any details of the decomposition. The *structure* of the equation (27) is all one needs. Notice, however, that the equivalence of (27) and (7) is no longer true, now. Instead, one can check that

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{\alpha} + i_v\hat{d}\hat{\alpha} = \hat{d}\hat{\beta} \tag{29}$$

(the term $\mathcal{L}_{\partial_t}\hat{\alpha}$ is new). Here $\hat{d}$ denotes the *spatial* exterior derivative. (In coordinate presentation – as if the variable t in components was *constant*.) So, the equation

$$\boxed{\mathcal{L}_{\partial_t}\hat{\alpha} + i_v\hat{d}\hat{\alpha} = \hat{d}\hat{\beta}} \tag{30}$$

is *the* equation that *time-dependent* forms $\hat{\alpha}$ and $\hat{\beta}$ are to satisfy in order that the integral of σ is to be relative integral invariant (in the new, more general, sense of encircling the common tube of solutions).

3.1. Non-stationary Euler equation

Retell Cartan's results in the context of hydrodynamics, i.e., for

$$\sigma = \hat{v} - \mathcal{B}dt \tag{31}$$

where, in usual coordinates $(\mathbf{r}, t)$ on $E^3\times\mathbb{R}$,

$$\hat{v} := \mathbf{v}\cdot d\mathbf{r} \equiv \mathbf{v}(\mathbf{r}, t)\cdot d\mathbf{r} \tag{32}$$

From (29) we get

$$i_\xi d\sigma = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_{\partial_t}\hat{v} + i_v\hat{d}\hat{v} = -\hat{d}\mathcal{B}. \tag{33}$$

One easily checks that the r.h.s. of (33) is nothing but the *complete, time-dependent*, Euler equation. Therefore the time-dependent Euler equation may also be written in remarkably succinct form

$$\boxed{i_\xi d\sigma = 0} \qquad \textit{Euler equation.} \tag{34}$$

Just looking at (28), (34), (31) and (32) shows that *Kelvin's theorem* is still true (the two loops c_1 and c_2 are usually in constant-time hyper-planes $t = t_1$ and $t = t_2$, so that the $\mathcal{B}dt$ term does not contribute).

3.2. Helmholtz theorem on frozen vortex lines – non-stationary case

Application of d on (34) results in

$$\mathcal{L}_\xi(d\sigma) = 0, \qquad \text{i.e.,} \qquad \Phi^*_\tau(d\sigma) = d\sigma \qquad \Phi_\tau \leftrightarrow \xi \tag{35}$$

So, $d\sigma$ is *invariant* w.r.t. the flow of the fluid.

Define the distribution $\mathcal{D}$ in terms of annihilation of as many as *two* exact forms:

$$\mathcal{D} \quad \leftrightarrow \quad i_w d\sigma = 0 = i_w dt. \tag{36}$$

The new distribution $\mathcal{D}$ is *integrable* as well. It is, however, also *invariant* w.r.t. the flow of the fluid. (Because of (35) and the trivial fact that $\mathcal{L}_\xi(dt) = 0$.) So, integral submanifolds (surfaces) *move with the fluid*.

What do they look like? Although it is not visible at first sight, they are nothing but vortex lines (see Ref. [10] or, in more detail, Ref. [4]). So, the *Helmholtz* theorem is also true in the non-stationary case: vortex lines *move with the fluid* (are *frozen into* the fluid).

Acknowledgment

This research was supported by VEGA project V-16-087-00.

References

[1] H. Poincaré, *Les Méthodes nouvelles de la Mécanique Céleste,* (*III, Invariants intégraux*). Gauthier-Villars et fils, 1899.

[2] E. Cartan, *Leçons sur les invariants intégraux*. Hermann, 1922.

[3] V. Arnold, *Mathematical Methods of Classical Mechanics*. Springer-Verlag, 1989.

[4] M. Fecko, *Modern geometry in not-so-high echelons of physics: Case studies*. Acta Physica Slovaca **63**, No.5 (2013), 261–359. (free from http://www.physics.sk/aps/; also arXiv:1406.0078 [physics.flu-dyn])

[5] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*. Pergamon Press, Second Edition, 1987.

[6] M. Fecko, *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2006.

[7] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*. Journal für die reine und angewandte Mathematik, 55 (1858), 25–55.

[8] J.Z. Wu, H.Y. Ma, M.D. Zhou, *Vorticity and Vortex Dynamics*. Springer-Verlag Berlin – Heidelberg, 2006.

[9] K.S. Thorne, R.D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*. Princeton University Press, 2017.

[10] M. Fecko, *Surfaces which behave like vortex lines*. XXXVI Workshop on Geometric Methods in Physics, Białowieza 2017, 223–228, (this volume).

Marián Fecko
Department of Theoretical Physics and Didactics of Physics
Comenius University in Bratislava
Mlynská dolina F2
SK-84248 Bratislava, Slovakia
e-mail: fecko@fmph.uniba.sk

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 383–388

Invitation to Hilbert C^*-modules and Morita–Rieffel Equivalence

Bartosz Kosma Kwaśniewski

Mathematics Subject Classification (2010). Primary 46L08; Secondary 46L55.

Keywords. Hilbert C^*-module, C^*-correspondence, Morita–Rieffel equivalence.

1. Introduction

Hilbert C^*-modules play a fundamental role in modern theory of operator algebras and related fields. From the present perspective, one could distinguish the following main areas of application, which were initiated respectively by Rieffel (1973), Kasparov (1981), Woronowicz (1991) and Pimsner (1997): (I) Induced representations and Morita equivalence; (II) KK-theory; (III) C^*-algebraic quantum groups; and (IV) Universal C^*-algebras.

Topics (I)–(III) are well established and thoroughly discussed in monographs: see [5] for (I), [2] for (II) and [3] for (III). The present notes form an extended abstract from a series of lectures, whose main aim was to introduce elements of the theory of Hilbert C^*-modules in a form suitable for further studies on modern approach to noncommutative dynamics and universal C^*-algebras (IV).

2. Hilbert C^*-modules and adjointable maps

Hilbert C^*-modules over commutative C^*-algebras appeared first in the work of Kaplansky (1953). The rudiments of the theory for general C^*-algebras were elaborated in the PhD thesis of Paschke (1972). The idea behind the notion is simple: "generalize Hilbert spaces by replacing complex numbers with a general C^*-algebra".

Namely, let A be a C^*-algebra. A (right) *pre-Hilbert A-module* is a (right) A-module X equipped with a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$ such that:

(1) $\langle x, ya + zb \rangle_A = \langle x, y \rangle_A \, a + \langle x, z \rangle_A \, b$ for any $x, y, z \in X$ and $a, b \in A$;
(2) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$ for any $x, y \in X$;
(3) $\langle x, x \rangle_A \geq 0$ for any $x \in X$ (positivity in A);
(4) $\langle x, x \rangle_A = 0$ implies $x = 0$ for any $x \in X$.

The map $\langle x, y\rangle_A$ is called an *A-valued inner-product*. Generalizing standard arguments one can show that defining

$$\|x\| := \sqrt{\|\langle x, x\rangle_A\|}, \qquad x \in X,$$

the function $d(x, y) = \|x - y\|$ is a metric on X. We say that X is a (right) *Hilbert A-module* if it is complete with respect to d.[1] Then using an approximate unit $\{\mu_\lambda\}$ in A one can show that the formula $\lambda x := \lim_\lambda x(\lambda\mu_\lambda)$, $\lambda \in \mathbb{C}$, $x \in X$, defines scalar multiplication on X. In this way X becomes a *complex Banach space* and $\langle\cdot,\cdot\rangle_A : X \times X \to A$ a *sesqui-linear form.*

Example (Hilbert spaces). Hilbert $\mathbb{C}$-modules are Hilbert spaces.

Example (C^*-algebras). A C^*-algebra A may be treated as a Hilbert A-module equipped with the following natural right multiplication and A-valued inner product:

$$x \cdot a := xa, \qquad \langle x, y\rangle_A := x^* y, \qquad \text{for } x, y, a \in A.$$

Hilbert A-submodules of A correspond to closed right ideals in A.

Example (Concrete Hilbert A-modules). Let H be a Hilbert space. Let $A \subseteq B(H)$ be a C^*-subalgebra and $X \subseteq B(H)$ a closed subspace such that $XA \subseteq X$ and $X^*X \subseteq A$. Then X with operations inherited from $B(H)$ is a Hilbert A-module. Every Hilbert A-module can be represented in this form.

Example (Hilbert $C(M)$-modules = Vector bundles). Let $H = (\{H_t\}_{t\in M}, \Gamma(H))$ be a continuous field of Hilbert spaces over a compact Hausdorff space M (i.e., $\{H_t\}_{t\in M}$ is a family of Hilbert spaces, $\Gamma(H)$ is a linear subspace of sections $M \ni t \mapsto x(t) \in H_t$ such that $M \ni t \mapsto \|x(t)\|$ is continuous, elements of $\Gamma(H)$ exhaust each space H_t, and $\Gamma(H)$ is maximal with these properties). Then $\Gamma(H)$ is a (right) Hilbert $C(M)$-module with the module action and a $C(M)$-valued sesqui-linear form given by:

$$(xa)(t) := a(t)x(t), \qquad \langle x, y\rangle_{C(M)}(t) := \langle x(t), y(t)\rangle,$$

$x \in \Gamma(H),\ a \in C(M),\ t \in M$. Every Hilbert $C(M)$-module is of the form described above.

Let X and Y be Hilbert A-modules. We say that a map $T : X \to Y$ is an *adjointable operator* if there exists a map $T^* : Y \to X$ such that

$$\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A, \qquad \text{for all } x \in X,\ y \in Y.$$

It follows then that both T and T^* are bounded $\mathbb{C}$-linear and A-linear operators. Moreover, T determines uniquely T^* and vice versa. In general, not every bounded (or even isometric) A-linear map is adjointable even when A is commutative. The set $\mathcal{L}(X, Y)$ of all adjointable operators from X to Y is a Banach space with respect to the operator norm. The space $\mathcal{L}(X) := \mathcal{L}(X, X)$ is a unital C^*-algebra

[1] Analogously one defines left Hilbert modules, as a left A-module equipped with an A-valued inner product which is A-linear with respect to the first variable.

with involution given by adjoint of an adjointable operator. For each $x \in X$, $y \in Y$, the map $\Theta_{x,y} : Y \to X$ defined by

$$\Theta_{x,y}(z) = x\langle y, z\rangle_A$$

is an adjointable operator with $\Theta^*_{x,y} = \Theta_{y,x}$. The elements of $\mathcal{K}(Y,X) := \overline{\mathrm{span}}\{\Theta_{x,y} : x \in X, Y \in Y\} \subseteq \mathcal{L}(Y,X)$ are called (generalized) *compact operators* from Y to X. The set $\mathcal{K}(Y,X)$ is a Banach space and $\mathcal{K}(X) := \mathcal{K}(X,X)$ is an ideal in $\mathcal{L}(X)$.

Example (Hilbert spaces). If $A = \mathbb{C}$, then X and Y are Hilbert spaces and $\mathcal{L}(X,Y) = B(X,Y)$ are bounded operators and $\mathcal{K}(X,Y) = K(X,Y)$ are usual compact operators.

Example (C^*-algebras). If we treat a C^*-algebra A as a Hilbert A-module, then $\mathcal{K}(A) \cong A$ where $\Theta_{x,y} \mapsto xy^*$, $x, y \in A$. In particular, if $A = B(H)$ then $\mathcal{K}(A) \cong B(H)$. This shows that, in general, compact operators in the sense of Hilbert modules are not compact as operators between Banach spaces.

Example (Multiplier C^*-algebras). The multiplier algebra $M(A)$ of a C^*-algebra A is as a maximal essential unitization of A. For any Hilbert C^*-module X we have $M(\mathcal{K}(X)) \cong \mathcal{L}(X)$. In particular, $\mathcal{L}(A) \cong M(A)$.

3. C^*-correspondences

Let A, B be C^*-algebras. A *C^*-correspondence from A to B* is a (right) Hilbert B-module X equipped with a homomorphism $\phi_X : A \to \mathcal{L}(X)$ – left action of A on X. We write $b \cdot x := \phi_X(b)x$. We will treat C^*-correspondences as "generalized morphisms" between C^*-algebras. In particular, to denote that X is a C^*-correspondence from A to B we write $A \xrightarrow{X} B$. We also say that X is *non-degenerate* if $\phi_X(A)X = X$.

Example (Representations). Representations $\pi : A \to B(H)$ of a C^*-algebra A may be identified with C^*-correspondences $A \xrightarrow{H_\pi} \mathbb{C}$ from A to $\mathbb{C}$.

Example (Homomorphisms). If $\alpha : A \to B$ is a $*$-homomorphism we may treat it as a non-degenerate C^*-correspondence $A \xrightarrow{X_\alpha} B$ where $X_\alpha := \alpha(A)B$ is equipped with operations $a \cdot x := \alpha(a)x$, $x \cdot b := xb$, $\langle x, y\rangle_B := x^*y$ for all $x, y \in X_\alpha$, $a \in A$, $b \in B$.

Example (Concrete C^*-correspondences). Let $X \subseteq B(H)$ be a closed linear space and $A, B \subseteq B(H)$ be C^*-subalgebras such that $XB \subseteq X$, $X^*X \subseteq B$, $AX \subseteq X$. Then X is naturally a C^*-correspondence from A to B. Every C^*-correspondence can be represented in this form.

Example (C^*-correspondences vs. graphs). Let V, W be sets (spaces with discrete topology). Let $G = (E, s, r)$ be a graph from V to W, i.e., E is a set of edges and $s : E \to V$ and $r : E \to W$ are source and range maps. We define C^*-

correspondence X_G from $A = C_0(W)$ to $B := C_0(V)$ by putting $X_G := \{x \in C_0(E) : V \ni v \longmapsto \sum_{e\in s^{-1}(v)} |x(e)|^2 \in \mathbb{C}$ is in $C_0(V)\}$, and

$$\langle x, y\rangle_A(v) := \sum_{e\in s^{-1}(v)} \overline{x(e)}y(e),$$

$$(a \cdot x)(e) := a(r(e))x(e), \qquad (x \cdot b)(e) := x(e)b(s(e)).$$

Every C^*-correspondence from $C_0(W)$ to $C_0(V)$ is of this form.

If $A \xrightarrow{X} B$ and $B \xrightarrow{Y} C$ are C^*-correspondences then there is a C^*-correspondence $A \xrightarrow{X\otimes_B Y} C$ defined as follows. The space $X \otimes_B Y = \overline{\operatorname{span}}\{x \otimes y : x \in X, y \in Y\}$ is the Hausdorff completion of the algebraic tensor product of X and Y with respect to the seminorm defined by the C-valued sesqui-linear form given by the formula

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_C := \langle y_1, \langle x_1, x_2\rangle_B \cdot y_2\rangle_C.$$

The left and right action on $X \otimes_B Y$ is defined in an obvious way: $a\cdot(x\otimes y)\cdot c := (a\cdot x)\otimes(y\cdot c)$ for $x \in X, y \in Y$, $a \in A$, $c \in C$. The C^*-correspondence $X \otimes_B Y$ is usually called the (*inner*) *tensor product* of X and Y. We encourage to think of it as a "composition" of C^*-correspondences X and Y.

Example (Induced representations). If $A \xrightarrow{X} B$ is a C^*-correspondence and $B \xrightarrow{H_\pi} \mathbb{C}$ is a representation of B, then $A \xrightarrow{X\otimes_B H_\pi} \mathbb{C}$ is a representation of A.

The latter representation is called the induced representation from π by X.

Example (Composition of homomorphisms). If $\alpha : A \to B$ and $\beta : B \to C$ are $*$-homomorphisms, then the C^*-correspondence $X_\alpha \otimes_B X_\beta$ is naturally isomorphic to the C^*-correspondence $X_{\beta\circ\alpha}$ associated to the $*$-homomorphism $\beta\circ\alpha : A \to C$.

Example (Concrete tensor products). Let $A, B, C \subseteq B(H)$ and $X, Y \subseteq B(H)$ be concrete C^*-correspondences $A \xrightarrow{X} B$ and $B \xrightarrow{Y} C$. Then $\overline{XY} = \overline{\operatorname{span}}\{xy : x \in X, y \in Y\} \subseteq B(H)$ is a concrete C^*-correspondence $A \xrightarrow{\overline{XY}} C$ which is naturally isomorphic to the C^*-correspondence $X \otimes_B Y$.

Example (Composition of graphs). Let $G = (E, s, r)$ a graph from V to W and $H = (F, s, r)$ a graph from W to U. We define the composite graph $H \circ G := (F \circ E, s, r)$, where $F \circ E := \{(f, e) \in F \times E : s(f) = r(e)\}$, $s(f, e) := s(e)$ and $r(f, e) = r(f)$. Then we have a natural isomorphism of C^*-correspondences $X_H \otimes_B X_G \cong X_{H\circ G}$.

Let us consider a "category" whose objects are C^*-algebras and morphisms are non-degenerate C^*-correspondences. Strictly speaking such a structure is not a category, but a bicategory because the associativity holds only up to a natural isomorphism. More specifically, if $A \xrightarrow{X} B$, $B \xrightarrow{Y} C$ and $C \xrightarrow{Z} D$ are C^*-categories, we have a natural isomorphism:

$$X \otimes_B (Y \otimes_C Z) \cong (X \otimes_B Y) \otimes_C Z.$$

C^*-algebras treated as Hilbert modules act as "identity morphisms": we have $X \otimes_B B \cong X$ and $(A \otimes_A X) \cong X$ (here is where we use non-degeneracy of X). In particular, a C^*-correspondence $A \xrightarrow{X} B$ is "invertible" if there is a C^*-correspondence $B \xrightarrow{X^\star} A$ such that

$$X^\star \otimes_A X \cong B, \qquad X \otimes_B X^\star \cong A.$$

A C^*-correspondence is "invertible" in the above sense if and only if it is a Morita–Rieffel equivalence bimodule – an object that we describe below.

4. The Morita–Rieffel equivalence

Let A, B be C^*-algebras. A *Hilbert A-B-bimodule* is a space X which is both a right Hilbert B-module and a left Hilbert A-module such that the respective inner products satisfy

$$_A\langle x, y\rangle z = x\langle y, z\rangle_B, \qquad x, y, z \in X.$$

Then $\langle X, X\rangle_B := \overline{\operatorname{span}}\{\langle x, y\rangle_B : x, y \in X\}$ is an ideal in B and $_A\langle X, X\rangle := \overline{\operatorname{span}}\{_A\langle x, y\rangle : x, y \in X\}$ is an ideal in A. We say that X is a (Morita–Rieffel) *equivalence bimodule* if in addition $\langle X, X\rangle_B = B$ and $_A\langle X, X\rangle = A$. If X is a Hilbert A-B-bimodule, and $X^\star$ is the adjoint Hilbert B-A-bimodule[2], then $X \otimes_B X^\star \cong \langle X, X\rangle_B$ and $(X^\star \otimes_A X) \cong {}_A\langle X, X\rangle$. Thus X is an equivalence bimodule if and only if it is "invertible". Two C^*-algebras A and B are *Morita equivalent* if there exists an equivalence Hilbert A-B-bimodule.

Remark. Every Hilbert A-B-bimodule X restricts to an equivalence $_A\langle X, X\rangle$-$\langle X, X\rangle_B$-bimodule. Every Hilbert A-B-bimodule X is a C^*-correspondence from A to B (the left action of A on X is necessarily given by adjointable operators). A C^*-correspondence $A \xrightarrow{X} B$ is a Hilbert A-B-bimodule if and only if the left action ϕ_X restricts to an isomorphism from an ideal J in A onto $\mathcal{K}(X)$ (then we necessarily have $\langle x, y\rangle_B = \phi_X|_J^{-1}(\Theta_{x,y})$).

Example (Compact operators). Every right Hilbert B-module is an equivalence $\mathcal{K}(X)$-$\langle X, X\rangle_B$-bimodule where $_{\mathcal{K}(X)}\langle x, y\rangle := \Theta_{x,y}$, $x, y \in X$. In particular, every Hilbert space H gives Morita equivalence between $\mathbb{C}$ and $K(H)$.

Example (Hereditary subalgebras and ideals). Let p be an element of a C^*-algebra C. The right ideal $X := pC$ is an equivalence bimodule establishing Morita equivalence between $A := pCp$ and $B := CpC$.

Example (Ternary rings of operators). A closed linear space $X \subseteq B(H)$ satisfying $XX^*X \subseteq X$ is called a (concrete) *ternary ring of operators.* Any such X is an equivalence bimodule from $A := \overline{XX^*}$ to $B := \overline{X^*X}$. Every equivalence A-B-bimodule can be represented in this form.

[2]It arises by exchanging the left and right structures.

Suppose that C^*-algebras A and B are embedded as corners into a C^*-algebra C, i.e., we have the decomposition $C = \left(\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right)$. Then the space X with operations inherited from C is a Hilbert A-B-bimodule. It is an equivalence bimodule if and only if A and B are full C^*-subalgebras of C (i.e., we have $CAC = C$ and $CBC = C$). In fact, any two C^*-algebras A and B are Morita equivalent if and only if they can be embedded into a C^*-algebra C as full and complementary corners, see [1]. The celebrated theorem of Brown, Green and Rieffel [1] states the following:

Theorem 1. *If A and B have countable approximate units then A and B are Morita equivalent if and only if A and B are stably isomorphic, i.e., $A \otimes K(H) \cong B \otimes K(H)$ where H is a Hilbert space.*

Morita equivalent C^*-algebras A and B share a vast list of properties, cf. [5]. For instance, they have: isomorphic lattices of ideals $\mathrm{Ideal}(A) \cong \mathrm{Ideal}(B)$; homeomorphic spectra $\widehat{A} \cong \widehat{B}$ (equivalence classes of irreducible representations equipped with Jacobson topology); isomorphic K-groups $K_i(A) \cong K_i(B)$, $i = 0, 1$. Moreover, A is nuclear (resp., liminal or postliminal) if and only if B is nuclear (resp., liminal or postliminal).

Comments on actions of C^*-correspondences: Group actions of Hilbert bimodules on C^*-algebras correspond to Fell bundles over groups. They generalize group actions by automorphisms and the associated crossed products model all group graded C^*-algebras. Inverse semigroup actions by Hilbert bimodules can be viewed as noncommutative groupoids. They model all regular C^*-inclusions and in particular noncommutative Cartan pairs. Semigroup actions of C^*-correspondences correspond to product systems. They model various variants and generalizations of Cuntz-Pimsner algebras [4].

References

[1] L.G. Brown, P. Green, M. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras*, Pacific J. Math., 71 (1977), 349–363.

[2] K.K. Jensen, K. Thomsen, Elements of KK-theory, Birkhäuser, 1991.

[3] E.C. Lance, Hilbert C^*-Modules: A Toolkit for Operator Algebraists. Cambridge University Press, Cambridge (1995).

[4] M.V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* $\mathbf{Z}$, in Free probability theory (Waterloo, ON, 1995), Amer. Math. Soc., Providence, RI, 1997, 189–212.

[5] I. Raeburn, D.P. Williams, Morita equivalence and continuous-trace C^*-algebras, Math. Surveys and Monographs, vol. 60, Amer. Math. Soc., Providence, RI, 1998.

Bartosz Kosma Kwaśniewski
Institute of Mathematics, University of Białystok
ul. K. Ciołkowskiego 1M, 15-245 Białystok
e-mail: bartoszk@math.uwb.edu.pl

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 389–401

After Plancherel Formula

Yury Neretin

Abstract. We discuss two topics related to Fourier transforms on Lie groups and on homogeneous spaces: the operational calculus and the Gelfand–Gindikin problem (program) about separation of non-uniform spectra. Our purpose is to indicate some non-solved problems of noncommutative harmonic analysis that definitely are solvable.

Mathematics Subject Classification (2010). 22E30, 22E46, 43A30, 43A80, 43A85.

Keywords. Unitary representations, Plancherel formula, differential-difference operators, Fourier transform on groups, operational calculus.

1. Abstract Plancherel theorem for groups

See, e.g., [2]. Let G be a type I locally compact group with a two-side invariant Haar measure dg. Denote by $\widehat{G}$ the set of all irreducible unitary representations of G (defined up to a unitary equivalence[1]). For $\rho \in \widehat{G}$ denote by H_ρ the space of the representation ρ. For $\rho \in \widehat{G}$ and $f \in L^1(G)$ we define the following operator in H_ρ:

$$\rho(f) := \int_G f(g)\, \rho(g)\, dg.$$

This determines a representation of the convolution algebra $L^1(G)$ in H_ρ,

$$\rho(f_1)\rho(f_2) = \rho(f_1 * f_2).$$

Consider a Borel measure ν on $\widehat{G}$ and the direct integral of Hilbert spaces H_ρ with respect to the measure ν. Consider the space $\mathcal{L}(\widehat{G}, \nu)$ of measurable functions Φ

Supported by the grant FWF, P28421.

[1] For a formal definition of type I groups see, e.g., [2, Sect. 7.2]. Connected semisimple Lie groups, connected nilpotent Lie groups, classical p-adic groups have type I. This condition implies a presence of the standard Borel structure on $\widehat{G}$ and a uniqueness of a decomposition of any unitary representation of G into a direct integral of irreducible representations.

on $\widehat{G}$ sending any $\rho \in G$ to a Hilbert–Schmidt operator in H_ρ and satisfying the condition

$$\int_{\widehat{G}} \operatorname{tr}\bigl(\Phi(\rho)^* \Phi(\rho)\bigr)\, d\nu(\rho) < \infty.$$

There exists a unique measure μ *on* $\widehat{G}$ (*the* Plancherel measure), *such that for any* f_1, $f_2 \in L^1 \cap L^2(G)$ *we have*

$$\langle f_1, f_2\rangle_{L^2(G)} = \int_{\widehat{G}} \operatorname{tr}\bigl(\rho(f_2)^* \rho(f_1)\bigr)\, d\mu(\rho)$$

and the map $f \mapsto \rho(f)$ *extends to a unitary operator from* $L^2(G)$ *to the space* $\mathcal{L}^2(\widehat{G}, \mu)$ (F.I. Mautner, I. Segal (1950), see, e.g., [2]).

2. An example. The group $\mathrm{GL}(2, \mathbb{R})$

Let $\mathrm{GL}(2, \mathbb{R})$ be the group of invertible real matrices of order 2. Let $\mu \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}_2$. We define the function $x^{\mu /\!\!/ \varepsilon}$ on $\mathbb{R} \setminus 0$ by

$$x^{\mu /\!\!/ \varepsilon} := |x|^\mu \operatorname{sgn}(x)^\varepsilon.$$

Denote $\Lambda := \mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C} \times \mathbb{Z}_2$. For each element $(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$ of Λ we define a representation $T_{\mu,\varepsilon}$ of $\mathrm{GL}_2(\mathbb{R})$ in the space of functions on $\mathbb{R}$ by

$$T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(t) = \varphi\left(\frac{b + td}{a + tc}\right) \cdot (a + tc)^{-1+\mu_1-\mu_2 /\!\!/ \varepsilon_1 - \varepsilon_2} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{1/2+\mu_2 /\!\!/ \varepsilon_2}.$$

This formula determines the *principal series of representations of* $\mathrm{GL}(2, \mathbb{R})$. If $\mu_1 - \mu_2 \notin \mathbb{Z}$, then representations $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}$ and $T_{\mu_2,\varepsilon_2;\mu_1,\varepsilon_1}$ are irreducible and equivalent (on representations of $\mathrm{SL}(2, \mathbb{R})$, see, e.g., [4, 39]).

If $\mu_1 = i\tau_1$, $\mu_2 = i\tau_2 \in i\mathbb{R}$, then a representation $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}$ is unitary in $L^2(\mathbb{R})$ (they are called representations the *unitary principal series*).

Next, we define *representations of the discrete series*. Let $n = 1, 2, 3, \ldots$ Consider the Hilbert space H_n of holomorphic functions φ on $\mathbb{C} \setminus \mathbb{R}$ satisfying

$$\int_{\mathbb{C}\setminus\mathbb{R}} |\varphi(z)|^2 |\operatorname{Im} z|^{n-1}\, d\operatorname{Re} z\, d\operatorname{Im} z < \infty.$$

In fact, φ is a pair of holomorphic functions determined on half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$. For $\tau \in \mathbb{R}$, $\delta \in \mathbb{Z}_2$ we define the unitary representation $D_{n,\tau,\delta}$ of $\mathrm{GL}_2(\mathbb{R})$ in H_n by

$$D_{n,\tau,\delta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(z) = \varphi\Bigl(\frac{b + zd}{a + zc}\Bigr)(a + zc)^{-1-n} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{1/2+n/2+i\tau /\!\!/ \delta}.$$

There exists also the complementary series of unitary representations, which does not participate in the Plancherel formula.

Remark. The expression for $D_{n,\tau,\delta}$ is contained in the family $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}$, but we change the space of the representations.

The Plancherel measure for $\mathrm{SL}(2,\mathbb{R})$ was explicitly evaluated in 1952 by Harish-Chandra, it is supported by the principal and discrete series. On the principal series the density is given by the formula (see, e.g., [39])

$$d\mathcal{P} = \frac{1}{16\pi^3}(\tau_1 - \tau_2)\tanh \pi(\tau_1 - \tau_2)/2\, d\tau_1\, d\tau_2, \qquad \text{if } \varepsilon_1 - \varepsilon_2 = 0;$$
$$d\mathcal{P} = \frac{1}{16\pi^3}(\tau_1 - \tau_2)\coth \pi(\tau_1 - \tau_2)/2\, d\tau_1\, d\tau_2 \qquad \text{if } \varepsilon_1 - \varepsilon_2 = 1.$$

On nth piece of the discrete series the measure is given by

$$d\mathcal{P} = \frac{n}{8\pi^3} d\tau.$$

3. Homogeneous spaces, etc.

The Plancherel formula for complex classical groups was obtained by I.M. Gelfand and M.A. Naimark [5] in 1948–50, for real semisimple groups by Harish-Chandra in 1965 (see, e.g., [11, 13]), there is also a formula for nilpotent groups (A.A. Kirillov [12], L. Pukanszky [37]).

During 1950–early 2000s there was obtained a big zoo of explicit spectral decompositions of L^2 on homogeneous spaces, of tensor products of unitary representations, of restrictions of unitary representations to subgroups. We present some references, which can be useful for our purposes [1, 5, 9, 11, 16, 23, 27, 38, 41]. Unfortunately, texts about groups of rank > 1 are written for experts and are heavy for exterior readers. See also the paper [29] on some spectral problems (deformations of L^2 on pseudo-Riemannian symmetric spaces), which apparently are solvable but are not solved.

However, a development of the last decades seems strange. The Plancherel formula for Riemannain symmetric spaces [7] (see, e.g., [10]) and Bruhat–Tits buildings [14] had a general mathematical influence (for instance to theory of special functions and to theory of integrable systems). Usually, Plancherel formulas are heavy results (with impressive explicit formulas) without further continuation even inside representation theory and noncommutative harmonic analysis.

4. Operational calculus for $\mathrm{GL}(2,\mathbb{R})$, see [33], 2017

Denote by Gr_4^2 the Grassmannian of all two-dimensional linear subspaces in $\mathbb{R}^4$. The natural action of the group $\mathrm{GL}(4,\mathbb{R})$ in $\mathbb{R}^4$ induces the action on Gr_4^2, therefore we have a unitary representation of the group $\mathrm{GL}(4,\mathbb{R})$ in L^2 on Gr_4^2 (this is an irreducible representation of a degenerate principal series) and the corresponding action of the Lie algebra $\mathfrak{gl}(4)$.

For $g \in \mathrm{GL}(2,\mathbb{R})$ its graph is a linear subspace in $\mathbb{R}^2 \oplus \mathbb{R}^2 = \mathbb{R}^4$. In this way we get an embedding

$$\mathrm{GL}(2,\mathbb{R}) \to \mathrm{Gr}_4^2.$$

The image of the embedding is an open dense subset in Gr_4^2. Thus we have an identification of Hilbert spaces

$$L^2\big(\mathrm{GL}(2,\mathbb{R})\big) \simeq L^2\big(\mathrm{Gr}_4^2\big)$$

(since natural measures on $\mathrm{GL}(2,\mathbb{R})$ and Gr_4^2 are different, we must multiply functions by an appropriate density to obtain a unitary operator). Therefore we get a canonical action of the group $\mathrm{GL}(4,\mathbb{R})$ in $L^2\big(\mathrm{GL}(2,\mathbb{R})\big)$. It is easy to see that the block diagonal subgroup $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}) \subset \mathrm{GL}(4,\mathbb{R})$ acts by left and right shifts on $\mathrm{GL}(2,\mathbb{R})$.

We wish to evaluate the action of the Lie algebra $\mathfrak{gl}(4)$ in the Fourier-image.

Consider the space $C_0^\infty\big(\mathrm{GL}(2,\mathbb{R})\big)$ of smooth compactly supported functions on $\mathrm{GL}(2,\mathbb{R})$. For any $F \in C_0^\infty\big(\mathrm{GL}(2,\mathbb{R})\big)$ consider the operator-valued function $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}(F)$ depending on $(\mu_1,\varepsilon_1;\mu_2,\varepsilon_2) \in \Lambda$. We write these operators in the form

$$T_{\mu_1,\varepsilon_1;\mu_2;\varepsilon_2}(F)\varphi(t) = \int_{-\infty}^{\infty} K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2)\,\varphi(s)\,ds.$$

The kernel K is smooth in t, s and holomorphic in μ_1, μ_2.

On the other hand we have the Hilbert space $\mathcal{L}^2\big(\widehat{\mathrm{GL}(2,\mathbb{R})}, d\mathcal{P}\big)$. The norm in this Hilbert space is given by

$$\begin{aligned} \|K\|^2 = \int \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big|K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2)\big|^2 dt\,ds\,d\mathcal{P}(\mu) + \\ + \Big\{\text{summands corresponding to the discrete series}\Big\}. \end{aligned} \tag{1}$$

We must write the action of the Lie algebra $\mathfrak{gl}(4)$. Denote by e_{kl} the standard generators of $\mathfrak{gl}(4)$ acting in smooth compactly supported functions on $\mathrm{GL}(2,\mathbb{R})$ and by E_{kl} the same generators acting in the space of functions of variables t, s, μ_1, ε_1, μ_2, ε_2. The action of the subalgebra $\mathfrak{gl}(2)\oplus\mathfrak{gl}(2)$ is clear from the definition of the Fourier transform, this Lie algebra acts by first-order differential operators. For instance

$$\begin{aligned} e_{12} &= -b\frac{\partial}{\partial a} - d\frac{\partial}{\partial b}, \qquad & E_{12} &= \frac{\partial}{\partial t};\\ e_{43} &= b\frac{\partial}{\partial a} + d\frac{\partial}{\partial c}, \qquad & E_{43} &= -s^2\frac{\partial}{\partial s} + (-1-\mu_1+\mu_2)s. \end{aligned}$$

Define shift operators V_1^+, V_1^-, V_2^+, V_2^- by

$$V_1^{\pm}K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2) = K(t,s|\mu_1\pm 1,\varepsilon_1+1;\mu_2,\varepsilon_2); \tag{2}$$

$$V_2^{\pm}K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2) = K(t,s|\mu_1,\varepsilon_1;\mu_2\pm 1,\varepsilon_2+1). \tag{3}$$

To be definite, we present formulas for two nontrivial generators e_{kl} and their Fourier images E_{kl}:

$$\begin{aligned}
e_{14} &= \frac{\partial}{\partial b} + \frac{c}{ad-bc},\\
E_{14} &= \frac{-1/2+\mu_1}{\mu_1-\mu_2}\,\frac{\partial}{\partial s}\,V_1^- + \frac{-1/2+\mu_2}{\mu_1-\mu_2}\,\frac{\partial}{\partial t}\,V_2^-,\\
e_{32} &= -\left(ac\frac{\partial}{\partial a} + ad\frac{\partial}{\partial b} + c^2\frac{\partial}{\partial c} + cd\frac{\partial}{\partial d}\right) - c,\\
E_{32} &= \frac{1/2+\mu_1}{\mu_1-\mu_2}\,\frac{\partial}{\partial t}\,V_1^+ + \frac{1/2+\mu_2}{\mu_1-\mu_2}\,\frac{\partial}{\partial s}\,V_2^+.
\end{aligned}$$

There is also a correspondence for operators of multiplication by functions. For instance, the operator of multiplication by c in $C_0^\infty\big(\mathrm{GL}(2,\mathbb{R})\big)$ corresponds to

$$\frac{1}{\mu_1-\mu_2}\left(\frac{\partial}{\partial t}V_1^+ + \frac{\partial}{\partial s}V_2^+\right)$$

in the Fourier-image. There are similar formulas for multiplications by a, b, d. The operator of multiplication by $(ad-bc)^{-1}$ corresponds to $V_1^- V_2^-$ (the last statement is trivial). The operator $\frac{\partial}{\partial b}$ corresponds to

$$\frac{\mu_1}{\mu_1-\mu_2}\,\frac{\partial}{\partial s}\,V_1^- + \frac{\mu_2}{\mu_1-\mu_2}\,\frac{\partial}{\partial t}\,V_2^-,$$

There are similar formulas for other partial derivatives.

We emphasize that **our formulas contain shifts in imaginary directions** (the shifts in (2)–(3) are transversal to the contour of integration in (1)).

5. Difference operators in imaginary direction and classical integral transforms

The operators iE_{kl} are symmetric in the sense of the spectral theory. The question about domains of self-adjointness is open.

There exist elements of spectral theory of self-adjoint difference operators in $L^2(\mathbb{R})$ of the type

$$Lf(s) = a(s)f(s+i) + b(s)f(s) + c(s)f(s-i), \qquad i^2 = -1, \tag{4}$$

see [8, 30]. Recall that several systems of classical hypergeometric orthogonal polynomials (Meixner–Polaszek, continuous Hahn, continuous dual Hahn, Wilson) are eigenfunctions of operators of this type. In the polynomial cases the problems are algebraic. The simplest nontrivial analytic example is the operator

$$Mf(s) = \frac{1}{is}\big(f(s+i) - f(s-i)\big)$$

in $L^2\big(\mathbb{R}_+, |\Gamma(is)|^{-2}ds\big)$. We define M on the space of functions f holomorphic in a strip $|\operatorname{Im} s| < 1+\delta$ and satisfying the condition

$$|f(s)| \leqslant \exp\{-\pi|\operatorname{Re} s|\}|\operatorname{Re} s|^{-3/2-\varepsilon}$$

in this strip. The spectral decomposition of M is given by the inverse Kontorovich–Lebedev integral transform. Recall that the direct Kontorovich–Lebedev transform

$$\mathcal{K}f(s) = \int_0^\infty K_{is}(x) f(x) \frac{dx}{x},$$

where K_{is} is the Macdonald–Bessel function, gives the spectral decomposition of a second-order differential operator, namely

$$D := \left(x\frac{d}{dx}\right)^2 - x^2, \qquad x > 0.$$

The transform $\mathcal{K}$ is a unitary operator $L^2(\mathbb{R}_+, dx/x) \to L^2\big(\mathbb{R}_+, |\Gamma(is)|^{-2}ds\big)$. It sends D to the multiplication by s^2, and $\mathcal{K}^{-1}$ sends M to the multiplication by $2/x$. So we get so-called *bispectral problem.*

Now there is a zoo of explicit spectral decompositions of operators (4). The similar bispectrality appears for some other integral transforms: the index hypergeometric transform (another names of this transform are: the Olevsky transform, the Jacobi transform, the generalized Mehler–Fock transform) [25], the Wimp transform with Whittaker kernel [30], for a continuous analog of expansion in Wilson polynomials proposed by W. Groenevelt [8], etc.

This subject is now a list of examples (which certainly can be extended), but there are no a priori theorems.

6. A general problem about overalgebras

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra. Let $H \subset G$ be a subgroup. Let σ be an irreducible unitary representation of G. Assume that we know an explicit spectral decomposition of restriction of ρ to a subgroup H. To write the action of the overalgebra $\mathfrak{g}$ in the spectral decomposition.

Remarks.

1) Above we have $G = \mathrm{GL}(4,\mathbb{R})$, its representation σ in L^2 on the Grassmannian Gr_4^2, and $H = \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$. The restriction problem is equivalent to the decomposition of regular representation of $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$ in $L^2\big(\mathrm{GL}(2,\mathbb{R})\big)$. The Fourier transform is the spectral decomposition of the regular representation.

2) It is important that similar overgroups exist for all 10 series of classical real Lie groups[2]. Moreover, a decomposition of L^2 on any classical symmetric space[3] G/M can be regarded as a certain restriction problem, see [24].
3) Next, consider a tensor product $\rho_1 \otimes \rho_2$ of two unitary representations of a group G. Then we have the action of $G \times G$ in the tensor product, so the problem of decomposition of tensor products can be regarded as a problem of a restriction from the group $G \times G$ to the diagonal subgroup G.

The question under the discussion was formulated in [30]. Several problems of this kind were solved [18–20, 30, 31, 33]. In all the cases we get differential-difference operators including shifts in imaginary direction. Expressions also include differential operators of high order, even for $\mathrm{SL}(2,\mathbb{R})$-problems we usually get operators of order 2.

Conjecture. *All problems of this kind are solvable* (*if we are able to write a spectral decomposition*).

7. The Gelfand–Gindikin problem, [3], 1977

The set $\widehat{H}$ of unitary representations of a semisimple group H naturally splits into different types (series).

Let H be a semisimple group, M a subgroup. Consider the space $L^2(H/M)$. Usually its H-spectrum contains different series. To write explicitly decomposition of L^2 into pieces with uniform spectrum.

A variant of the problem: let G be a Lie group, $H \subset G$ a semisimple subgroup, ρ is a unitary representation of G. Answer to the same question.

8. Example: separation of series for the one-sheet hyperboloid

Consider the space $\mathbb{R}^3$ equipped with an indefinite inner product

$$\langle u, v\rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Consider the pseudo-orthogonal group preserving the form $\langle\cdot,\cdot\rangle$, denote by $\mathrm{SO}_0(2,1)$ its connected component. Recall that $\mathrm{SO}_0(2,1)$ is isomorphic to the quotient $\mathrm{PSL}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ by the center $\{\pm 1\}$.

Consider a one-sheet hyperboloid H defined by $x_1^2 - x_2^2 - x_3^2 = 1$. It is an $\mathrm{SO}_0(2,1)$-homogeneous space admitting a unique (up to a scalar factor) invariant measure. Decomposition of $L^2(H)$ into irreducible representations of $\mathrm{SO}_0(2,1)$ is well known. The spectrum is a sum of all representations of the discrete series

[2] More precisely, an overgroup $\widetilde{G}$ exists for $G = \mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{H})$, $\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$, $\mathrm{SO}^*(2n)$ (and not for $\mathrm{SL}(n,\cdot)$, $\mathrm{SU}(p,q)$). For instance, for $g \in \mathrm{Sp}(2n,\mathbb{R})$ its graph is a Lagrangian subspace in $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$, this determines a map from $\mathrm{Sp}(2n,\mathbb{R})$ to the Lagrangian Grassmannian with an open dense image. We set $\widetilde{G} := \mathrm{Sp}(4n,\mathbb{R})$.

[3] The groups G, M must be from the list of the previous footnote, M must be a symmetric subgroup in G.

of $\mathrm{PSL}(2,\mathbb{R})$ and the integral over the whole principal series with multiplicity 2. The separation of series was proposed by V.F. Molchanov [15] in 1980 (we use a modification from [22]).

Denote by $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ the Riemann sphere, by $\overline{\mathbb{R}} = \mathbb{R} \cup \infty$ denote the real projective line, $\overline{\mathbb{R}} \subset \overline{\mathbb{C}}$. Consider the diagonal action of $\mathrm{SL}(2,\mathbb{R})$ on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$,

$$(x_1, x_2) \mapsto \left(\frac{b+dx_1}{a+cx_1}, \frac{b+dx_2}{a+cx_2}\right).$$

Consider the subset H' in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ consisting of points x_1, x_2 such that $x_1 \neq x_2$. It is easy to verify that H' is an orbit of $\mathrm{SL}(2,\mathbb{R})$, it is equivalent to the hyperboloid H as a homogeneous space[4]. It is easy to verify that the invariant measure on H' is given by the formula

$$d\nu(x_1, x_2) = |x_1 - x_2|^{-2}\, dx_1\, dx_2.$$

We identify the space $L^2(H', d\nu)$ with the standard $L^2(\mathbb{R} \times \mathbb{R})$ by the *unitary* operator

$$Jf(x_1, x_2) = f(x_1, x_2)(x_1 - x_2)^{-1}.$$

Now our representation in $L^2(H)$ transforms to the following unitary representation in the standard $L^2(\mathbb{R}^2)$:

$$Q\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x_1, x_2) = f\left(\frac{b+dx_1}{a+cx_1}, \frac{b+dx_2}{a+cx_2}\right)(a+cx_1)^{-1}(a+cx_2)^{-1}. \tag{5}$$

Next, consider a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ in $L^2(\mathbb{R})$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = f\left(\frac{b+xd}{a+xc}\right)(a+xc)^{-1}.$$

Obviously, we have $Q = T \otimes T$. The representation T is contained in the unitary principal series and it is a unique reducible element of this series (see, e.g., [4]).

Denote by $\Pi_\pm$ the upper and lower half-planes in $\overline{\mathbb{C}}$. The Hardy space $H^2(\Pi_+)$ consists of functions F_+ holomorphic in Π_+ that can be represented in the form

$$F_+(x) = \int_0^\infty \varphi(t) e^{itx}\, dt, \qquad \text{where } \varphi(t) \in L^2(\mathbb{R}_+).$$

Obviously, F is well defined also on $\mathbb{R}$ and is contained in L^2. The space $H^2(\Pi_-)$ consists of functions F_- holomorphic in Π_- of the form

$$F_-(x) = \int_{-\infty}^0 \varphi(t) e^{itx}\, dt, \qquad \text{where } \varphi(-t) \in L^2(\mathbb{R}_+).$$

Evidently,

$$L^2(\mathbb{R}) = H^2(\Pi_+) \oplus H^2(\Pi_+).$$

It can be shown that the subspaces $H^2(\Pi_\pm) \subset L^2(\mathbb{R})$ are invariant with respect to operators $T(\cdot)$, and therefore T splits into two summands $T_+ \oplus T_-$ (one of them

[4] Two families of lines on the hyperboloid correspond to two families of lines $x_1 = \mathrm{const}$ and $x_2 = \mathrm{const}$ on $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

has a highest weight, another a lowest weight). Hence $Q = (T_+ \oplus T_-) \otimes (T_+ \oplus T_-)$ splits into 4 summands. It can be shown that this is the desired decomposition:

- the space $H^2(\Pi_+) \otimes H^2(\Pi_+)$ consists of functions in $L^2(\mathbb{R}^2)$ continued holomorphically to the domain $\Pi_+ \times \Pi_+$; the representation $T_+ \otimes T_+$ in $H^2(\Pi_\pm) \subset L^2(\mathbb{R})$ is a direct sum of all highest weight representations of representation of $\mathrm{PSL}(2, \mathbb{R})$;
- $T_- \otimes T_-$ is a direct sum of all lowest weight representations;
- in $T_+ \oplus T_-$ we have the direct integral of all representations of the principal series (and the same integral in $T_- \otimes T_+$).

Remark. S.G. Gindikin [6] used a similar argument (restriction from a reducible representation of an overgroup) for multi-dimensional hyperboloids.

9. Splitting off the complementary series, see [35]

Consider the pseudo-orthogonal group $\mathrm{O}(1, q)$ consisting of operators preserving the following indefinite inner product in $\mathbb{R}^{1+q}$,

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_q y_q.$$

We write elements of this group as block $(1 + q) \times (1 + q)$ matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Denote by $\mathrm{SO}_0(1, q)$ its connected component, it consists of matrices satisfying two additional conditions $\det g = +1$, $a > 0$. Denote by S^{q-1} the unit sphere in $\mathbb{R}^n$. The group $\mathrm{O}(1, q)$ acts on S^{q-1} by conformal transformations $x \mapsto (a + xc)^{-1}(b + xd)$ (they preserve the sphere), the coefficient of a dilation equals to $(a + xc)^{-1}$.

For $\lambda \in \mathbb{C}$ we define a representation $T_\lambda = T^q_\lambda$ of $\mathrm{SO}_0(1, q)$ in a space of functions on S^{q-1} by

$$T_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = (a + xc)^{-(q-1)/2+\lambda} f\big((a + xc)^{-1}(b + xd)\big).$$

If $\lambda = i\sigma \in i\mathbb{R}$, then our representation is unitary in $L^2(S^{q-1})$, in this case $T_{i\sigma}$ is called a representation of the *unitary spherical principal series*, representations $T_{i\sigma}$ and $T_{-i\sigma}$ are equivalent (on these representations see, e.g., [40]). If $0 < s < (q - 1)/2$, then T_s is unitary in the Hilbert space H_s with the inner product

$$\langle f_1, f_2 \rangle_s = \int_{S^{q-1}} \int_{S^{q-1}} \frac{f_1(x_1)\, \overline{f_2(x_2)}\, dx_1\, dx_2}{\|x_1 - x_2|^{(q-1)/2-s}}.$$

More precisely, $\langle , \cdot, \cdot \rangle$ determines a positive definite Hermitian form on the space $C^\infty(S^{q-1})$ (this is not obvious), we get a pre-Hilbert space and consider its completion H_s. Such representations form the *spherical complementary series*. The spaces H_s are Sobolev spaces[5].

[5]In the standard notation, H_s is the Sobolev space $H^{-s,2}(S^{q-1})$. Notice that Sobolev spaces $H^{\sigma,2}(\cdot)$ are Hilbert spaces but inner product are defined not canonically. In our case the inner products are uniquely determined from the $\mathrm{SO}_0(1, q)$-invariance. For semisimple groups of rank > 1 complementary series are realized in functional Hilbert spaces that are not Sobolev spaces.

Consider a restrictions of $T_{i\sigma}$ to the subgroup $\mathrm{SO}_0(1,q-1)$. The group $\mathrm{SO}_0(1,q-1)$ has the following orbits on S^{q-1}: the equator $Eq = S^{q-2}$ defined by the equation $x_q = 0$, the upper hemisphere H_+ and the lower hemisphere H_-. The equator has zero measure and can be forgotten. Therefore $L^2(S^{q-1}) = L^2(H_+) \oplus L^2(H_-)$. On the other hand, hemispheres as homogeneous spaces are equivalent to $\mathrm{SO}_0(1,q-1)/\mathrm{SO}(q-1)$, i.e., to the $(q-1)$-dimensional Lobachevsky space. The decomposition of L^2 is a classical problem, in each summand $L^2(H_\pm)$ we get a multiplicity-free direct integral over the whole spherical principal series.

The restriction of a representation T_s of the complementary series is more interesting, it contains several summands of the complementary series and is equivalent to

$$\bigoplus\nolimits_{k:\, s-k>1/2} T^{q-1}_{s-k} \bigoplus L^2(H_+) \bigoplus L^2(H_-). \tag{6}$$

This spectrum was obtained by Ch. Boyer (1973), our purpose is to visualize summands of the complementary series.

According to the *trace theorems* Sobolev spaces of negative order can contain distributions supported by submanifolds. Denote by δ_{Eq} the delta-function of the equator, $\delta_{Eq} := \delta(x_q)$. Let φ be a smooth function on Eq.

$$\|\varphi\delta_{Eq}\|^2_s = \langle \varphi\delta_{Eq}, \varphi\delta_{Eq}\rangle_s = \int_{S^{q-2}} \int_{S^{q-2}} \frac{\varphi(y_1)\,\overline{\varphi(y_2)}\,dy_1\,dy_2}{\|y_1-y_2|^{-(q-1)/2+s}}.$$

If $s > 1/2$ the integral converges and $\varphi\delta_{Eq} \in H_s$. The representation of $\mathrm{SO}_0(1,q)$ in the space of such functions is T^{q-1}_s.

Denote by $\frac{\partial}{\partial n}\delta_{Eq} := \delta'(x_q)$ the derivative of δ_{Eq} in the normal direction. Similar arguments show that for $s > 3/2$ and smooth ψ we have $\psi\frac{\partial}{\partial n}\delta_{Eq} \in H_s$. The space of functions of the form

$$\varphi\delta_{Eq} + \psi\frac{\partial}{\partial n}\delta_{Eq}$$

again is invariant. It contains the subspace T^{q-1}_s and we get the representation T^{q-1}_{s+1} in the quotient. Since our representation is unitary, T^{q-1}_{s+1} must be direct summand, etc...

Next, we consider the operator $J : H_s \mapsto L^2(S^{q-1})$ given by

$$Jf(x) = |x_q|^{(q-1)/2-s} f(x).$$

It intertwines restrictions of T_s and T_0, the kernel of J consists of distributions supported by Eq and the image is dense[6]. This gives us (6).

[6]More precisely, we consider this operator as an operator on smooth functions compactly supported outside Eq, take the closure Γ of its graph in $H_s \oplus L^2$, and examine projection operators $\Gamma \to H_s$, $\Gamma \to L^2$.

10. The modern status of the problem

We mention the following works:

a) G.I. Olshanski [36] (1990) proposed a way to split off highest weight and lowest weight representations.

b) The author in [21] (1986) proposed a way to split off complementary series (see proofs and further examples in [35], the paper [28] contains an example with separation of direct integrals of different complementary series).

c) S.G. Gindikin [6] (1993) and V.F. Molchanov [17] (1998) obtained a separation of spectra for multi-dimensional hyperboloids.

These old works had continuations, in particular there were many further works with splitting off highest weight representations (for more references, see [32]).

The recent paper [32] (2017) contains formulas for projection operators separating spectrum for L^2 on pseudo-unitary groups $\mathrm{U}(p,q)$. In this case we can consider separation into series (if we fix the number r of continuous parameters of a representation, $r \leqslant \min(p,q)$), subsubseries (if we fix all discrete parameters of a representation) and intermediate subseries. All these question are solvable. The solution was obtained by a summation of all characters corresponding to a given type of spectrum, certainly this way must be available for all semisimple Lie groups.

In [34] the problem was solved for L^2 on pseudo-Riemannian symmetric spaces $\mathrm{GL}(n,\mathbb{C})/\mathrm{GL}(n,\mathbb{R})$. The calculation is based on an explicit summation of spherical distributions. Apparently, this can be extended to all symmetric spaces of the form $G_\mathbb{C}/G_\mathbb{R}$, where $G_\mathbb{C}$ is a complex semisimple Lie group and $G_\mathbb{R}$ is a real form of $G_\mathbb{C}$ (on Plancherel formulas for such spaces, see [1, 9, 38]).

For arbitrary semisimple symmetric spaces the problem does not seem to be well formulated, see a discussion of multi-dimensional hyperboloids in [17].

References

[1] Bopp, N., Harinck P. *Formule de Plancherel pour* $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(p,q)$. J. Reine Angew. Math. 428 (1992), 45–95.

[2] Folland G.B. *A course in abstract harmonic analysis.* Second edition, CRC Press, Boca Raton, FL, 2016.

[3] Gelfand I.M., Gindikin S.G. *Complex manifolds whose spanning trees are real semisimple Lie groups, and analytic discrete series of representations.* Funct. Anal. and Appl., 1977, 11:4, 258–265.

[4] Gelfand I.M., Graev, M.I., Vilenkin, N.Ya. *Generalized functions. Vol. 5: Integral geometry and representation theory.* Academic Press, New York-London 1966.

[5] Gelfand I.M., Naimark M.A. *Unitary representations of the classical groups.* (Russian) Trudy Mat. Inst. Steklov., vol. 36, Moscow-Leningrad, 1950. German transl.: Akademie-Verlag, Berlin, 1957.

[6] Gindikin S., *Conformal analysis on hyperboloids*, J. Geom. Phys., 10, 175–184 (1993).

[7] Gindikin S.G., Karpelevich F.I., *An integral associated with Riemannian symmetric spaces of non-positive curvature*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat., 30:5 (1966), 1147–1156.

[8] Groenevelt, W. *The Wilson function transform.* Int. Math. Res. Not. 2003, no. 52, 2779–2817.

[9] Harinck P. *Fonctions orbitales sur $G_{\mathbb{C}}/G_{\mathbb{R}}$. Formule d'inversion des intégrales orbitales et formule de Plancherel.* J. Funct. Anal. 153 (1998), no. 1, 52–107.

[10] Helgason S. *Geometric analysis on symmetric spaces.* American Mathematical Society, Providence, RI, 2008.

[11] Herb R.A., Wolf J.A. *The Plancherel theorem for general semisimple groups.* Compositio Math. 57 (1986), no. 3, 271–355.

[12] Kirillov A.A. *Plancherel's measure for nilpotent Lie groups.* Funct. Anal. Appl., 1:4 (1967), 330–331.

[13] Knapp A.W. *Representation theory of semisimple groups. An overview based on examples.* Princeton University Press, Princeton, NJ, 1986.

[14] Macdonald I.G. *Spherical functions on a group of p-adic type.* Publications of the Ramanujan Institute, Madras, 1971.

[15] Molchanov V.F. *Quantization on the imaginary Lobachevski plane.* Funct. Anal. and Appl., 1980, 14:2, 142–144.

[16] Molchanov V.F. *Plancherel's formula for hyperboloids*, Proc. Steklov Inst. Math., 147 (1981), 63–83.

[17] Molchanov V.F. *Separation of series for hyperboloids.* Funct. Anal. Appl., 31:3 (1997), 176–182.

[18] Molchanov, V.F. *Canonical representations and overgroups for hyperboloids of one sheet and Lobachevsky spaces.* Acta Appl. Math. 86 (2005), no. 1–2, 115–129.

[19] Molchanov, V.F. *Canonical representations and overgroups for hyperboloids.* Funct. Anal. Appl. 39 (2005), no. 4, 284–295

[20] Molchanov, V.F. *Canonical representations on Lobachevsky spaces: an interaction with an overalgebra.* Acta Appl. Math. 99 (2007), no. 3, 321–337.

[21] Neretin, Yu.A. *Representations of complementary series entering discretely in tensor products of unitary representations*, Funct. Anal. Appl., 20:1 (1986), 68–70

[22] Neretin, Yu.A. *Restriction of functions holomorphic in a domain to curves lying on the boundary of the domain, and discrete* $\mathrm{SL}_2(\mathbb{R})$*-spectra.* Izv. Math. 62 (1998), no. 3, 493–513.

[23] Neretin, Yu.A. *Matrix analogues of the B-function, and the Plancherel formula for Berezin kernel representations.* Sb. Math. 191 (2000), no. 5–6, 683–715

[24] Neretin, Yu.A. *Pseudo-Riemannian symmetric spaces: uniform realizations, and open embeddings into Grassmanians.* J. Math. Sci. (N.Y.), 2001, 107:5, 4248–4264

[25] Neretin, Yu.A. *The index hypergeometric transform and an imitation of the analysis of Berezin kernels on hyperbolic spaces.* Sb. Math. 192 (2001), no. 3–4, 403–432

[26] Neretin, Yu.A. *The action of an overalgebra in the Plancherel decomposition and shift operators in an imaginary direction.* Izv. Math. 66 (2002), no. 5, 1035–1046.

[27] Neretin, Yu.A. *Plancherel formula for Berezin deformation of* L^2 *on Riemannian symmetric space.* J. Funct. Anal. 189 (2002), no. 2, 336–408.

[28] Neretin, Yu.A. *On the Separation of Spectra in the Analysis of Berezin Kernels.* Funct. Anal. Appl., 2000, 34:3, 197–207

[29] Neretin, Yu.A. *Notes on Stein–Sahi representations and some problems of non-L^2-harmonic analysis*, J. Math. Sci. (N.Y.), 141:4 (2007), 1452–1478

[30] Neretin, Yu.A. *Difference Sturm-Liouville problems in the imaginary direction.* J. Spectr. Theory 3 (2013), no. 3, 237–269.

[31] Neretin, Yu.A. *Restriction of representations of* $\mathrm{GL}(n+1,\mathbb{C})$ *to* $\mathrm{GL}(n,\mathbb{C})$ *and action of the Lie overalgebra.* Preprint, arXiv:1510.03611

[32] Neretin, Yu.A. *Projections Separating Spectra for* L^2 *on Pseudounitary Groups* $\mathrm{U}(p,q)$. Int. Math. Res. Not., 2017, DOI: 10.1093/imrn/rnx268

[33] Neretin, Yu.A. *The Fourier Transform on the Group* $\mathrm{GL}_2(R)$ *and the Action of the Overalgebra* $\mathfrak{gl}_4$. J. Fourier Anal. and Appl., DOI: 10.1007/s00041-017-9589-8

[34] Neretin, Yu.A. *Projectors separating spectra for* L^2 *on symmetric spaces* $\mathrm{GL}(n,\mathbb{C})/\mathrm{GL}(n,\mathbb{R})$. Preprint, arXiv:1704.00049

[35] Neretin, Yu.A., Olshanski G.I. *Boundary values of holomorphic functions, singular unitary representations of* $\mathrm{O}(p,q)$*, and their limits as* $q\to\infty$. J. Math. Sci. (N. Y.), 1997, 87:6, 3983–4035.

[36] Olshanski, G.I. *Complex Lie semigroups, Hardy spaces and the Gelfand–Gindikin program.* Differential Geom. Appl. 1 (1991), no. 3, 235–246.

[37] Pukanszky, L. *On the characters and the Plancherel formula of nilpotent groups.* J. Funct. Anal. 1 1967, 255–280.

[38] Sano, Sh. *Distributions sphériques invariantes sur les espaces symétriques semi-simples* $G_{\mathbb{C}}/G_{\mathbb{R}}$. J. Math. Kyoto Univ. 31 (1991), no. 2, 377–417.

[39] Varadarajan, V.S. *An introduction to harmonic analysis on semisimple Lie groups.* Cambridge University Press, Cambridge, 1989.

[40] Vilenkin, N.Ja. *Special functions and the theory of group representations.* American Mathematical Society, Providence, R. I. 1968

[41] Wallach, N.R. *Real reductive groups. II.* Academic Press, Inc., Boston, MA, 1992.

Yury Neretin
Math. Dept., University of Vienna
and
Institute for Theoretical and Experimental Physics (Moscow)
MechMath Dept., Moscow State University
Institute for Information Transmission Problems
URL: http://mat.univie.ac.at/∼neretin/
e-mail: yurii.neretin@univie.ac.at

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 403–409

A Glimpse of Noncommutative Geometry

Andrzej Sitarz

Mathematics Subject Classification (2010). Primary 58B34; Secondary 46L85, 46L87, 81T75.

Keywords. Connes' noncommutative geometry, spectral triples.

1. Introduction

We present a short introduction to noncommutative geometry and spectral triples. We start the journey with C^* algebras and noncommutative differential, briefly mentioning K-theory, K-homology and cyclic (co)homology to finish with the notion of spectral triples, their benefits and examples.

2. What is noncommutative geometry?

The story of noncommutative geometry begins with the classical (differential) geometry and extends into the realm of abstract algebras and operators using the language Hilbert spaces and operators on them. One may, of course, say that *noncommutative geometry* studies the geometry of *quantum spaces* – or, to be more explicit – the geometry of noncommutative algebras. Clearly, the word *quantum*, although at first only superficially related to *quantum mechanics* or *quantum field theory* might be the right one – both physics and mathematics are involved in many examples and there is a huge interplay between them. However, the notion of *quantum spaces* is a delicate one since the objects that noncommutative geometry attempts to study are (usually) not spaces – they cannot be visualized.

Why study noncommutative geometry and why have an interest in it? First of all, it seems to be a natural and rich extension of the concept of spaces, one that can admit the notion of *geometry* in its various aspects. Moreover, within noncommutative geometry one can have on the same footing various objects, which, at first sight, are completely different. Last not least one should mention that many basic examples do arise from physics: the phase space in quantum mechanics, the

Supported by NCN grant 2016/21/B/ST1/02438.

Brillouin zone in the Quantum Hall Effect, the geometry of finite spaces in the noncommutative description of the Standard Model, quantum groups in integrable models or the quantized target space of string theory.

Let us attempt to place the subject matter of noncommutative geometry in relation to some other subjects in physics and mathematics. It certainly lies situated between operator algebra and functional analysis and differential geometry, with many links to abstract algebra, rings and modules, homological algebra, topology, probability, measure theory and algebraic geometry. In physics, it has most of the applications in classical field theory, gauge theories, but with a view on quantum field theory, renormalization, quantum mechanics as well as gravity, cosmology and even string theory. In this short review, which is based on three lectures, we can only present a very basic and superficial overview of the mathematical ideas behind Noncommutative Geometry.

3. From spaces to algebras (and back)

The space is nothing but a collection of points, however, when in mathematics we start to think about a space the first and basic idea that arises is that of *a topological space* – that is a space, which allows us to distinguish (in a very basic way) whether two points are "close" to each other, without being specific about actually measuring it with some numbers.

The fundamental idea that might have been the origin of noncommutative geometry is already present in the following two theorems from the last century:

Theorem 1 (Gelfand–Naimark). *Every commutative unital C^*-algebra is an algebra of continuous functions on a compact Hausdorff space.*

Theorem 2 (Gelfand–Naimark–Segal). *Every C^*-algebra is isomorphic to a complex, involutive, normed-closed algebra of bounded operators on a Hilbert space.*

Skipping details of the precise formulation of the above statements and their proofs let us concentrate on their significance. First, thanks to the Gelfand–Naimark's theorem, we can use noncommutative C^* algebras as the definition of *noncommutative Hausdorff compact spaces* and then, the GNS construction allows us to make a precise recipe to construct a C^* algebra, which became then very concrete – subalgebras of the operators on a Hilbert space.

The Gelfand–Naimark theorem is just a good starting point for noncommutative topology and towards many other notions like *measurable functions*, for instance. To summarize this section let us quote the dictionary, which establishes parallel notions between *standard* and *noncommutative* topology:

TOPOLOGY	ALGEBRA
(locally compact) topological space	commutative C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact space	unital C^*-algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone–Čech compactification	multiplier algebra
Cartesian product	tensor product

4. From topology to geometry (noncommutative way)

Having started with topology we have established a good point for the discussion of noncommutative spaces. However, we are still very far from geometry as topology does not distinguish between a ball and a cube!

We shall skip all constructions and theorems that extend the notions of vector bundles, connections, homology and cohomology – just to name the most important ones. Instead, we shall carry out the parallels built up for C^*-algebras, while concentrating on the differential calculi.

4.1. Differential Calculi

In the course of differential geometry one begins with the notion of a smooth manifold, C^∞ functions and vector fields. This is, however, reserved for a purely commutative world, as some simple algebras, like $M_n(\mathbb{C})$, do not admit any outer derivations, which is the algebraic characterisation of a vector field. Can this be cured? Not directly, however, a good answer is that one should rather pass to differential forms and differential algebras.

Definition 3. A differential graded algebra (DGA) over an algebra $\mathcal{A}$ is an $\mathbb{N}$-graded algebra, not necessarily finite, such that the 0th grade is isomorphic with $\mathcal{A}$ and that is equipped with a degree 1 linear map (grade increasing), which obeys the graded Leibniz rule:

$$d(\rho\,\omega) = d\rho\,\omega + (-1)^{|\rho|}\rho\,d\omega,$$

for any elements ω, ρ, where $|\rho|$ denotes the degree of the form ρ.

There is, unfortunately, no unique way to construct the DGAs in noncommutative world and we can have too many of them even in the commutative case. The canonical one, the *universal differential algebra* is completely uninteresting, as it carries no cohomological information and is infinite-dimensional even in the simplest case.

4.2. How to represent differential algebras?

As with the C^*-algebras came the natural representation on the Hilbert space, let us consider a specific way of obtaining differential graded algebras – connected with representations and commutators. Let $\mathcal{A}$ be an algebra and let π be its representation on a vector space (not necessarily finite-dimensional). Let F be an endomorphism (a linear operator, in other words) of this vector space.

Lemma 4. *If π is a representation of the algebra $\mathcal{A}$, then for each linear operator F the following gives a representation of the universal differential algebra $\Omega_u(\mathcal{A})$:*

$$\pi_F\,(a_0\,da_1\,da_2,\ldots,da_n) = \pi(a_0)\,[F,\pi(a_1)][F,\pi(a_2)]\ldots[F,\pi(a_n)].$$

Of course, if we do not assume anything about F we have just a representation of the algebra, and neither grading nor the external derivative could be represented. While dealing with infinite-dimensional representations on a Hilbert space we need to be careful as the commutators might be (in principle) unbounded. Therefore, it is natural to assume that all operators $\pi(a)$ and the commutators $[F,\pi(a)]$ are bounded for all $a \in \mathcal{A}$.

There exists a canonical way to obtain a differential graded algebra through π_F: we have to take $\mathcal{J} = \ker\pi_F + d(\ker\pi_F)$. This is a differential ideal within $\Omega_u(\mathcal{A})$ and then $\Omega^u(\mathcal{A})/\mathcal{J}$ will be a differential algebra. However, it might not have a representation on the Hilbert space.

A very particular situation happens if we assume more about F, for example if we take $F^2 = 1$, which means that (as seen on a Hilbert space) F is a sign operator with eigenvalues being $+1$ and -1. We then have:

Lemma 5. *Let and $F^2 = 1$ be an operator on the Hilbert space $\mathcal{H}$ and let π be the representation of $\mathcal{A}$ as bounded operators on $\mathcal{H}$. Then π_F defined in Lemma* 3.2 *is a representation of the differential algebra, with:*

$$\pi_F(d\omega) = F\pi_F(\omega) - (-1)^{|w|}\pi_F(\omega)F,$$

for any universal form ω of degree $|\omega|$.

The above construction is not just another way of obtaining differential graded algebras but has a deep geometric meaning and some equivalence classes of such constructions (called Fredholm modules) over an algebra are building blocks of K-homology.

5. Spectral triples and how to use them

In this last part of these notes we use (and probably overuse) the word *spectral.* Its sense will be described in the definition of properties of spectral triples – a concise proposition for *noncommutative spin manifolds.* The clue is that (almost) everything is set by the Dirac operator and it, in turn, is defined through its set of discrete eigenvalues with multiplicities. We very briefly describe the idea which links the theory with physics: the construction of gauge theories and the spectral action principle.

5.1. What the Dirac operator is good for

The Laplace operator on a Riemannian compact manifolds encodes a lot of information about the geometry. The same is true for its nontrivial square root, the Dirac operator (in case we have a spin manifold). The Dirac operator on a compact spin manifolds is indeed a very elegant object: an unbounded, self-adjoint operator, with a discrete spectrum and with the growth of eigenvalues governed by the dimension of the manifold. It encodes also the topological and geometrical information about the manifold, in particular about the differential algebra and the metric. Spectral triples just mimic this construction in the noncommutative world, assuming that it is the basic data that makes the noncommutative approach *the geometry.*

Definition 6. Let us have an algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint unbounded operator D with compact resolvent, such that

$$\forall a \in \mathcal{A}, \quad [D, \pi(a)] \in B(\mathcal{H}),$$

then we call $(\mathcal{A}, \pi, D)$ a spectral triple.

Since the definition is very basic, we shall need (in most cases) some additional structures. We say that the spectral triple is *even* if there exists an operator γ such that $\gamma = \gamma^\dagger$, $\gamma\pi(a) = \pi(a)\gamma$ and $\gamma D + D\gamma = 0$. We say that the spectral triple is finitely summable if the operator $|D|^{-1}$ has eigenvalues growing like n^s for some $s \geq 0$. If the growth of eigenvalues of $|D|^{-1}$ is exactly of the order $n^{\frac{1}{p}}$, we say that the spectral triple is of metric dimension p.

Spectral triples allow for more "decorations" and conditions like reality structure, which we omit here. Instead, let us quote the most important result, which establishes (precisely) the relation of spectral triples to classical differential geometry.

Theorem 7. *If $\mathcal{A} = C^\infty(M)$, M is a spin Riemannian compact manifold, S is a spinor bundle over M, $\mathcal{H} = L^2(S)$ (summable sections of spinor bundle) and D is the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure) and metric dimension* $\dim(M)$.

Even more interesting is the reconstruction theorem, which roughly states the inverse and (with several additional assumption) was demonstrated by Connes.

5.2. Differential forms and fluctuations

Since $[D, \pi(a)]$ is bounded we can easily apply what we have learned about differential algebras – and we can call all elements of the type $\sum_i \pi(a_i)[D, \pi(b_i)]$ first-order differential forms. Since both the algebra and the forms ale embedded in $B(\mathcal{H})$ the one forms have naturally the structure of a bimodule over $\mathcal{A}$ and generate the algebra corresponding to the sections of the Clifford bundle. The one forms play an important role as possible bounded perturbations of the Dirac operator. For every $A = \sum_i \pi(a_i)[D, \pi(b_i)]$, which is selfadjoint, we consider $D_A = D + A$, which satisfies all conditions for the Dirac operator thus giving us a family of *fluctuations* of the original Dirac operator.

5.3. Integration for spectral triples

Cutting a very long story short let us recall that for the finitely summable spectral triples one make a nice use of the Dirac operator. Using the ζ-function regularisation of the trace we can define a noncommutative integral:

$$\fint T = \mathrm{Res}_{z=0} \,\mathrm{Tr} \left(T|D|^{-z} \right).$$

This exists for all operators T, which are products of $\pi(a)$, powers of D and their commutators with D and $|D|$ (assuming regularity of the spectral triple).

If the spectral triple has metric dimension p then it could be shown that $\fint \pi(a)|D|^{-p}$ defines a trace on the algebra $\mathcal{A}$. In particular, for a compact, p-dimensional spin manifold M with a true Dirac operator $\fint |D|^{-p} \sim \mathrm{vol}(M)$.

5.4. Measuring spectral geometries

Using the integration defined above we can measure the volume, however, there is much more that could be extracted. In the classical case a simple formula allows us to recover distances on the manifold:

$$d(x,y) = \sup_{||[D,a]|| \leq 1} |x(a) - y(a)|, \qquad x, y \in M, a \in C^\infty(M).$$

As in the noncommutative situation there may be no points, we need to extend it, replacing the points with the states on the algebra, thus making the space of states equipped with a metric. If this metrizes the weak-* topology then we are dealing with quantum metric spaces, which then allows us to study convergence of spaces in the Rieffel–Gromov–Hausdorff distance.

5.5. Towards (noncommutative) physics

Suppose we accept that spectral triples do describe *noncommutative manifolds*. Is there any physical contents in them? Can we use them to describe some noncommutative physics? The answer is yes and, indeed, we shall be able to provide – at least – some partial answers.

Again, assume that we have a spectral triple, that is $(\mathcal{A}, \pi, D)$, and consider the family of all allowed Dirac operators as the physical degrees of freedom. That includes not only some possible rescaling, changes of the metric but also the fluctuations of the gauge type as described before. Next, we define a functional on the space of all admissible Dirac operators:

$$S(D) = \mathrm{Tr}\, f(D^2),$$

where f is a suitable cut-off function, which, for instance, vanishes for arguments bigger than a certain number Λ. This idea appeared for the first time (in a similar phrasing) in the work of Sakharov in 1965 to describe the gravity action and possible corrections.

5.6. The Standard Model and gravity

The story of spectral action becomes interesting when we apply it to geometries of the type $M \times F$, where M is a Riemannian manifold and F is a discrete geometry. It is like a Kaluza–Klein model but with the extra dimensions being in fact of (classical dimension) zero. In fact, if the discrete geometry is $F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ then, in addition to classical gravity and the Einstein–Hilbert term, the spectral action yields all Yang–Mills action terms together with the Higgs field (as a doublet) and the correct Higgs potential.

5.7. Where can you learn more?

In these very short note we have tried to give a glimpse of noncommutative geometry – a theory, which, motivated by examples, extends the notion of *geometry* into the algebraic world. What we still need to supply is a word about prospects: first of learning (where to learn more) but also the prospects of the field (why learn it). Interested student should go to http://bit.ly/2zfCB7v where the author maintains a basic list of recommended material for further reading.

Andrzej Sitarz
Institute of Physics
Jagiellonian University
prof. Stanislawa Lojasiewicza 11
PL-30-348 Kraków, Poland

and

Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8
PL-00-950 Warszawa, Poland
e-mail: andrzej.sitarz@uj.edu.pl

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 411–418

An Example of Banach and Hilbert Manifold: The Universal Teichmüller Space

Alice Barbara Tumpach

Mathematics Subject Classification (2010). 58B20, 58B25.

Keywords. Universal Teichmüller space, Beltrami coefficients, diffeomorphisms of the circle, Siegel disc.

1. Motivations

***H^s*-Diffeomorphisms groups of the circle.** For $s > 3/2$, the group $\mathrm{Diff}^s(S^1)$ of Sobolev class H^s diffeomorphisms of the circle is a $\mathcal{C}^\infty$-manifold modeled on the space of H^s-section of the tangent bundle TS^1 ([1]), or equivalently on the space of real H^s-function on S^1. It is a topological group in the sense that the multiplication $(f, g) \mapsto f \circ g$ is well defined and continuous, the inverse $f \mapsto f^{-1}$ is continuous, the left translation L_γ by $\gamma \in \mathrm{Diff}^s(S^1)$ applying f to $\eta \circ f$ is continuous, and the right translation R_γ by $\gamma \in \mathrm{Diff}^s(S^1)$ applying f to $f \circ \eta$ is smooth. These results are consequences of the Sobolev Lemma which states that for a compact manifold of dimension n, the space of H^s-sections of a vector bundle E over M is contained, for $s > k+n/2$, in the space of $\mathcal{C}^k$-sections, and that the injection $H^s(E) \hookrightarrow \mathcal{C}^k(E)$ is continuous. In particular, for $s > 3/2$, $\mathrm{Diff}^s(S^1)$ is the intersection of the space of $\mathcal{C}^1$-diffeomorphisms of the circle with the space $H^s(S^1, S^1)$ of H^s maps from S^1 into itself. Hence $\mathrm{Diff}^s(S^1)$ is an open set of $H^s(S^1, S^1)$.

For the same reasons, the subgroup of $\mathrm{Diff}^s(S^1)$ preserving three points in S^1, say $-1, -i$ and 1, is, for $s > 3/2$, a $\mathcal{C}^\infty$ manifold and a topological group modeled on the space of H^s-vector fields which vanish on $-1, -i$ and 1.

One may ask what happens for the critical value $s = 3/2$ and look for a group with some regularity and a manifold structure such that the tangent space at the identity is isomorphic to the space of $H^{\frac{3}{2}}$-vector fields vanishing at $-1, -i$ and 1 (or equivalently on any codimension 3 subspace of $H^{\frac{3}{2}}$). The universal Teichmüller space $T_0(1)$ defined below will verify these conditions.

$\mathrm{Diff}^+(S^1)$ **as a group of symplectomorphisms.** Consider the Hilbert space $\mathcal{V} = H^{\frac{1}{2}}(S^1,\mathbb{R})/\mathbb{R}$ of real-valued $H^{\frac{1}{2}}$ functions with mean-value zero. Each element $u \in \mathcal{V}$ can be written as

$$u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \quad \text{with} \quad u_0 = 0,\ u_{-n} = \overline{u_n} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |n||u_n|^2 < \infty.$$

Endow $\mathcal{V}$ with the symplectic form

$$\Omega(u,v) = \frac{1}{2\pi}\int_{S^1} u(x)\partial_x v(x)dx = -i\sum_{n\in\mathbb{Z}} n u_n \overline{v_n},$$

The group of orientation preserving $\mathcal{C}^\infty$-diffeomorphisms of the circle acts on $\mathcal{V}$ by

$$\varphi\cdot f = f\circ\varphi - \frac{1}{2\pi}\int_{S^1} f\circ\varphi,$$

preserving the symplectic form Ω. Note that the previous action is well defined for any orientation preserving homeomorphism of S^1. Therefore one may ask what is the biggest subgroup of the orientation preserving homeomorphisms of the circle which preserves $\mathcal{V}$ and Ω. The answer is the group of quasisymmetric homeomorphisms of the circle defined below (Theorem 3.1 and Proposition 4.1 in [3]).

Teichmüller spaces of compact Riemann surfaces. Consider a compact Riemann surface Σ. The Teichmüller space $\mathcal{T}(\Sigma)$ of Σ is defined as the space of complex structures on Σ modulo the action by pull-back of the group of diffeomorphisms which are homotopic to the identity. It can be endowed with a Riemannian metric, called the Weil–Petersson metric, which is not complete. A point beyond which a geodesic cannot be continued corresponds to the collapsing of a handle of the Riemann surface ([6]), hence yields to a Riemann surface with lower genus. One can ask for a Riemannian manifold in which all the Teichmüller spaces of compact Riemann surfaces with arbitrary genus inject isometrically. The answer will be the universal Teichmüller space endowed with a Hilbert manifold structure and its Weil–Petersson metric ([5]).

2. The universal Teichmüller space

Quasiconformal and quasisymmetric mappings. Let us give some definitions and basic facts on quasiconformal and quasisymmetric mappings.

Definition 1. An orientation preserving homeomorphism f of an open subset A in $\mathbb{C}$ is called quasiconformal if the following conditions are satisfied.

- f admits distributional derivatives $\partial_z f$, $\partial_{\bar z} f \in L^1_{\mathrm{loc}}(A,\mathbb{C})$;
- there exists $0 \leq k < 1$ such that $|\partial_{\bar z} f(z)| \leq k|\partial_z f(z)|$ for every $z \in A$.

Such an homeomorphism is said to be K-quasiconformal, where $K = \frac{1+k}{1-k}$.

Example 1. For example, $f(z) = \alpha z + \beta\bar z$ with $|\beta| < |\alpha|$ is $\frac{|\alpha|+|\beta|}{\alpha-|\beta|}$-quasiconformal.

Denote by $L^\infty(A,\mathbb{C})$ the complex Banach space of bounded complex-valued functions on an open subset $A \subset \mathbb{C}$.

Theorem 2 ([2]). *An orientation preserving homeomorphism f defined on an open set $A \subset \mathbb{C}$ is quasiconformal if and only if it admits distributional derivatives $\partial_z f$, $\partial_{\bar{z}} f \in L^1_{\mathrm{loc}}(A,\mathbb{C})$ which satisfy*

$$\partial_{\bar{z}} f(z) = \mu(z)\partial_z f(z), \quad z \in A$$

for some $\mu \in L^\infty(A,\mathbb{C})$ with $\|\mu\|_\infty < 1$.

The function μ appearing in the previous theorem is called the Beltrami coefficient or the complex dilatation of f. Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$.

Theorem 3 (Ahlfors–Bers). *Given $\mu \in L^\infty(\mathbb{D},\mathbb{C})$ with $\|\mu\|_\infty < 1$, there exists a unique quasiconformal mapping $\omega_\mu : \mathbb{D} \to \mathbb{D}$ with Beltrami coefficient μ, extending continuously to $\overline{\mathbb{D}}$, and fixing $1, -1, i$.*

Definition 2. An orientation preserving homeomorphism η of the circle S^1 is called quasisymmetric if there is a constant $M > 0$ such that for every $x \in \mathbb{R}$ and every $|t| \leq \frac{\pi}{2}$

$$\frac{1}{M} \leq \frac{\tilde{\eta}(x+t) - \tilde{\eta}(x)}{\tilde{\eta}(x) - \tilde{\eta}(x-t)} \leq M,$$

where $\tilde{\eta}$ is the increasing homeomorphism on $\mathbb{R}$ uniquely determined by $0 \leq \tilde{\eta}(0) < 1$, $\tilde{\eta}(x+1) = \tilde{\eta}(x) + 1$, and the condition that it projects onto η.

Theorem 4 (Beurling–Ahlfors extension Theorem). *Let η be an orientation preserving homeomorphism of S^1. Then η is quasisymmetric if and only if it extends to a quasiconformal homeomorphism of the open unit disc $\mathbb{D}$ into itself.*

$T(1)$ as a Banach manifold. One way to construct the universal Teichmüller space is the following. Denote by $L^\infty(\mathbb{D})_1$ the unit ball in $L^\infty(\mathbb{D},\mathbb{C})$. By Ahlfors–Bers theorem, for any $\mu \in L^\infty(\mathbb{D})_1$, one can consider the unique quasiconformal mapping $w_\mu : \mathbb{D} \to \mathbb{D}$ which fixes $-1, -i$ and 1 and satisfies the Beltrami equation on $\mathbb{D}$

$$\frac{\partial}{\partial \bar{z}}\omega_\mu = \mu \frac{\partial}{\partial z}\omega_\mu.$$

Therefore one can define the following equivalence relation on $L^\infty(\mathbb{D})_1$. For μ, $\nu \in L^\infty(\mathbb{D})_1$, set $\mu \sim \nu$ if $w_\mu|S^1 = w_\nu|S^1$. The universal Teichmüller space is defined by the quotient space

$$T(1) = L^\infty(\mathbb{D})_1/\sim.$$

Theorem 5 ([2]). *The space $T(1)$ has a unique structure of complex Banach manifold such that the projection map $\Phi : L^\infty(\mathbb{D})_1 \to T(1)$ is a holomorphic submersion.*

The differential of Φ at the origin $D_0\Phi\ : L^\infty(\mathbb{D},\mathbb{C}) \to T_{[0]}T(1)$ is a complex linear surjection and induces a splitting of $L^\infty(\mathbb{D},\mathbb{C})$ into ([5]) :

$$L^\infty(\mathbb{D},\mathbb{C}) = \operatorname{Ker} D_0\Phi \oplus \Omega_\infty(\mathbb{D}),$$

where $\Omega^\infty(\mathbb{D})$ is the Banach space defined by

$$\Omega_\infty(\mathbb{D}) := \left\{\mu \in L^\infty(\mathbb{D},\mathbb{C}) \mid \mu(z) = (1-|z|^2)^2\overline{\phi(z)},\ \ \phi \text{ holomorphic on } \mathbb{D}\ \right\}.$$

$T(1)$ **as a group.** By the Beurling–Ahlfors extension theorem, a quasiconformal mapping on $\mathbb{D}$ extends to a quasisymmetric homeomorphism on the unit circle. Therefore the following map is a well-defined bijection

$$\begin{array}{ccc} T(1) & \to & \mathrm{QS}(S^1)/PSU(1,1) \\ [\mu] & \mapsto & \left[w_\mu|S^1\right]. \end{array}$$

The coset $\mathrm{QS}(S^1)/PSU(1,1)$ inherits from its identification with $T(1)$ a Banach manifold structure. Moreover the coset $\mathrm{QS}(S^1)/PSU(1,1)$ can be identified with the subgroup of quasisymmetric homeomorphisms fixing $-1, i$ and 1. This identification allows to endow the universal Teichmüller space with a group structure. With respect to this differential structure, the right translations in $T(1)$ are biholomorphic mappings, whereas the left translations are not even continuous in general. Consequently $T(1)$ is not a topological group.

The WP-metric and the Hilbert manifold structure on $\boldsymbol{T(1)}$. The Banach manifold $T(1)$ carries a Weil–Petersson metric, which is defined only on a distribution of the tangent bundle ([4]). In order to resolve this problem the idea in [5] is to change the differentiable structure of $T(1)$.

Theorem 6 ([5]). *The universal Teichmüller space $T(1)$ admits a structure of Hilbert manifold on which the Weil–Petersson metric is a right-invariant strong hermitian metric.*

For this Hilbert manifold structure, the tangent space at $[0]$ in $T(1)$ is isomorphic to

$$\Omega_2(\mathbb{D}) := \left\{\mu(z) = (1-|z|^2)^2\overline{\phi(z)},\quad \phi \text{ holomorphic on } \mathbb{D},\quad \|\mu\|_2 < \infty\ \right\},$$

where $\|\mu\|_2^2 = \int\int_{\mathbb{D}} |\mu|^2\rho(z)d^2z$ is the L^2-norm of μ with respect to the hyperbolic metric of the Poincaré disc $\rho(z)d^2z = 4(1-|z|^2)^{-2}d^2z$. The Weil–Petersson metric on $T(1)$ is given at the tangent space at $[0] \in T(1)$ by

$$\langle \mu, \nu\rangle_{WP} := \iint_{\mathbb{D}} \mu\,\overline{\nu}\,\rho(z)d^2z$$

With respect to this Hilbert manifold structure, $T(1)$ admits uncountably many connected components. For this Hilbert manifold structure, the identity component $T_0(1)$ of $T(1)$ is a topological group. Moreover the pull-back of the Weil–Petersson metric on the quotient space $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$ is given at [Id] by

$$h_{WP}([\mathrm{Id}])([u],[v]) = 2\pi\sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v_n}.$$

Hence the identity component $T_0(1)$ of $T(1)$ can be seen as the completion of $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$ for the $H^{3/2}$-norm. This metric make $T(1)$ into a strong Kähler–Einstein–Hilbert manifold, with respect to the complex structure given at [Id] by the Hilbert transform (see below where the definition of the Hilbert transform is recalled). The tangent space at [Id] consists of Sobolev class $H^{3/2}$ vector fields modulo $\mathfrak{psu}(1,1)$. The associated Riemannian metric is given by

$$\mathrm{g}_{WP}([\mathrm{Id}])([u],[v]) = \pi \sum_{n\neq -1,0,1} |n|(n^2-1)u_n\overline{v_n},$$

and the imaginary part of the Hermitian metric is the two-form

$$\omega_{WP}([\mathrm{Id}])([u],[v]) = -i\pi \sum_{n\neq -1,0,1} n(n^2-1)u_n\overline{v_n}.$$

Note that ω_{WP} coincides with the Kirillov–Kostant–Souriau symplectic form obtained on $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$ when considered as a coadjoint orbit of the Bott–Virasoro group.

3. The restricted Siegel disc

The Siegel disc. Let $\mathcal{V} = H^{\frac{1}{2}}(S^1,\mathbb{R})/\mathbb{R}$ be the Hilbert space of real-valued $H^{\frac{1}{2}}$ functions with mean-value zero. The Hilbert inner product on $\mathcal{V}$ is given by

$$\langle u,v\rangle_{\mathcal{V}} = \sum_{n\in\mathbb{Z}} |n|u_n\overline{v_n}.$$

Endow the real Hilbert space $\mathcal{V}$ with the following complex structure (called the Hilbert transform)

$$\mathrm{J}\left(\sum_{n\neq 0} u_n e^{inx}\right) = i\sum_{n\neq 0}\mathrm{sgn}(n)u_n e^{inx}.$$

Now $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and J are compatible in the sense that J is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{V}}$. The associated symplectic form is defined by

$$\Omega(u,v) = \langle u, J(v)\rangle_{\mathcal{V}} = \frac{1}{2\pi}\int_{S^1} u(x)\partial_x v(x)dx = -i\sum_{n\in\mathbb{Z}} nu_n\overline{v_n}.$$

Let us consider the **complexified Hilbert space** $\mathcal{H} := H^{1/2}(S^1,\mathbb{C})/\mathbb{C}$ and the complex linear extensions of J and Ω still denoted by the same letters. Each element $u\in\mathcal{H}$ can be written as

$$u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \quad \text{with} \quad u_0 = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |n||u_n|^2 < \infty.$$

The eigenspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of the operator J are the following subspaces

$$\mathcal{H}_+ = \left\{ u\in\mathcal{H} \,\middle|\, u(x) = \sum_{n=1}^{\infty} u_n e^{inx} \right\} \text{ and } \mathcal{H}_- = \left\{ u\in\mathcal{H} \,\middle|\, u(x) = \sum_{n=-\infty}^{-1} u_n e^{inx} \right\},$$

and one has the Hilbert decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into the sum of closed orthogonal subspaces. **The Siegel disc** associated with $\mathcal{H}$ is defined by

$$\mathfrak{D}(\mathcal{H}) := \{Z \in L(\mathcal{H}_-, \mathcal{H}_+) \mid \Omega(Zu, v) = \Omega(Zv, u), \forall\, u, v \in \mathcal{H}_- \text{ and } I - Z\bar{Z} > 0\},$$

where, for $A \in L(\mathcal{H}_+, \mathcal{H}_+)$, the notation $A > 0$ means $\langle A(u), u\rangle_{\mathcal{H}} > 0$, for all $u \in \mathcal{H}_+, u \neq 0$ and where for $B \in L(\mathcal{H}_-, \mathcal{H}_+)$, define

$$\overline{B}(u) := \overline{B(\bar{u})}, \qquad B^T := (\bar{B})^*.$$

It follows easily that $\mathfrak{D}(\mathcal{H})$ can be written as

$$\mathfrak{D}(\mathcal{H}) := \{Z \in L(\mathcal{H}_-, \mathcal{H}_+) \mid Z^T = Z, \forall\, u, v \in \mathcal{H}_- \quad \text{and} \quad I - Z\bar{Z} > 0\}.$$

The restricted Siegel disc associated with $\mathcal{H}$ is by definition

$$\mathfrak{D}_{\mathrm{res}}(\mathcal{H}) := \{Z \in \mathfrak{D}(\mathcal{H}) \mid Z \in L^2(\mathcal{H}_-, \mathcal{H}_+)\},$$

where $L^2(\mathcal{H}_-, \mathcal{H}_+)$ denotes the space of Hilbert–Schmidt operators from $\mathcal{H}_-$ to $\mathcal{H}_+$.

The restricted Siegel disc as an homogeneous space. Consider the symplectic group $\mathrm{Sp}(\mathcal{V}, \Omega)$ of bounded linear maps on $\mathcal{V}$ which preserve the symplectic form Ω

$$\mathrm{Sp}(\mathcal{V}, \Omega) = \{a \in \mathrm{GL}(\mathcal{V}) \mid \Omega(au, av) = \Omega(u, v), \text{ for all } u, v \in \mathcal{V}\}.$$

The restricted symplectic group $\mathrm{Sp}_{\mathrm{res}}(\mathcal{V}, \Omega)$ is by definition the intersection of the symplectic group with the restricted general linear group defined by

$$\mathrm{GL}_{\mathrm{res}}(\mathcal{H}, \mathcal{H}_+) = \left\{g \in \mathrm{GL}(\mathcal{H}) \mid [d, g] \in L^2(\mathcal{H})\right\},$$

where $d := i(p_+ - p_-)$ and $p_\pm$ is the orthogonal projection onto $\mathcal{H}_\pm$. Using the block decomposition with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, one gets

$$\mathrm{Sp}_{\mathrm{res}}(\mathcal{V}, \Omega) := \left\{\begin{pmatrix} g & h \\ \bar{h} & \bar{g} \end{pmatrix} \in \mathrm{GL}(\mathcal{H}) \middle| h \in L^2(\mathcal{H}_-, \mathcal{H}_+), gg^* - hh^* = I, gh^T = hg^T\right\}.$$

Proposition 7. *The restricted symplectic group acts transitively on the restricted Siegel disc by*

$$\mathrm{Sp}_{\mathrm{res}}(\mathcal{V}, \Omega) \times \mathfrak{D}_{\mathrm{res}}(\mathcal{H}) \longrightarrow \mathfrak{D}_{\mathrm{res}}(\mathcal{H}), \quad \left(\begin{pmatrix} g & h \\ \bar{h} & \bar{g} \end{pmatrix}, Z\right) \longmapsto (gZ + h)(\bar{h}Z + \bar{g})^{-1}.$$

The isotropy group of $0 \in \mathfrak{D}_{\mathrm{res}}(\mathcal{H})$ *is the unitary group* $\mathrm{U}(\mathcal{H}_+)$ *of* $\mathcal{H}_+$*, and the restricted Siegel disc is diffeomorphic as Hilbert manifold to the homogeneous space* $\mathrm{Sp}_{\mathrm{res}}(\mathcal{V}, \Omega)/U(\mathcal{H}_+)$.

On the space $\{A \in L^2(H_-, H_+) \mid A^T = A\}$ consider the following Hermitian inner product

$$\mathrm{Tr}(V^*U) = \mathrm{Tr}(\bar{V}U).$$

Since it is invariant under the isotropy group of $0 \in \mathfrak{D}_{\mathrm{res}}(\mathcal{H})$, it extends to an $\mathrm{Sp}_{\mathrm{res}}(\mathcal{V}, \Omega)$-invariant Hermitian metric $h_{\mathfrak{D}}$.

Remark 8. In the construction above, replace $\mathcal{V}$ by $\mathbb{R}^2$ endowed with its natural symplectic structure. The corresponding Siegel disc is nothing but the open unit disc $\mathbb{D}$. The action of $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$ is the standard action of $\mathrm{SU}(1,1)$ on $\mathbb{D}$ given by

$$z \in \mathbb{D} \longmapsto \frac{az+b}{\bar{b}z+\bar{a}} \in \mathbb{D}, \quad |a|^2 - |b|^2 = 1,$$

and the Hermitian metric obtained on $\mathbb{D}$ is given by the hyperbolic metric

$$h_{\mathfrak{D}}(z)(u,v) = \frac{1}{(1-|z|^2)^2} u\bar{v}.$$

Therefore, $\mathfrak{D}_{\mathrm{res}}(\mathcal{H})$ can be seen as an infinite-dimensional generalization of the Poincaré disc.

4. The period mapping

The following theorems answer the second question addressed in the first section.

Theorem 9 (Theorem 3.1 in [3]). *For ϕ a orientation preserving homeomorphism and any $f \in \mathcal{V}$, set by $V_\phi f = f \circ \varphi - \frac{1}{2\pi}\int_{S^1} f \circ \varphi$. Then V_ϕ maps $\mathcal{V}$ into itself iff ϕ is quasisymmetric.*

Theorem 10 (Proposition 4.1 in [3]). *The group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle acts on the right by symplectomorphisms on*

$$\mathcal{H} = H^{1/2}(S^1, \mathbb{C})/\mathbb{C}$$

by

$$V_\phi f = f \circ \varphi - \frac{1}{2\pi}\int_{S^1} f \circ \varphi,$$

$\varphi \in \mathrm{QS}(S^1)$, $f \in \mathcal{H}$.

Consequently this action defines a map $\Pi : \mathrm{QS}(S^1) \to \mathrm{Sp}(\mathcal{V}, \Omega)$. Note that the operator $\Pi(\varphi)$ preserves the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ iff φ belongs to $\mathrm{PSU}(1,1)$. The resulting map (Theorem 7.1 in [3]) is an injective equivariant holomorphic immersion

$$\Pi \ : T(1) = \mathrm{QS}(S^1)/\,\mathrm{PSU}(1,1) \to \mathrm{Sp}(\mathcal{V}, \Omega)/\,\mathrm{U}(H_+) \simeq \mathfrak{D}(\mathcal{H})$$

called the **period mapping** of $T(1)$. The Hilbert version of the period mapping is given by the following

Theorem 11 ([5]). *For $[\mu] \in T(1)$, $\Pi([\mu])$ belongs to the restricted Siegel disc if and only if $[\mu] \in T_0(1)$. Moreover the pull-back of the natural Kähler metric on $\mathfrak{D}_{\mathrm{res}}(\mathcal{H})$ coincides, up to a constant factor, with the Weil–Petersson metric on $T_0(1)$.*

References

[1] Ebin, D.G. [1968], *The manifold of Riemannian metrics*, 1970 Global Analysis (Proc. Sympos. Pure Math., Vol XV, Berkeley, Calif., 1968) pp. 11–40 Amer. Math. Soc., Providence, R.I.

[2] Lehto, O. [1987], *Univalent functions and Teichmüller spaces*, Springer-Verlag, New York.

[3] Nag, S. and D. Sullivan [1995], Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, *Osaka J. Math.*, **32**, no. 1, 1–34.

[4] Nag, S. and A. Verjovsky [1990], $\mathrm{Diff}(S^1)$ and the Teichmüller Spaces, *Comm. Math. Phys.*, **130**, no. 1, 123–138.

[5] Takhtajan, L. and L.-P. Teo [2004], Weil–Petersson metric on the universal Teichmüller space, *Mem. Amer. Math. Soc.*, **183**, no.861, viii+119pp.

[6] Tromba, A.J. [1992], *Teichmüller theory in Riemannian geometry*, Lectures in Mathematics, ETH Zürich, Birkhäuser.

Alice Barbara Tumpach
UMR CNRS 8524
UFR de Mathématiques
Laboratoire Paul Painlevé
F-59655 Villeneuve d'Ascq Cedex, France
e-mail: barbara.tumpach@math.univ-lille1.fr

Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 419–425

Extensions of Symmetric Operators and Evolution Equations on Singular Spaces

Andrei Shafarevich

Mathematics Subject Classification (2010). Primary 37K25; Secondary 53C22, 57R45, 11P21, 11P21.

Keywords. Hybrid spaces, polyhedral surfaces, extension theory, propagation of quasi-particles, properties of geodesic flows, integral points in polyhedra, theory of abstract primes.

1. Introduction

Differential operators on varieties with singularities were studied in a great number of papers. General theory of such operators is rather complicated and the majority of general results are rather implicit because singularities can have very complicated structure. However, certain spaces with simplest singularities can demonstrate clear behavior of differential equations and formulas for their solutions as well as for spectral characteristics of corresponding operators can be much more explicit then in general situation. From the other hand, such spaces appear in different applications, so the properties mentioned above seem to be interesting. Further we discuss two examples of spaces with singularities – hybrid spaces and polyhedral surfaces. In particular, we study spectral properties of Laplacians on hybrid spaces and polyhedral 2D surfaces as well as behavior of localized solutions of Schrödinger equations. One of the main tools for studying differential operators on such spaces is the theory of extensions of symmetric operators. The proofs of some of the results can be found in [1–3] (see also [4]); the proofs of the remaining results will be published in a separate paper.

1.1. Hybrid spaces

Let $M_1, \dots, M_k$ be smooth compact geodesically complete oriented Riemannian manifolds of dimension at most 3 and let $\gamma_1, \dots, \gamma_s$ be segments endowed with parameterization. The *hybrid space* M is a topological space obtained by gluing

The work was supported by the Russian Scientific Foundation (grant 16-11-10069).

the endpoints of the segments to certain points $q_1, \ldots, q_{2s}$ on the manifolds; we assume that different endpoints are glued to different points q_j. Such spaces appear as models for nanostructures, electronic devices and even for transport motion.

1.2. Polyhedral surfaces

We will also consider polyhedral surfaces – compact 2D oriented surfaces M, glued from a finite number of flat polygons in a usual manner. The surfaces will be not necessary embedded in $\mathbb{R}^3$; the total angles $\beta_1, \ldots, \beta_M$ at the vertices can be less or greater than 2π – the unique condition is the Gauss-Bonnet relation $\sum_{j=1}^{M}(1-\frac{\beta_j}{2\pi}) = \chi(M)$, where χ denotes Euler characteristics.

Remark 1. Each polyhedron admits a natural complex structure. Namely, if P is a point of a face, then the natural complex coordinate is $z = x_1 + ix_2$, where (x_1, x_2) are standard Euclidean coordinate on $\mathbb{R}^2$. The same states for the points, lying on edges – one can unfold the vicinity of such a point to the plane and then introduce the same coordinate. If P is a vertex with total angle β then the vicinity of P can be unfolded to the plane angle of volume β; the natural coordinate on M near P is $\zeta = z^{2\pi/\beta}$, where $z = x_1 + ix_2$ is a standard coordinate on the plane. This complex structure, in particular induces the smooth structure of each polyhedral surface.

Remark 2. The natural metric on a polyhedral surface has the form $ds^2 = dzd\bar{z}$ outside the vertices; near the vertex it has the form $ds^2 = (\frac{\beta}{2\pi})^2|\zeta|^{2(\frac{\beta}{2\pi}-1)}d\zeta d\bar{\zeta}$ and has singularities at vertices. In particular, the wave equation in coordinates (y_1, y_2), $\zeta = y_1 + iy_2$ has the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{2\pi}{\beta}\left(y_1^2 + y_2^2\right)^{1-\frac{\beta}{2\pi}}\left(\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}\right).$$

The velocity of waves vanishes (if $\beta < 2\pi$) or becomes infinite (if $\beta > 2\pi$) at vertices; such a situation appear, in particular, when long waves meet small obstacles (islands or narrow hollows).

2. Definitions of Laplacians

Further we discuss properties of Laplacians and Schrödinger equations on hybrid spaces and polyhedral surfaces; in order to define the corresponding operators one has to state boundary conditions in singular points (points of gluing for hybrid spaces or vertices for polyhedra). These conditions can be defined by the following natural arguments. 1. The Laplacian must be self-adjoint. 2. On the “regular” part of the space the Laplacian must coincide with the usual one. The formal definitions have the following forms.

2.1. Hybrid spaces

Consider the direct sum $\oplus_j \Delta_j \oplus_l \frac{d^2}{dz_l^2}$, where Δ_j are Laplace–Beltrami operators on M_j and $\frac{d^2}{dz_l^2}$ are second derivatives on γ_l with Neumann boundary conditions. Let us restrict this operator to the set of functions, vanishing at the points of gluing (note that, as $\dim M_j \leq 3$, the domain of the direct sum consists of continuous functions); we will denote this restriction by Δ_0. Evidently, Δ_0 is a symmetric but not self-adjoint operator in $L^2(M)$.

Definition 3. Laplace operator on a hybrid space M is a self-adjoint extension of Δ_0.

Remark 4. The Laplacian is not unique; different operators are defined by different boundary conditions at the points of gluing. Namely, the explicit description of the corresponding domains has the following form. For each point of gluing q consider the pair (u', u), where u is the limit at q of a function on the segment, and u' is the limit of the derivative in the direction entering q. For the function on the manifold consider its asymptotics at the point q; this asymptotics has the form $u(x) = aF(x, q) + b + o(1)$, where a, b are constants and $F = -1/(4\pi d)$ if $\dim M_k = 3$ and $F = \log d/2\pi$, if $\dim M_k = 2$ (here d is the geodesic distance from x to q). Let us collect the constants (u, u', a, b) for all points of gluing and consider a vector $v = (u'_1, a_1, \ldots, u'_{2s}, a_{2s}, u_1, b_1, \ldots, u_{2s}, b_{2s}) \in \mathbb{C}^{4s} \oplus \mathbb{C}^{4s}$. Let us fix in the latter space a plane L, Lagrangian with respect to the standard skew-Hermitian form. The boundary conditions have the form $v \in L$; they can be written explicitly in terms of the unitary matrix, defining L. Further we consider only local boundary conditions – this means that the plane L supposed to be a direct sum of $2D$ planes, corresponding to different points of gluing.

2.2. Polyhedral surfaces

Consider the non-compact smooth Riemannian manifold $M_0 = M \backslash \{P_1, \ldots, P_M\}$, where P_j are vertices. Consider the usual Laplace–Beltrami operator $\tilde{\Delta}$ on $C_0^\infty(M_0)$ and let Δ_0 denote the closure of this operator with respect to the graph norm $\|\circ\|_\Delta$: $\|u\|_\Delta^2 = \|u\|^2 + \|\tilde{\Delta} u\|^2$, where $\|\circ\|$ denotes the L^2-norm. Clearly, Δ_0 is a symmetric operator in $L^2(M)$.

Definition 5. The Laplace operator on a polyhedral surface M is a self-adjoint extension of Δ_0.

Remark 6. The boundary conditions for Laplacians on polyhedral surfaces can be formulated analogous to those on hybrid spaces; the vector, defining the asymptotics of the corresponding function in vertices, must lie in a fixed Lagrangian plane.

3. Spaces of harmonic functions

Now we describe the kernel of the Laplacian Δ^L, corresponding to the Lagrangian plane L. Formulation of the result is similar for hybrid spaces and polyhedra.

Theorem 7. *The kernel of the operator Δ^L is isomorphic to the intersection $L \cap L_0$ where Lagrangian plane L_0 is defined by the singular space itself.*

Remark 8. In general position the intersection is zero, so there are no nontrivial harmonic functions. However, for hybrid spaces A.A. Tolchennikov ([1]) introduced special Laplacians with natural boundary conditions of "continuous type"; for these Laplacians he proved the estimates $b_0 \le \dim Ker\Delta^L \le b_0 + b_1$, where b_0, b_1 are Betti numbers of the graph, obtained from M by contracting all manifolds M_k to points.

For polyhedral surfaces, the plane L_0 can be expressed in terms of the Mittag-Leffler problem, corresponding to the Riemannian surface M. The kernel of the Friedrichs extension is one-dimensional and contains constants; for spherical polyhedra and Laplacians with local boundary conditions harmonic functions can be described explicitly.

4. Time-dependent Schrödinger equations on hybrid spaces. Propagation of quasi-particles

Consider the following Cauchy problem for the time-dependent Schrödinger equation on a hybrid space M

$$ih\frac{\partial\psi}{\partial t} = \Delta\psi, \quad \psi|_{t=0} = A_0(z)e^{\frac{i}{h}((z-z_0)+q_0(z-z_0)^2)},$$

where z_0 is a point of a segment γ, $h \to 0$ is a semi-classical small parameter, $q_0 \in \mathbb{C}$, $\Im q_0 > 0$, $A_0 \in C_0^\infty(\gamma)$ is a smooth cut-off function. The initial function has the form of a narrow peak, concentrated in a small vicinity of a point z_0. Asymptotics as $h \to 0$ of this problem was considered in [2, 3]; the behaviour of solution on segments is following. Consider the geodesic on γ, starting from z_0 with the fixed unit velocity. At some instant of time the geodesic meets one of the points of gluing q. At this instant consider *all* geodesics on the corresponding manifold, starting from this point with unit velocities as well as the geodesic on the initial segment, starting from q in the direction, opposite to the direction of the initial geodesic. At certain instants the geodesics meet points of gluing; we consider all geodesics (on manifolds and on segments), starting from all these points with unit velocities. Clearly, for arbitrary instant of time we will have a set of points on the segments, propagating along the geodesics, and certain surfaces on the manifolds (union of geodesic spheres), moving along the geodesics. The asymptotic solution of the Cauchy problem has the form of a number of narrow peaks, concentrated near these sets. We suppose that for arbitrary time t the number of points, appearing on the segments, is finite and denote this number by $N(t)$.

Definition 9. The number $N(t)$ is called *the number of quasi-particles* on the edges of M.

Our aim is to describe the asymptotics of $N(t)$ as $t \to \infty$.

4.1. The counting function for geodesics

The behavior of the function $N(t)$ depends essentially on the properties of the geodesic flow on M. Namely, for each pair (q_i, q_j) of the points of gluing, lying on the same manifold M_l, consider the number $m_{ij}(t)$ of different lengths of geodesics on M_l, connecting q_i and q_j and such that these lengths are at most t (let us remind, that we assume that this number is finite for arbitrary t). Note that the points q_i and q_j can coincide. Let us denote by $m(t)$ the sum of $m_{ij}(t)$ for all pairs of points, lying on the same manifolds. The asymptotics of $N(t)$ is defined by the asymptotics of $m(t)$; the latter is defined by the properties of the geodesic flows on the manifolds M_j. We will consider three different situations.

5. The finite number of geodesic lengths

The simplest situation takes place if the total number of times of geodesics is finite (such situation appears, for example, if all M_j are Euclidean or hyperbolic spaces or spheres). We denote by $L_1, \dots, L_p$ these lengths and by $l_1, \dots, l_s$ the lengths of the segments.

Theorem 10. *Let the set $L_1, \dots, L_p, l_1, \dots, l_s$ be linearly independent over the field $\mathbb{Q}$. Then the number of quasi-particles $N(t)$ has the following asymptotics as $t \to \infty$*

$$N(t) = Ct^{p+s-1} + o(t^{p+s-1}), \quad C = \frac{\sum_{j=1}^{s} l_s}{2^{2s-2}(p+s-1)! \prod_{j=1}^{s} l_j \prod_{i=1}^{p} L_i}. \tag{1}$$

Remark 11. The main step in the proof of this theorem is following: the problem of the computation of $N(t)$ can be reduced to the problem of computation of the number of lattice points in certain growing polyhedra.

6. Case of the polynomial growth of $m(t)$

Suppose that the number of geodesics $m(t)$ grows polynomially as $t \to \infty$. Such a situation takes place for the manifolds with not very complicated geodesic flow. Note that there are popular classes of such manifolds; in particular, so-called uniformly secure ones (the manifold is called uniformly secure, if there exists an integer R, such that for arbitrary pair of points all geodesics, connecting these points, can be blocked by an R-point obstacle). In this case the number of quasi-particles grows in a sub-exponential way.

Theorem 12. *Let $m(t) = c_0 t^{\gamma}(1 + O(t^{-\varepsilon}))$, $\gamma > 0, \varepsilon > 0$. Let the set of lengths L_j, l_j be linearly independent over $\mathbb{Q}$ (i.e., any finite subset of lengths is linearly independent). Then*

$$\log N(t) = (\gamma+1)\left(\frac{c_0 \Gamma(\gamma+1)\zeta(\gamma+1)}{\gamma^{\gamma}}\right)^{\frac{1}{\gamma+1}} t^{\frac{\gamma}{\gamma+1}}(1+o(1)) \tag{2}$$

Here $\Gamma(x)$ and $\zeta(x)$ are the Γ-function and the Riemann ζ-function.

7. Exponential growth of m

Finally we suppose that the function $m(t)$ grows exponentially. Note that this case is typical for geodesic flows with positive topological entropy: if M is a compact Riemannian manifold, then the topological entropy H of the geodesic flow equals

$$H = \lim_{t\to\infty} \frac{1}{t} \log \int_{M\times M} m_{x,y}(t)dxdy,$$

where $m_{x,y}(t)$ denotes the number of geodesics with the lengths at most t, connecting the points x and y. Moreover, if M does not have conjugate points (this is the case, for example, for compact surfaces of constant negative curvature), then for arbitrary pair of points x, y $H = \lim_{t\to\infty} \frac{1}{t} \log m_{x,y}(t)$.

Theorem 13. *Let* $\log m(t) = Ht(1+t^{-\varepsilon})$, $\varepsilon > 0$. *Let the set of lengths* L_j, l_j *be linearly independent over* $\mathbb{Q}$ *(i.e., any finite subset of lengths in linearly independent). Then*

$$\log N(t) = Ht(1 + o(1)). \tag{3}$$

8. Abstract prime numbers distributions

The main steps in the proofs of the theorems of Sections 6, 7 are following: the problem of the computation of $N(t)$ can be reduced to certain problem of the analytic number theory. Namely, consider an arithmetic semigroup $G = \oplus_{j\in J}\mathbb{Z}_+$, where J is a countable set, and a homomorphism $\rho : G \to \mathbb{R}_+$, such that for arbitrary $t \in \mathbb{R}_+$ the set of elements $g \in G$ with $\rho(g) \le t$ is finite. We can identify elements $j \in J$ with the corresponding generators of $\mathbb{Z}_+$. Consider two functions $m(t) = \sharp\{j \in J|\rho(j) \le t\}$, $N(t) = \sharp\{g \in G, \rho(g) \le t\}$. The direct (inverse) problem of abstract primes distribution is the following question. If one knows the asymptotics of N (m), how to compute the asymptotics of m (N)?

Remark 14. If J is the set of primes and $\rho(j)=\log j$, then $m(\log t)$ is the distribution function of primes and $N(t)$ is the integral part of t.

If J is the set of integers and $\rho(j) = j$ then $N(t)$ is the number of partitions of integer t and $m(t) = t$.

The result of the previous sections follow from the results which give the solution of the inverse problem of abstract primes distribution for the cases of polynomial and exponential growths of $m(t)$.

References

[1] Tolchennikov A.A. *The kernel of Laplace–Beltrami operators with zero-radius potential or on decorated graphs.* Sbornik: Mathematics(2008), **199**(7):1071

[2] Chernyshev V.L., Shafarevich A.I. 2013 *Statistics of gaussian packets on metric and decorated graphs*, Philosophical transactions of the Royal Society A., Volume: **372**, Issue: 2007, Article number: 20130145.

[3] Chernyshev V.L., Shafarevich A.I. *Semiclassical asymptotics and statistical properties of Gaussian packets for the nonstationary Schrödinger equation on a geometric graph.* Russ. J. Math. Phys., 2008, **15**, 25–34.

[4] Tolchennikov A.A, Chernyshev V.L., Shafarevich A.I. *Behavior of Quasi-particles on Hybrid Spaces. Relations to the Geometry of Geodesics and to the Problems of Analytic Number Theory.* Regular and Chaotic Dynamics, Vol. **21**, No. 5, 2016, 531–538.

Andrei Shafarevich
"M.V. Lomonosov" Moscow State University
Leninskie Gory 1
119992 Moscow, Russia
e-mail: shafarev@yahoo.com